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THE NON POLYNOMIAL STRUCTURE OF SUPERSYMMETRIC CHIRAL ANOMALIES

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ABSTRACT

Supersymmetric chiral anomalies cannot be expressed as polynomials in the superconnection coefficients and suitable generalizations thereof which transform like the adjoint representation under global complex transformations.

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I. INTRODUCTION

Recently, several forms of the supersymmetric chiral anomalies have been proposed using different methods^{1),2),3),4)}. Their derivations fall essentially into two categories : explicit calculations which manifestly fulfill supersymmetry^{1),2)} and algebraic constructions which solve the consistency condition^{3),4)}.

We are guaranteed that both methods, if properly applied, should give the anomaly thanks to the known existence and uniqueness results previously established⁵⁾.

So far, however, all proposed formulae are rather complicated, in particular they have a non-polynomial structure in the (dimension zero) prepotential superfield $V(x, \theta, \bar{\theta})$ of the chiral gauge group. On the other hand one may hope that the non polynomial character of the presently known forms of the anomalies can be eliminated if the result is expressed in terms of different geometrical objects such as the connection superfields and generalizations thereof.

The purpose of the present paper is to show that this is not the case.

In section II we fix the notations, recall the main definitions, and precisely state the no-go theorem to be proved.

The proof to follow is divided into two parts described respectively in sections III and IV.

An appendix is devoted to the derivation of an explicit form for the anomaly correct to order V^3 .

II. NOTATIONS, STATEMENT OF THE MAIN THEOREM

Let G be the structure group which we assume to be a compact, semi-simple, Lie group ; $\text{Lie } G$ its Lie algebra ; V the $\text{Lie } G$ valued real superfield in terms of which the supersymmetric gauge theory is formulated. The corresponding gauge group $\mathcal{G}$ is the chiral complex gauge group. Its Lie algebra $\text{Lie } \mathcal{G}$ is parametrized by $\text{Lie } G$ valued chiral, (resp. antichiral), superfields ω , (resp. $\bar{\omega}$). Associated to the gauge superfield V are the Maurer Cartan forms:

$$X = W^{-1} D_{\gamma} W \quad , \quad \bar{X} = D_{\bar{\gamma}} W W^{-1} \quad (\text{II.1})$$

where

$$W = e^V \quad (\text{II.2})$$

and

$$\dot{} = \alpha, \dot{\bar{}}, (\alpha \dot{\alpha}). \quad (\text{II.3})$$

$$D_\gamma = D_\alpha, \bar{D}^{\dot{\alpha}}, \partial_{\alpha \dot{\alpha}}. \quad (\text{II.4})$$

The chiral anomaly $\mathcal{A}$ is of the form

$$\mathcal{A}(\omega, \bar{\omega}) = \int d^4x d^4\theta \alpha(\omega, \bar{\omega}, \cdot) \quad (\text{II.5})$$

where α is linear in $\omega, \bar{\omega}$, local - in superspace - in its dependence on V denoted by the dot:

$$\mathcal{A}(\omega, \bar{\omega}) = \int d^4x d^4\theta \left[\omega^a \alpha_a(\cdot) + \bar{\omega}^{\bar{a}} \bar{\alpha}_{\bar{a}}(\cdot) \right] \quad (\text{II.6})$$

The consistency condition reads

$$s \mathcal{A} = 0 \quad (\text{II.7})$$

where s is induced by

$$\begin{aligned} s \omega &= -\frac{1}{2} [\omega, \omega] & s \bar{\omega} &= -\frac{1}{2} [\bar{\omega}, \bar{\omega}] \\ s W &= W \omega - \bar{\omega} W \end{aligned} \quad (\text{II.8})$$

Condition (II.7) means that $\mathcal{A}$ is an integrated local cocycle of Lie $\mathfrak{g}$ with values in the space of local functionals of V . $\mathcal{A}$ is furthermore known to be defined up to the gauge variations of local functionals:

$$\mathcal{A} \text{ defined modulo } \mathcal{A}^{\text{trivial}} = s T^{\text{loc}}(\cdot) = s \int d^4x d^4\theta \mathcal{T}^{\text{loc}}(\cdot) \quad (\text{II.9})$$

Recall that $X_\alpha, \bar{X}^{\dot{\alpha}}$ are components of superconnections and transform accordingly purely with $\omega, \bar{\omega}$, but that the other components transform under mixed action of $\omega, \bar{\omega}, WD_\gamma \omega W^{-1}, W^{-1}D_\gamma \bar{\omega} W$. On the other hand, the non gradient terms reduce to the adjoint action of ω on X , $\bar{\omega}$ on $\bar{X}$.

We shall now consider possible anomalies for which the integrands $\mathcal{A}(\cdot), \bar{\mathcal{A}}(\cdot)$ are polynomials in $X, \bar{X}$ and eventually show that all such are trivial, at least when the usual power counting conditions are met; namely, we shall prove the following

Theorem.

All anomalies (Eq. II.6) which fulfill the consistency conditions (eq. II.7), which are local and polynomial in $X, \bar{X}$ and their derivatives, of canonical dimension 2, are trivial. (The canonical dimension of a monomial is computed with $\dim V = 0, \dim D_\alpha = \dim \bar{D}^{\dot{\alpha}} = \frac{1}{2}, \dim \partial_{\alpha\dot{\alpha}} = 1$, e.g. $\dim X_\alpha = \dim X_{\dot{\alpha}} = \dim \bar{X}^{\dot{\alpha}} = \dim \bar{X}^\alpha = \frac{1}{2}$).

The proof will be divided in two parts. First the polynomial form of the anomaly will be reduced to a simple form through iterated use of the Whitehead cohomological lemma⁶⁾ which states the vanishing of the first cohomology of a semi-simple Lie algebra with values in a finite dimensional representation space. This reduction is performed in section III. The remaining necessary work has already been performed, as it turns out, in the course of the proof of the Piguet-Sibold theorem⁵⁾. This is explained in section IV and concludes the proof of the theorem.

III. TWO COHOMOLOGICAL LEMMAS

The first two steps are supersymmetric extensions of part of the analysis of ordinary chiral anomalies⁷⁾.

Lemma 1.

By a trivial redefinition (II.9), which respects polynomiality, $\mathcal{A}(\omega, \bar{\omega})$ can be made to vanish for constant complex $\omega, \bar{\omega}$.

Proof⁷⁾

For $\omega, \bar{\omega}$ constant complex, $\mathcal{A}(\omega, \bar{\omega})$ is a cocycle for $\text{Lie } G^C$, where G^C is the complexified of G , with values in the finite dimensional space of integrated local polynomials of dimension 2 in $X, \bar{X}$ and their derivatives, which transform under the adjoint action of $\text{Lie } G^C$. By Whitehead's lemma⁶⁾

$$\mathcal{A}(\omega, \bar{\omega}) = s_{\text{global}} \Gamma^{\text{loc.}}(X, \bar{X}) \quad (\text{III.1})$$

$\omega, \bar{\omega}$ complex, constant

where s_{global} is the restriction of s to $\text{Lie } G^c$ and $\Gamma^{\text{loc.}}$ is an integrated polynomial in $X, \bar{X}$ and their derivatives, of dimension 2.

Define

$$\tilde{\mathcal{A}}(\omega, \bar{\omega}) = \mathcal{A}(\omega, \bar{\omega}) - s \Gamma^{\text{loc.}} \quad (\text{III.2})$$

Obviously $\tilde{\mathcal{A}}$ vanishes for $\omega, \bar{\omega}$ constant complex. More precisely, its dependence on $\omega, \bar{\omega}$ is only due to the gradient terms occurring in $sX, s\bar{X}$, so that, after integrations by parts one finds $\tilde{\mathcal{A}}$ of the form

$$\begin{aligned} \tilde{\mathcal{A}}(\omega, \bar{\omega}) = & \int d^4x d^4\theta \left[D_\mu \omega^a \alpha_a^\mu(x, \bar{X}) + \bar{D}^{\dot{\alpha}} \bar{\omega}_{\dot{a}} \bar{\alpha}_{\dot{a}}^{\dot{\alpha}}(x, \bar{X}) + \right. \\ & \left. + (W D_\mu \omega W^{-1})^{\bar{a}} \mathcal{B}_{\bar{a}}^\mu(x, \bar{X}) + (W^{-1} \bar{D}^{\dot{\alpha}} \bar{\omega} W)_a \bar{\mathcal{B}}_a^{\dot{\alpha}}(x, \bar{X}) \right] \end{aligned}$$

(III.3)

Where the α 's and $\mathcal{B}$'s are polynomials of dimension 2 in $X, \bar{X}$ and their derivatives

Lemma 2.

The densities α and $\mathcal{B}$ of eq.(III.3) can be chosen to transform under the adjoint of $\text{Lie } G^c$.

Proof⁷⁾

One writes the consistency relation for one set of local parameters and one set of global parameters. It just says that, under global transformation, $\mathcal{A}(\omega, \bar{\omega})$ is invariant if $\omega, W^{-1}\bar{\omega}W, X$ transform under the adjoint, $\bar{\omega}, W\omega W^{-1}, \bar{X}$ under the complex conjugate adjoint. Since these adjoint actions commute with integration $\int d^4x d^4\theta$ and the densities can be fully decomposed into an invariant part and a non invariant part, it follows from Schur's lemma that the integrated

non invariant part vanishes.

Now, invariance of the integrand allows to get rid of the explicit W dependence in front of the $\mathcal{B}$ terms and we end up with the following form for the anomaly:

$$\tilde{\mathcal{A}}(\omega, \bar{\omega}) = \int d^4x d^4\theta \left[\mathcal{J}(D_\alpha \omega, X) + \bar{\mathcal{J}}(\bar{D}^{\dot{\alpha}} \bar{\omega}, \bar{X}) \right] \quad (\text{III.4})$$

where $\mathcal{J}$, $\bar{\mathcal{J}}$ are invariant under the action of $\text{Lie } G^C$, $\text{Lie } \bar{G}^C$ respectively, polynomial of dimension $3/2$ in X and its derivatives, $\bar{X}$ and its derivatives respectively.

IV. TRIVIALITY OF THE CONSISTENT POLYNOMIAL ANOMALY

In the present section we conclude the proof of the theorem stated in section II, i.e. that the only consistent anomaly which is polynomial in the objects X 's and their conjugates $\bar{X}$ is trivial, i.e. is the gauge variation of a local functional.

In order to show this we only need to write for the consistent anomaly as defined by eq. (III.4) the invariants which are linear in the d -symbols of the structure group G . The reason is the following : in order that the supersymmetric anomaly be non-trivial it is necessary and sufficient, due to the uniqueness theorem⁵⁾, that it contains, for the vector component $A_\mu(x)$ of the gauge-prepotential, the canonical chiral anomaly⁸⁾, which is obtained by setting $\omega(x, \theta) = \omega(x)$ with $\omega(x)$ real. The canonical chiral anomaly is indeed, up to gauge variations of local functionals of $A_\mu(x)$, linear in the d -symbol. Now it is straightforward to show that any invariant expression $\mathcal{J}(D_\alpha \omega, X)$ not containing a d -symbol, cannot give rise, when expanded in power series of the V -prepotential, to terms quadratic and cubic in V which contain the d -symbol. Those terms are in fact the only terms which contribute to the canonical chiral A_μ anomaly.

We can conclude that the only polynomials in X 's which may possibly give rise to a consistent anomaly are the following:

$$d_{abc} D_\alpha \omega^a \left({}^\alpha \mathcal{J}_2^{bc} + {}^\alpha \mathcal{J}_3^{bc} \right) \quad (\text{IV.1})$$

where ${}^\alpha \mathcal{J}_2^{bc}$ is quadratic in the X 's :

$$\mathfrak{J}_2^{bc} = a \bar{D}^{\dot{\alpha}} X_{\dot{\alpha}}^b X^{\alpha c} + b X_{\dot{\alpha}}^b \bar{D}^{\dot{\alpha}} X^{\alpha c} + c D^{\alpha} X_{\dot{\alpha}}^b X^{\dot{\alpha} c}, \quad (\text{IV.2})$$

$$\mathfrak{J}_3^{bc} = X_{\dot{\alpha}}^b f^{cde} X_d^{\dot{\alpha}} X_e^{\alpha} \quad (\text{IV.3})$$

Symmetry in the b, c indices and R-symmetry⁵⁾ uniquely fix the possible structure of the $\mathfrak{J}_3^{bc}$ functional. It is obvious that the part quadratic in A_{μ} of the consistent anomaly can be conveniently fixed since we know that a consistent abelian anomaly can be fully expressed in terms of the X 's (see appendix). The non-trivial part is the cubic term $d_{abc} \epsilon_{\mu\nu\rho\sigma} \partial_{\mu} \omega^a A_{\nu}^b [A_{\rho}, A_{\sigma}]^c$. We will show that such a term cannot be contained in $\mathfrak{J}_2^{bc}$ or $\mathfrak{J}_3^{bc}$.

To find such a term it is sufficient to expand $X_{\alpha}, X_{\dot{\alpha}}$ up to order V^2 i.e.

$$\begin{aligned} X_{\alpha} &= D_{\alpha} V - \frac{1}{2} [V, D_{\alpha} V] + O(V^3) \\ X_{\dot{\alpha}} &= \bar{D}_{\dot{\alpha}} V - \frac{1}{2} [V, \bar{D}_{\dot{\alpha}} V] + O(V^3) \end{aligned} \quad (\text{IV.4})$$

A term containing the Levi-Civita ϵ -symbol can only come, in superspace, by a full trace of four σ_{μ} matrices. This immediately implies that the only possible term which may give rise to an ϵ -term trilinear in V can be obtained by the second term in $\mathfrak{J}_2^{bc}$. All other terms contain traces of two σ -matrices. However the second term in $\mathfrak{J}_2^{bc}$ can be simplified by partial-integration to:

$$\bar{D}^{\dot{\alpha}} (X_{\dot{\alpha}}^b X^{\alpha c})$$

which finally gives

$$\int d^4x d^4\theta \partial_{\alpha\dot{\alpha}} \omega^a X^{\dot{\alpha} b} X^{\alpha c} d_{abc} \quad (\text{IV.5})$$

The term trilinear in V contained in (IV.5) is

$$d_{abc} \int d^4x d^4\theta \partial_{\alpha\dot{\alpha}} \omega^a \left(\bar{D}^{\dot{\alpha}} V^b [V, D^{\alpha} V]^c + [V, \bar{D}^{\dot{\alpha}} V]^b D^{\alpha} V^c \right) \quad (\text{IV.6})$$

By invariance of the $\mathcal{J}$ functional this can be rewritten as

$$\int d^4x d^4\theta d_{abc} [\partial_{\alpha\dot{\alpha}} \omega, V]^a \bar{D}^{\dot{\alpha}} V^b D^{\alpha} V^c \quad (\text{IV.7})$$

which finally shows the symmetry in the last two indices. Since the space-time derivative acts on the chiral parameter ω^a the V -terms contribute only to A_{μ}^- terms without derivative which therefore cannot be totally antisymmetric. In fact the A_{μ}^3 -term coming from expression (IV.7) can be shown to be completely absent because of the global invariance of the $\mathcal{J}$ functional.

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APPENDIX

In the present appendix we show that the abelian anomaly (i.e. the quadratic part of the non-abelian anomaly) can always be chosen in such a way that it reduces to a total superspace-derivative of an expression containing only the (linearized) X_α 's, $\bar{X}_\alpha$'s. The cubic term, giving the full A_μ chiral anomaly, can be computed by means of the method of ref.⁵⁾. We give the expression of such a term. All terms of higher order in V do not contribute to the canonical chiral A_μ anomaly.

From ref. 5) we know that the abelian part of the consistent anomaly can be chosen to be

$$d_{abc} \int d^4x d^4\theta \left(\omega^a D^\alpha V^b \bar{D}\bar{D} D_\alpha V^c - \bar{\omega}^a \bar{D}_\alpha V^b D D \bar{D}^\alpha V^c \right) \quad (A.1)$$

By adding the (abelian) gauge variation of the following local functional

$$A^{\text{loc}} = \frac{1}{2} \int d^4x d^4\theta d_{abc} V^a V^b D^\alpha \bar{D}\bar{D} D_\alpha V^c \quad (A.2)$$

the quadratic part of the consistent anomaly can be written as a total superspace-divergence

$$\begin{aligned} A_2 = d_{abc} \int d^4x d^4\theta & \left(D^\alpha \omega^a \left[\bar{D}^2 V^b D_\alpha V^c + 2 \bar{D}_\alpha V^b \bar{D}^\alpha D_\alpha V^c \right] - \right. \\ & \left. - \bar{D}_\alpha \bar{\omega}^a \left[D^2 V^b \bar{D}^\alpha V^c + 2 D^\alpha V^b D_\alpha \bar{D}^\alpha V^c \right] \right) \end{aligned} \quad (A.3)$$

The cubic term in V , $\mathcal{A}_3$, of the anomaly which makes it consistent up to this order is obtained by solving the consistency condition

$$S_1 \mathcal{A}_2 + S_0 \mathcal{A}_3 = 0 \quad (A.4)$$

where

$$S V = \omega - \bar{\omega} + \frac{1}{2} [V, \omega + \bar{\omega}] + \dots \equiv S_0 V + S_1 V + \dots$$

$$S \omega = -\frac{1}{2} [\omega, \omega] \equiv S_1 \omega$$
(A.5)

One finds

$$\begin{aligned} \mathcal{A}_3 = d_{abc} \int d^4x d^4\theta \left(\frac{1}{4} [\omega + \bar{\omega}, V]^a [D^\alpha V^b \bar{D}^2 D_\alpha V^c + \right. \\ \left. + \bar{D}_\alpha V^b D^2 \bar{D}^\alpha V^c - V^b D^\alpha \bar{D}^2 D_\alpha V^c - D^\alpha \bar{D}^2 D_\alpha (V^b V^c)] \right. \\ \left. - \frac{3}{4} \omega^a \bar{D}^2 D^\alpha V^b [V, D_\alpha V]^c - \frac{3}{4} \bar{\omega}^a D^2 \bar{D}_\alpha V^b [V, \bar{D}^\alpha V] \right) \end{aligned}$$
(A.6)

For the canonical anomaly, which is obtained by setting

$$V(x, \theta, \bar{\theta}) = 2 \theta \sigma^\mu \bar{\theta} A_\mu(x), \quad \omega(x, \theta) = \omega(x), \quad \omega \text{ real}$$

one finds the correct result

$$\begin{aligned} \mathcal{A}(\omega; A) = i 512 d_{abc} \varepsilon^{\mu\nu\rho\sigma} \int d^4x \partial_\mu \omega^a \left\{ A_\nu^b \partial_\rho A_\sigma^c + \right. \\ \left. + \frac{i}{4} A_\nu^b [A_\rho, A_\sigma]^c \right\} \end{aligned}$$
(A.7)

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