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QUANTUM VORTICES

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E R R A T U M

Page 17 : θ should read $\exp i\theta$ in the last seven lines.

Page 26 : The equation at the head of the page should be numbered (5.46).

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A B S T R A C T

We make the contribution of vortices to the partition function of the Abelian Higgs model explicit without resorting to approximation. The formulation is a gauge invariant quantization that generalizes to non-Abelian gauge theories. We also calculate an effective potential that is of physical significance because it is guaranteed gauge invariant.

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1. - INTRODUCTION

One of the main difficulties of continuum quantum field theory is how to take account of non-perturbative effects. Many theories with gauge symmetry contain topologically non-trivial field configurations which cannot contribute to a perturbation expansion because, by definition, they cannot be built up from a vacuum configuration by continuously varying the fields. The topological structure guarantees the existence of non-trivial solutions to the classical equations of motion, for example vortices in the Ginzburg-Landau model, magnetic monopoles in the non-Abelian Higgs model and instantons in pure Yang-Mills theories in four dimensions. When these solutions are known they can be made the basis for a semiclassical quantization of the appropriate theory in which the contribution of each homotopy class of fields is represented by the neighbourhood of the corresponding classical solution. This is only an approximation which may not be relevant to the physics and may also be intractable, nonetheless the non-trivial configurations exist and should be taken account of somehow. We describe a method for making explicit their contribution to the functional integral of the Abelian Higgs model without approximation. Our approach can readily be extended to an arbitrary gauge theory. It is based on a treatment of the degrees of freedom corresponding to gauge transformations in which they are integrated out of the functional integral. This results in a formulation entirely in terms of gauge invariant fields which correspond to the physical excitations of the theory.

This is not the usual approach of the Faddeev-Popov method¹⁾ which consists of replacing the integral over all gauge field configurations by one in which each gauge equivalence class is represented by a single member chosen so as to satisfy a gauge condition. The resulting formulation loses manifest gauge invariance and may also cease to be manifestly Lorentz-invariant or require the introduction of non-physical ghost fields. Because these symmetries are no longer obvious, any further manipulation, for example the introduction of a regulator, must be checked to see that it preserves all the required symmetries of the theory. Furthermore whilst the gauge condition may choose a unique representative of each gauge equivalence class locally, the choice may not be unique globally, leading to the Gribov ambiguity²⁾. Of course these problems are not as severe for the Abelian model we discuss here as they are for non-Abelian gauge theories although our approach may also be applied to the latter. Indeed we have chosen the model because despite its mathematical simplicity it resembles the non-Abelian case in two important respects. Firstly there is an interaction with the scalar field that is quadratic in the gauge field. This mimics the self-interaction of a non-Abelian gauge theory which may be written as a term quadratic in

the gauge field and linear in the field strength. Secondly both theories have field configurations with non-trivial topology.

The main advantage of our approach is that it is a formulation in terms of gauge invariant fields. The variables in which the usual formalism is couched do not necessarily correspond to the physical excitations of the theory, only gauge invariant quantities are physically observable but these must be constructed out of the fields appearing in the functional integral and so arise as secondary objects which makes the physical content of the theory difficult to unravel. For example, the particle spectrum generated by the Higgs mechanism is obvious at the Lagrangian level only for a particular choice of gauge condition. Another of the examples we will discuss is the calculation of effective potentials. This was first done for the Abelian Higgs model by Coleman and Weinberg³⁾, however, it was pointed out by Jackiw⁴⁾ that their result was gauge dependent and so its physical interpretation was unclear. Because our formalism is trivially gauge invariant we are able to calculate a gauge invariant effective potential.

In three dimensions the classical Abelian Higgs model has particle-like solutions that are stable because they are topologically non-trivial. In condensed matter physics these are the vortices of the Ginzburg-Landau phenomenological model of superconductivity⁵⁾. As was pointed out by Nielsen and Olesen⁶⁾ in four dimensions these configurations become strings. Thus the model is of interest both from the point of view of soliton quantization and also because the strings emerge from a renormalizable quantum field theory which must somehow cure their well-known pathologies. As we have said, these configurations will not contribute to perturbation theory. A semiclassical approximation is not possible because we do not have a complete set of solutions to the equations of motion. For certain values of the couplings some solutions are known⁷⁾ together with the number of parameters on which the complete set depends⁸⁾. The number of parameters equals the number of position co-ordinates needed to specify the vortices and it is physically reasonable that these co-ordinates could be taken as the parameters. In principle a semiclassical treatment would reduce the functional integral to one over these positions. As we will see this greatly underplays the role of the vortex positions because they do not just describe the classical configurations but rather they parametrize the topology of arbitrary field configurations.

Because our method of treating the gauge degrees of freedom results in a formalism expressed only in terms of gauge invariant fields, which are directly related to the physical excitations of the theory, the vortex parameters appear

automatically. The functional integral is rewritten as an integral over these parameters without resorting to any approximation. Of course, to make further progress we may have to employ some approximation scheme but our exact treatment already enables us to see how Nielsen-Olesen vortices may overcome some of the problems of the dual string.

We begin by formulating scalar QED in terms of the field strength rather than the gauge potential. In Section 3 we discuss the approximation that the scalar field ϕ may be expanded about some constant value enabling us to write the functional integral description of the Higgs mechanism entirely in terms of gauge invariant fields. In Section 4 we define an effective potential that is of physical interest because in our formalism it is guaranteed to be gauge invariant. We discuss in Section 5 the general case in which ϕ cannot be assumed to be close to a constant value everywhere, but may vanish on a set of surfaces associated with topologically non-trivial configurations of its phase, and we are able to make the vortex degrees of freedom explicit.

2. - FIELD STRENGTH FORMULATION OF SCALAR QED

The model we will discuss consists of a complex self-interacting scalar field coupled to electromagnetism in D Euclidean dimensions. The action is

$$S = \int d^D x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \overline{D_\mu \phi} D_\mu \phi + V(\bar{\phi}\phi) \right) \quad (2.1)$$

where

$$D_\mu \phi \equiv (\partial_\mu - ie A_\mu) \phi \quad (2.2)$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.3)$$

For

$$V(\bar{\phi}\phi) = m^2 \bar{\phi}\phi + \frac{\lambda}{4!} (\bar{\phi}\phi)^2; \quad m^2 < 0, \lambda > 0 \quad (2.4)$$

this is essentially the Abelian Higgs model⁹⁾ in D = 4 dimensions and the Ginzburg-Landau phenomenological theory of superconductivity⁵⁾ for D = 3. The classical equations of motion implied by the action are

$$\partial_\mu F_{\mu\nu} = ie (\bar{\varphi} D_\nu \varphi - \overline{D_\nu \varphi} \varphi) \equiv ie \bar{\varphi} \vec{D}_\nu \varphi \quad (2.5)$$

$$D_\mu D_\mu \varphi = \varphi V'(\bar{\varphi} \varphi) \quad (2.6)$$

The action and equations of motion are invariant under the gauge transformation specified by the continuous function χ

$$A \rightarrow A + \partial \chi \quad (2.7)$$

$$\varphi \rightarrow e^{ie\chi} \varphi, \quad \bar{\varphi} \rightarrow e^{-ie\chi} \bar{\varphi} \quad (2.8)$$

The quantum theory may be defined by the partition function

$$Z[J, \kappa] = \int \mathcal{D}A \mathcal{D}\bar{\varphi} \mathcal{D}\varphi \exp(-S - S_{J, \kappa}) \quad (2.9)$$

where $S_{J, \kappa}$ contains source terms coupled to the fields so as to preserve gauge invariance

$$S_{J, \kappa} = \int d^4x (\kappa_\mu A_\mu + J \bar{\varphi} \varphi), \quad \partial_\mu \kappa_\mu = 0 \quad (2.10)$$

The functional integration over the fields includes variations in the configuration due to gauge transformations. Since they leave the integrand unchanged these modes contribute a source independent factor to Z . This factor is infinite and must be isolated from the partition function to make the latter well-defined. Usually this is done by gauge fixing. The purpose of this paper is to describe an alternative method which enables the gauge degrees of freedom to be explicitly separated out of the functional integral. The method is sufficiently general to be applied to non-Abelian gauge theories, however these are more complicated algebraically so here we will concentrate on the model (2.1).

Our aim is to rewrite the theory in terms of variables which directly parametrize those changes in the field configuration due to gauge transformations and those that are physical. For the physical variables we take the field strength and the modulus of ϕ because these are both gauge invariant. For the

remaining variable we take the phase of ϕ , θ as this has a simple transformation under χ :

$$\varphi = \rho e^{i\theta}$$

$$\rho \rightarrow \rho, \quad \theta \rightarrow \theta + e\chi. \quad (2.11)$$

There is one subtlety associated with taking θ as an integration variable. Although the path integral formulation (2.9) presupposes that the field configurations $(A, \bar{\phi}, \phi)$ are continuous, and hence that ρ is continuous, θ may have singularities if ρ vanishes without spoiling the continuity of ϕ . These singularities can contain gauge invariant, and thus physical information. Since we only allow gauge transformations, χ , that are continuous it is clear that any continuous variation in θ may be eliminated by an appropriate choice of χ but that discontinuities may not. Thus the functional integration over the phase of the field must be split into two parts, one of which is an integration over the singular part of θ , the other is over the continuous variations in θ which are essentially the gauge degrees of freedom, χ , and may be factored out of the integral. We will postpone a proper discussion of this problem to Section 5 and concentrate here on transforming the integration over the gauge field A to one over the field strength.

Many authors have attempted to construct field strength formulations of gauge theories¹⁰⁾, particularly for Yang-Mills theories, by inverting the relation (2.3). The problem is that it is non-local (and for a non-Abelian theory non-linear) and of course does not give a unique solution because F is unchanged by gauge transforming A . A clever choice of gauge condition¹¹⁾ can remove the non-linearity in the non-Abelian case but still the result is non-local and in any case we need to be able to change variables in the functional integral without gauge-fixing. Rather than invert (2.3) we choose to invert the equation of motion (2.5). This has several advantages. The equation is local and linear in A_μ and remains so in the non-Abelian case. Inverting it is just a matter of algebra and the corresponding operation in the quantum theory is straightforward, as we will see later. The solution is

$$A_\nu = (\partial_\mu F_{\mu\nu} - ie \bar{\phi} \vec{\partial}_\nu \phi) / (2e^2 \bar{\phi} \phi) \quad (2.12)$$

To obtain a classical action in terms of F rather than A we must start with a formulation in which F enters as a variable rather than as the definition (2.3)

$$S_1 = \int d^4x \left(F_{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) / 2 - F_{\mu\nu} F_{\mu\nu} / 4 + \overline{D_\mu \varphi} D_\mu \varphi + V(\bar{\varphi} \varphi) \right) . \quad (2.13)$$

The Euler-Lagrange equations obtained by varying F are (2.3) the others are as before, in particular we again have (2.12) which we use to eliminate the A from S_1 in favour of the dynamical variable F to obtain:

$$S_1 = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \partial_\mu \varphi \partial_\mu \varphi + V(\varphi^2) - \partial_\mu F_{\mu\nu} \partial_\lambda F_{\lambda\nu} / (4e^2 \varphi^2) - \frac{i}{2e} F_{\mu\nu} \partial_\mu \left(\frac{\bar{\varphi} \overleftrightarrow{\partial}_\nu \varphi}{\bar{\varphi} \varphi} \right) \right), \quad (2.14)$$

where we have dropped a total divergence

$$\partial_\mu \left(F_{\mu\nu} \partial_\lambda F_{\lambda\nu} / (2e^2 \bar{\varphi} \varphi) \right), \quad (2.15)$$

and set

$$\varphi^2 = \bar{\varphi} \varphi . \quad (2.16)$$

The final term in (2.14) couples the field strength to singularities in the phase of φ , it vanishes in regions where the phase is non-singular and continuous because then

$$F_{\mu\nu} \partial_\mu \left(\frac{\bar{\varphi} \overleftrightarrow{\partial}_\nu \varphi}{\bar{\varphi} \varphi} \right) = 2i F_{\mu\nu} \partial_\mu \partial_\nu \theta = 0 . \quad (2.17)$$

We will discuss this term further in Section 5.

It is straightforward to carry this argument over to the quantum theory. Insert into the partition function (2.9) the following identity which is valid for any A

$$\int \mathcal{D}f \exp \left(- \int d^4x \frac{i}{4} (f_{\mu\nu} - i[\partial_\mu A_\nu - \partial_\nu A_\mu]) (f_{\mu\nu} - i[\partial_\mu A_\nu - \partial_\nu A_\mu]) \right) = 1 , \quad (2.18)$$

$f_{\mu\nu}$ is an antisymmetric tensor. The expression for Z becomes

$$\begin{aligned}
 Z[J, K] = & \int \mathcal{D}f \mathcal{D}A \mathcal{D}\bar{\phi} \mathcal{D}\phi \exp \left(- \int d^4x \left[\frac{1}{4} f_{\mu\nu} f_{\mu\nu} \right. \right. \\
 & + \partial_\mu \bar{\phi} \partial_\mu \phi + i A_\mu (\partial_\lambda f_{\lambda\mu} + e \bar{\phi} \vec{\partial}_\mu \phi + \kappa_\mu) + e^2 A_\mu A_\mu \bar{\phi} \phi \\
 & \left. \left. + V(\bar{\phi} \phi) + J \bar{\phi} \phi \right) \right]
 \end{aligned} \tag{2.19}$$

where we have ignored a total derivative. Integrating over A gives (again with the neglect of surface terms)

$$\begin{aligned}
 Z[J, K] = & \int \mathcal{D}f \mathcal{D}\bar{\phi} \mathcal{D}\phi \Gamma \exp - \int d^4x \left(\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{1}{4e^2} \partial_\lambda f_{\lambda\mu} \partial_\nu f_{\nu\mu} \right. \\
 & - \frac{1}{2e} f_{\lambda\mu} \partial_\lambda (\bar{\phi} \vec{\partial}_\mu \phi / (\bar{\phi} \phi)) + (\partial_\mu \varphi)^2 + V(\varphi^2) + \kappa^2 / (4e^2 \varphi^2) \\
 & \left. + J \varphi^2 + \kappa_\mu (\partial_\lambda f_{\lambda\mu}) / (2e^2 \varphi^2) \right),
 \end{aligned} \tag{2.20}$$

where Γ is a product over all space-time points

$$\Gamma = \prod_x \varphi^{-D} \tag{2.21}$$

This series of Gaussian integrations is essentially the Stratonovitch-Hubbard transformation¹²⁾ of statistical mechanics.

The integrand of (2.20) is manifestly gauge invariant because it is written in terms of gauge invariant variables. To complete our programme we have to isolate from $\mathcal{D}\bar{\phi} \mathcal{D}\phi$ the variables that parametrize gauge transformations - essentially the non-singular part of the phase of ϕ . We will do this in two stages. Firstly we will discuss the approximation that ϕ may be expanded about some constant, non-zero, value. This simplification ignores the subtleties mentioned earlier but enables us to treat the Higgs mechanism and to generalize the effective potential of Coleman and Weinberg. Secondly we will give a more careful treatment of the integration over θ that shows how the contribution to the partition function of topologically non-trivial field configurations can be made manifest without resorting to an approximation.

3. - THE HIGGS MECHANISM

If in (2.1) we take the potential (2.4) appropriate to spontaneous symmetry breakdown

$$V(\rho^2) = m^2 \rho^2 + \frac{\lambda}{4!} \rho^4 \quad ; \quad m^2 < 0 \quad , \quad \lambda > 0 \quad , \quad (3.1)$$

then we can simplify our functional integral (2.20) by assuming that ρ lies close to the position independent minimum of V , ρ_0 . Since ρ_0 is non-vanishing

$$\rho_0 = \sqrt{-12m^2/\lambda} \quad (3.2)$$

the phase θ of ϕ cannot have any singularities and we can neglect the term

$$f_{\lambda\mu} \partial_\lambda ((\bar{\phi} \vec{\partial}_\mu \phi) / (\bar{\phi} \phi)) \quad (3.3)$$

in the exponent of (2.20). Furthermore

$$d\bar{\phi}(x) d\phi(x) = \rho(x) d\rho(x) d\theta(x) \quad (3.4)$$

which implies that we can write

$$\mathcal{D}\bar{\phi} \mathcal{D}\phi = \mathcal{D}\rho \mathcal{D}\theta \prod_x \rho \quad (3.5)$$

For the time being we will drop source terms so that Z becomes

$$Z = \int \mathcal{D}\theta \mathcal{D}f \mathcal{D}\rho \tilde{F} \exp \left(- \int d^4x \left(\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \partial_\lambda f_{\lambda\mu} \partial_\nu f_{\nu\lambda} \frac{1}{4e^2 \rho^2} + (\partial\rho)^2 + V(\rho^2) \right) \right) \quad (3.6)$$

The integral over θ can now be done trivially, it gives an infinite constant which can be factored out of Z . We have arrived at a formulation of scalar QED entirely in terms of gauge invariant fields

$$\begin{aligned} Z &= \int \mathcal{D}f \mathcal{D}\rho \tilde{F} \exp - \int d^4x \left(\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{1}{4e^2 \rho^2} \partial_\lambda f_{\lambda\mu} \partial_\nu f_{\nu\lambda} + (\partial\rho)^2 + V(\rho^2) \right) \\ &\equiv \int \mathcal{D}f \mathcal{D}\rho \tilde{F} \exp - S[f, \rho] \quad , \quad \tilde{F} \equiv \prod_x \rho^{-2} \end{aligned} \quad (3.7)$$

The only approximation we have made has been to neglect singularities in the phase of ϕ . Changing variables appropriate to an expansion about ρ_0

$$\varphi \rightarrow \varphi_0 + \varphi \quad f \rightarrow f / (2e\varphi_0)$$

gives

$$Z = \int \mathcal{D}f \mathcal{D}\varphi \tilde{F} \exp - \int d^4x (\mathcal{L}_0 + \mathcal{L}_I)$$

$$\mathcal{L}_0 = \partial_\mu f_{\mu\nu} \partial_\nu f_{\rho\sigma} + \frac{M^2}{2} f_{\mu\nu} f_{\mu\nu} + (\partial\varphi)^2 + \mu^2 \varphi^2$$

$$M = e\varphi_0 \sqrt{2} \quad \mu^2 = m^2 + \frac{\lambda}{4} \varphi_0^2$$

$$\mathcal{L}_I = \frac{\lambda\varphi_0}{3!} \varphi^3 + \frac{\lambda}{4!} \varphi^4 + \sum_{n=1}^{\infty} \left(-\frac{\varphi}{\varphi_0}\right)^n (n+1) \partial_\mu f_{\mu\nu} \partial_\nu f_{\lambda\sigma} \quad (3.8)$$

The quadratic part of the Lagrangian density describes a scalar particle φ with mass μ . The f field has acquired a mass since its equation of motion derived from $\mathcal{L}_0$ is

$$\partial_\mu \partial_\nu f_{\rho\sigma} - \partial_\nu \partial_\rho f_{\sigma\mu} - M^2 f_{\mu\nu} = 0 \quad (3.9)$$

Taking the divergence gives

$$(\partial^2 - M^2) \partial_\mu f_{\mu\nu} = 0 \quad (3.10)$$

so the vector field $\partial_\mu f_{\mu\nu}$ has mass M . The remaining part of $f_{\mu\nu}$ which does not contribute to $\partial_\mu f_{\mu\nu}$ does not propagate and furthermore does not couple to any other fields or sources.

Thus $\mathcal{L}_0$ displays the Higgs mechanism. The present formalism makes the particle content manifest because it deals with gauge invariant fields, it does not require a special choice of gauge condition to be made.

It is interesting to compare our formalism with the usual gauge fixing approach when the gauge condition chosen is that which makes the particle content of (2.1) manifest

$$\varphi = \bar{\varphi} \quad (3.11)$$

If we apply the Faddeev-Popov method to (2.9) by inserting the identity

$$\int \mathcal{D}\alpha \delta[e^{-i\alpha}\varphi - e^{i\alpha}\bar{\varphi}] \prod_x \rho = 1 \quad (3.12)$$

and changing variables in the $\bar{\varphi}, \varphi$ integrations to $e^{i\alpha}\bar{\varphi}, e^{-i\alpha}\varphi$ then because S and $S_{J,K}$ are gauge invariant we obtain

$$\begin{aligned} \mathcal{Z} = & \int \mathcal{D}\alpha \mathcal{D}A \mathcal{D}\bar{\varphi} \mathcal{D}\varphi \delta[\varphi - \bar{\varphi}] \prod_x \rho \cdot \exp - \int d^4x \left(\frac{i}{4} F_{\mu\nu} F_{\mu\nu} \right. \\ & \left. + \overline{D_\mu \varphi} D_\mu \varphi + V(\bar{\varphi}\varphi) + K_\mu A_\mu + J \bar{\varphi}\varphi \right). \end{aligned} \quad (3.13)$$

The α integration is trivial and leads to an infinite factor. The $\bar{\varphi}$ integration may be accomplished using the δ -function to give

$$\begin{aligned} \mathcal{Z} = & \int \mathcal{D}A \mathcal{D}\varphi \prod_x \rho \exp - \int d^4x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (\partial\varphi)^2 \right. \\ & \left. + e^2 A_\mu A_\mu \varphi^2 + V(\varphi^2) + K_\mu A_\mu + J\varphi^2 \right) \end{aligned} \quad (3.14)$$

This is of exactly the same form as the expression we would obtain from (3.7) (with the source terms reinstated) if we introduce a vector field A by inserting

$$\begin{aligned} & \int \mathcal{D}A \prod_x \rho^2 \exp - \int d^4x e^2 \rho^2 \left(A_\mu + \frac{i}{2e^2 \rho^2} [\partial_\lambda f_{\lambda\mu} + K_\mu] \right) \\ & = 1 \quad \times \left(A_\mu + \frac{i}{2e^2 \rho^2} [\partial_\nu f_{\nu\mu} + K_\mu] \right) \end{aligned} \quad (3.15)$$

and then integrating over f .

Our formulation of scalar QED is formally equivalent to gauge fixing in the unitarity gauge (3.11), however it offers two significant advantages. Our method works with fields ρ and f that are explicitly gauge invariant, the gauge fixing method does not, although it turns out from (3.14) that ρ and ϕ coincide. The two approaches must yield the same result when gauge invariant quantities are being calculated as in (3.14), however many problems require as intermediate steps the evaluation of quantities that are not gauge invariant and then the two approaches may differ. It is not always obvious how to extract the gauge invariant physical information from these quantities, for example Coleman and Weinberg³⁾ calculated the effective potential of scalar QED by coupling ϕ to a constant external source J for a functional integral defined in the Landau gauge. Of course the coupling

$$\int d^4x \mathcal{J}\varphi \quad (3.16)$$

is not gauge invariant and so neither is Coleman and Weinberg's expression for the effective potential, as was pointed out by Jackiw⁴⁾. In our approach, however, we may add to (3.7) the natural coupling to a constant external source

$$\int d^4x \mathcal{J}\varphi \quad (3.17)$$

calculate the effective potential and be sure that it will be gauge invariant and contain information about the physics of the original theory (2.1). Even though our method is formally equivalent to the theory expressed in the unitarity gauge (3.14) we could not have been certain, a priori, that by using the gauge variant coupling (3.17) we would have obtained a physically relevant result. Secondly the unitarity gauge cannot be implemented when the phase of ϕ has singularities whereas our formalism may be, as we will show in Section 5.

4. - AN EFFECTIVE POTENTIAL

We define an effective potential, V , by adding the source term (3.17) into the exponent of our functional integral (3.7), so that in four dimensions

$$\begin{aligned} & \exp - \int d^4x g(\mathcal{J}) \\ & = \int \mathcal{D}\varphi \mathcal{D}\varphi \tilde{\Gamma} \exp - (S[\varphi, \varphi] + \int d^4x \mathcal{J}\varphi) \end{aligned} \quad (4.1)$$

from which V is obtained as

$$V(\varphi_c) = g(\mathcal{J}) - \mathcal{J}\varphi_c \quad (4.2)$$

where

$$\varphi_c = \frac{\partial g}{\partial \mathcal{J}} \quad (4.3)$$

The physical interpretation of the effective potential is that it is the expectation value of the energy density minimized over states for which the expectation value of the field $\varphi(x)$ equals φ_c . As we have stressed this is a

gauge invariant constraint so that $V(\rho_c)$ is a physically relevant quantity. It is straightforward to calculate the effective potential to one loop by expanding the functional integral (4.1) about a saddle point. Now

$$\tilde{S}[f, \rho] \equiv \int d^4x \left(\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{1}{4e^2 \rho^2} \partial_\lambda f_{\lambda\mu} \partial_\nu f_{\nu\mu} + (\partial \rho)^2 \right. \\ \left. + m^2 \rho^2 + \frac{\lambda}{4!} \rho^4 + J \rho \right) \quad (4.4)$$

is stationary for

$$f = 0, \quad 2m^2 \rho + \frac{\lambda}{3!} \rho^3 + J = 0 \quad (4.5)$$

Denote the solution of this equation by ρ_0 and expand the action to second order

$$\tilde{S}[f, \rho_0 + \rho] = \tilde{S}[0, \rho_0] + \tilde{S}_2[f, \rho] + O(\rho^3)$$

$$\tilde{S}_2[f, \rho] = \int d^4x \left(\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{1}{4e^2 \rho_0^2} \partial_\lambda f_{\lambda\mu} \partial_\nu f_{\nu\mu} \right. \\ \left. + (\partial \rho)^2 + (m^2 + \frac{\lambda}{4} \rho_0^2) \rho^2 \right). \quad (4.6)$$

To one-loop accuracy we ignore higher orders so that

$$\exp - \int d^4x g(J) = \\ \exp(-\tilde{S}[0, \rho_0]) \cdot \int \mathcal{D}f \mathcal{D}\rho \prod_x \rho_0^{-3} \exp - \tilde{S}_2[f, \rho]. \quad (4.7)$$

$\tilde{f}$ takes its value at the saddle point, fluctuations in $\tilde{f}$ do not contribute to this order as may be seen by reinstating Planck's constant and remembering that the loop expansion is an expansion in powers of $\hbar$. The integrations over f and ρ give rise to determinants in the usual way so that if we write Δ_f for the infinite product of eigenvalues λ where

$$\frac{1}{2e^2 \rho_0^2} (\partial_\mu \partial_\nu f_{\rho\sigma} - \partial_\nu \partial_\sigma f_{\rho\mu}) - f_{\mu\nu} = -\lambda f_{\mu\nu}, \quad (4.8)$$

and set

$$\Delta_\rho = \text{Det} (-\partial^2 + m^2 + \frac{\lambda}{4} \rho_0^2) / M^2 \quad (4.9)$$

(where M is an arbitrary mass inserted to make the determinant dimensionless) then the one-loop expression for $g(J)$ is

$$\int d^4x g(J) = \tilde{S}[0, \rho_0] + \frac{1}{2} \ln \Delta_f + \frac{1}{2} \ln \Delta_g + 3 \int d^4x \delta^4(0) \ln \left(\frac{\sqrt{2} e \rho_0}{M} \right) \quad (4.10)$$

The last term comes from $\prod_x \rho_0^{-3}$ with constants introduced to make the logarithm dimensionless. To define the determinants and delta function we restrict momentum space integrals to the inside of a sphere of radius Λ , thus

$$\delta^4(0) = \int_{|k| < \Lambda} \frac{d^4k}{(2\pi)^4} = \Lambda^4 / (32\pi^2) \quad (4.11)$$

$$\begin{aligned} \ln \text{Det} (-\partial^2 + \mu^2) / M^2 &= \int d^4x \int_{|k| < \Lambda} \frac{d^4k}{(2\pi)^4} \ln \left(\frac{k^2 + \mu^2}{M^2} \right) \\ &= \int d^4x \frac{1}{16\pi^2} \left(\mu^2 \Lambda^2 + \mu^4 \left[\ln \frac{\Lambda}{M} - \frac{1}{4} \right] + \Lambda^4 \left[\ln \frac{\Lambda}{M} - \frac{1}{4} \right] + O\left(\frac{1}{\Lambda^2}\right) \right) \end{aligned} \quad (4.12)$$

Δ_f can be brought into the form (4.12) by noticing that any antisymmetric $f_{\mu\nu}$ can be written as

$$f_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) + g_{\mu\nu} \quad , \quad g_{\mu\nu} = -g_{\nu\mu} \quad (4.13)$$

where $g_{\mu\nu}$ can be chosen to be orthogonal to the bracketed term using a natural metric:

$$0 = \int d^4x g_{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) = -2 \int d^4x A_\nu \partial_\mu g_{\mu\nu}$$

which is guaranteed by

$$\partial_\mu g_{\mu\nu} = 0 \quad (4.14)$$

Substituting this decomposition into the eigenvalue equation (4.8) gives

$$\begin{aligned} \frac{1}{2e^2 \rho_0^2} \partial^2 (\partial_\mu A_\nu - \partial_\nu A_\mu) - (\partial_\mu A_\nu - \partial_\nu A_\mu) - g_{\mu\nu} = \\ - \lambda (\partial_\mu A_\nu - \partial_\nu A_\mu) - \lambda g_{\mu\nu} \end{aligned} \quad (4.15)$$

For $A = 0$ we have an eigenfunction for each independent function $g_{\mu\nu}$ satisfying (4.14) belonging to the eigenvalue $\lambda = 1$. The product of these eigenvalues is again 1 and so does not contribute to $\ln \Delta_c$. For $g = 0$ we obtain a solution to

(4.15) whenever A satisfies

$$\partial^2 A_\mu = \tilde{\lambda} A_\mu \quad (4.16)$$

for constant $\tilde{\lambda}$, and the corresponding value of λ is

$$\lambda = 1 - \tilde{\lambda} / (2 e^2 \rho_0^2) . \quad (4.17)$$

There are three independent solutions to (4.16) that give independent functions $f_{\mu\nu}$ in (4.13) because two A_μ differing by a gauge transformation

$$A' = A + \partial X \quad (4.18)$$

yield the same value for $f_{\mu\nu}$. We conclude that

$$\ln \Delta_f = 3 \ln \text{Det} [(-\partial^2 + 2e^2 \rho_0^2) / (2e^2 \rho_0^2)] . \quad (4.19)$$

As we only work to one-loop accuracy we can replace ρ_c by ρ_0 in the definition of the effective potential (4.2). Putting all this together gives

$$\begin{aligned} V(\rho_0) = & \left(m^2 \rho_0^2 + \frac{\lambda}{4!} \rho_0^4 \right) + (B \rho_0^2 + C \rho_0^4 + D) \\ & + \frac{1}{32\pi^2} \left[\Lambda^2 \left(m^2 + \frac{\lambda}{4} \rho_0^2 + 6e^2 \rho_0^2 \right) + \Lambda^4 \left(2 \ln \frac{\Lambda^2}{M^2} - 1 \right) \right. \\ & + 6e^4 \rho_0^4 \left(\ln \frac{2e^2 \rho_0^2}{\Lambda^2} - \frac{1}{2} \right) \\ & \left. + \frac{1}{2} \left(m^2 + \frac{\lambda}{4} \rho_0^2 \right)^2 \left(\ln \left[\frac{m^2 + \frac{\lambda}{4} \rho_0^2}{\Lambda^2} \right] - \frac{1}{2} \right) \right] . \quad (4.20) \end{aligned}$$

The terms in the first bracket constitute the classical result. The second bracket contains counterterms introduced to cancel those terms in the square bracket that become divergent as $\Lambda \rightarrow \infty$, and we have included a constant, D , which will allow us to choose the minimum of the potential. Note that the $\delta^4(0)$ in (4.10) has already cancelled with part of Δ_f .

Following Coleman and Weinberg³⁾ we consider the case of zero mass and so impose

$$m^2 = \frac{1}{2} V''(0) = 0 \quad (4.21)$$

which fixes B to be

$$B = - \frac{\Lambda^2}{32\pi^2} \left(\frac{\lambda}{4} + 6e^2 \right). \quad (4.22)$$

For $V(0)$ to be finite

$$D = - \frac{\Lambda^4}{32\pi} \left(2 \ln \frac{\Lambda^2}{M^2} - 1 \right) + \delta D \quad (4.23)$$

where δD is finite. The remaining divergences may be removed by taking

$$C = \frac{1}{32\pi^2} \left(6e^4 \ln \frac{\Lambda^2}{2e^2 M^2} + \frac{\lambda^2}{32} \ln \frac{4\Lambda^2}{\lambda M^2} \right) + \delta C \quad (4.24)$$

where δC is finite. With these conditions $V(\rho)$ becomes

$$\begin{aligned} V(\rho) = & \left(\frac{\lambda}{4!} + \delta C \right) \rho^4 + \delta D \\ & + \frac{1}{32\pi^2} \left(6e^4 + \frac{\lambda^2}{32} \right) \rho^4 \left(\ln \frac{\rho^2}{M^2} - \frac{1}{2} \right). \end{aligned} \quad (4.25)$$

δC and δD remain to be fixed by renormalization conditions which, by analogy with Ref. 3) we take to be that λ equals the fourth derivative of V evaluated at some arbitrary value of ρ (which for convenience we choose to be M),

$$V''''(M) = \lambda. \quad (4.26)$$

This implies that

$$\delta C = - \frac{11}{96\pi^2} \left(6e^4 + \frac{\lambda^2}{32} \right). \quad (4.27)$$

Similarly we fix δD by requiring

$$V(0) = 0, \text{ so } \delta D = 0 \quad (4.28)$$

so that finally

$$V(\rho) = \frac{\lambda}{4!} \rho^4 + \frac{1}{32\pi^2} \left(6e^4 + \frac{\lambda^2}{32} \right) \rho^4 \left(\ln \frac{\rho^2}{M^2} - \frac{25}{6} \right). \quad (4.29)$$

After taking our different conventions into account this differs from Coleman and Weinberg's result in the numerical coefficient of λ^2 . We have already discussed the source of this discrepancy, our potential is manifestly gauge invariant; theirs is not. We may rewrite $V(\rho)$ by eliminating the scale M in favour of the value of ρ that minimizes V , $\tilde{\rho}$

$$V(\varphi) = \tilde{\alpha} \varphi^4 \left(\ln \frac{\varphi^2}{\tilde{\rho}^2} - \frac{1}{2} \right), \quad \tilde{\alpha} = \frac{1}{32\pi^2} (6e^4 + \frac{\lambda^2}{32}). \quad (4.30)$$

The same physical system may of course be described by different choices of scale M provided the couplings λ are suitably adjusted, but the value of $\tilde{\rho}$ will not be altered. $\tilde{\rho}$ is thus a natural choice for the scale M , and for this value (4.26) becomes

$$\frac{11}{4\pi^2} (6e^4 + \frac{\lambda^2}{32}) = \lambda. \quad (4.31)$$

Thus λ is of order e^4 with this convention³⁾ and so the discrepancy in $\tilde{\alpha}$ is of order e^8 . Jackiw⁴⁾ has shown that the gauge dependence of Coleman and Weinberg's result is a term of order λe^2 in $\tilde{\alpha}$ which is clearly larger than the discrepancy. The virtue of our result is that because it is gauge invariant it has the simple interpretation given at the beginning of this section whereas a gauge variant result would not.

5. - VORTICES

It is well known that the action

$$S = \int d^D x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \overline{D_\mu \varphi} D_\mu \varphi + \frac{\lambda}{4!} (\bar{\varphi} \varphi - a^2)^4 \right) \quad (5.1)$$

allows vortex-like solutions which are stable because they are characterized by a topological invariant associated with the homotopy group $\pi_1(S^1) = \mathbb{Z}$. We have considered the theory defined in D Euclidean dimensions. Take the D^{th} dimension to be Euclidean time and let this be of finite duration whilst the other $D-1$ dimensions correspond to space and are of infinite extent. Since the action is a sum of squares any finite action continuous field configuration must behave at spatial infinity as

$$F \sim 0, \quad D\varphi \sim 0, \quad \bar{\varphi} \varphi \sim a^2. \quad (5.2)$$

A region of space in which these relations hold has been called a Higgs vacuum¹³⁾. Any finite action solution (apart from the trivial one) will consist

of a region of space where ϕ is non-trivial surrounded by (an approximate) Higgs vacuum. Let c be a closed curve lying entirely within the latter, so that on c

$$\varphi = a e^{i\theta} \quad (5.3)$$

The vanishing of $D\phi$ on c implies that

$$\int_{\Sigma} d\Sigma_{\mu\nu} F_{\mu\nu} = \oint_c dx_{\mu} A_{\mu} = \oint_c dx_{\mu} \partial_{\mu} \theta / e = \frac{1}{e} [\theta] \quad (5.4)$$

The left-hand side is the magnetic flux through the area Σ spanned by c and the right-hand side is the change in θ as c is traversed once, divided by e . ϕ is, of course, single-valued but θ defined by (5.3) need not be, we only require that

$$[\theta] = 2\pi n, \quad n \in \mathbb{Z} \quad (5.5)$$

which leads to the famous property of flux quantization in superconductivity. From (5.3) ϕ/a is a map from c , which has the topology of a circle, to $U(1)$ which also has the topology of a circle. The flux (5.4) is a topological invariant unchanged by continuous deformations of the fields and the winding number n characterizes the homotopy classes of such maps. The time evolution of a field configuration consists of such deformations so if n is initially non-zero it must remain so and the non-trivial configuration cannot evolve into a trivial one. Gauge transformations are also continuous deformations so it follows that n is gauge invariant. Outside of the Higgs vacuum we define the phase of ϕ , as before, by

$$\varphi = \rho e^{i\theta} \quad (5.6)$$

If c is shrunk continuously it will eventually leave the Higg's vacuum but if we use (5.6) to define θ then we still have

$$\oint dx_{\mu} \partial_{\mu} \theta = 2\pi n \quad (5.7)$$

with the same integer n as before if θ is continuous. Eventually we can shrink c to zero and then we have a contradiction, if we still insist on the continuity of θ , because the left-hand side of (5.7) will vanish whilst the right-hand side will not. So θ must become singular at least on some $(D-2)$ dimensional surface that loops through c . The original field ϕ is continuous so it follows that θ can only be singular when ρ vanishes. The vanishing of ρ is of course gauge invariant.

Previously we took as gauge invariant physical variables f and ρ . Now we see that there exist field configurations for which ρ may vanish on a set of surfaces. Each surface has a topologically conserved integer associated with it. Some of these configurations are solutions to the classical equations of motion, however the functional integral (2.20) involves an integration over all configurations and it is this that makes the topological character of n so significant. For each continuous configuration we integrate over, the total topological charge, taken as the sum of the winding numbers n , is the same at the final Euclidean time as it was at the initial time. When the field configuration is a classical solution the surfaces on which ρ vanishes and for which the winding number is non-zero lie at the centre of vortices. For $D = 3$ these are Ginzburg-Landau vortices, for $D = 4$ they are the Nielsen-Olesen vortices that resemble the dual string model. However, these surfaces are also present in configurations which are not classical solutions and yet contribute to the functional integral. These vortex-like configurations are of course more general. The surfaces on which ρ vanishes are world surfaces of what we will call bare vortices, the associated winding numbers we will term the charges of the vortices. The purpose of the present section is to make explicit the dependence of the functional integral on the bare vortices and their charges. We will treat the two cases of $D = 3$ and $D = 4$ separately.

5.1. Ginzburg-Landau vortices

In three dimensions the world surfaces of the bare vortices are just curves so in that respect the vortices resemble particles. We have seen that if we want to describe the field configuration in terms of gauge invariant variables then in addition to $f_{\mu\nu}$ and ρ we must specify the winding numbers, $\{n^i\}$, associated with the world lines of the bare vortices

$$x_\mu = x_\mu^i(s) . \quad (5.8)$$

We parametrize the world lines by arc length measured along them from some arbitrary point in one or other of the two possible directions. To fix the sign of n^i we must attach a direction to the world line in question and we must choose this to coincide with the direction of increasing arc-length, s . A typical configuration of world lines intersecting a plane of constant Euclidean time is shown in the Figure. The arrows denote the direction used to define the integers n^i . Because the world lines are either closed curves or begin and end on the

initial and final time surfaces the total topological charge, Q , passing through the $x_0 = \text{constant}$ plane is independent of the value of x_0

$$Q = \sum_i n^i \varepsilon \left(\frac{dx_0^i}{ds} \right) . \quad (5.9)$$

ε is the usual sign function

$$\varepsilon(x) = x / |x| \quad (5.10)$$

and is included in the formula for Q to ensure that closed world lines make no net contribution.

Let us now return to the functional integral, (2.20), of Section 2. With the external sources suppressed this is

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}f \mathcal{D}\bar{\phi} \mathcal{D}\phi \Gamma \exp - (S + S_g) , \\ S &= \int d^3x \left(\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{1}{4e^2 p^2} \partial_\lambda f_{\lambda\mu} \partial_\nu f_{\nu\mu} + (\partial f)^2 + \frac{\lambda}{4!} (f^2 - a^2)^2 \right) \\ S_g &= -\frac{1}{2e} \int d^3x f_{\lambda\mu} \partial_\lambda (\bar{\phi} \vec{\partial}_\mu \phi / (\bar{\phi} \phi)) . \end{aligned} \quad (5.11)$$

S_g is the only term that cannot be written entirely in terms of f and ϕ . Furthermore its integrand vanishes whenever the phase of ϕ is non-singular, that is whenever ρ is non-zero, so it only contributes along the world lines of the bare vortices. Consider one such world line and surround it by a tube, Σ , of very small radius, a . Since f is assumed to be continuous we may neglect its variation over the width of the tube so that to leading order the contribution of this world line to S_g is

$$\begin{aligned} \delta S_g &\simeq -\frac{1}{2e} \int d^3x \partial_\lambda \{ f_{\lambda\mu} \bar{\phi} \vec{\partial}_\mu \phi / (\bar{\phi} \phi) \} \\ &= -\frac{1}{2e} \int_\Sigma d\Sigma_\lambda f_{\lambda\mu} (\bar{\phi} \vec{\partial}_\mu \phi) / (\bar{\phi} \phi) . \end{aligned} \quad (5.12)$$

We may parametrize Σ by s , the arc-length along the world line, and an angle, ξ , that measures rotation about it. Assuming that the tube has a circular cross-section the surface element may be taken as

$$d\Sigma_\lambda = \varepsilon_{\lambda\mu\nu} \hat{\xi}_\mu \frac{dx_\nu}{ds} a ds d\xi \quad (5.13)$$

with ξ a unit vector along the direction of increasing ξ . Introducing the dual of f by

$$f_{\lambda\nu} = \varepsilon_{\lambda\nu\rho} B_\rho \quad (5.14)$$

enables us to write δS_g as

$$\delta S_g \simeq -\frac{1}{2e} \int dx_\rho B_\rho \left(\oint d\xi \frac{\bar{\varphi} \overleftrightarrow{\partial}_\xi \varphi}{\bar{\varphi} \varphi} \right) + \frac{a}{2e} \int ds d\xi \hat{\xi}_\rho B_\rho \left(\frac{\bar{\varphi} \overleftrightarrow{\partial}_s \varphi}{\bar{\varphi} \varphi} \right) \quad (5.15)$$

where

$$\partial_\xi \equiv \frac{\partial}{\partial \xi} \quad , \quad \partial_s \equiv \frac{\partial}{\partial s} \quad (5.16)$$

The second term vanishes as $a \rightarrow 0$. On Σ we can write ϕ as in (5.6) because ρ does not vanish there so that

$$\oint d\xi \frac{\bar{\varphi} \overleftrightarrow{\partial}_\xi \varphi}{\bar{\varphi} \varphi} = 2i [\theta] = 4\pi n i \quad , \quad (5.17)$$

where n is the charge of the vortex. In the limit as $a \rightarrow 0$ we are left with

$$\delta S_g = -\frac{2\pi n}{e} i \int dx_\rho B_\rho \quad (5.18)$$

If we define the vortex strength by

$$g \equiv \frac{2\pi n}{e} \quad , \quad (5.19)$$

then for a configuration consisting of an arbitrary number of vortices

$$S_g = - \sum_j i \int dx_\mu^j g^j B_\mu \quad (5.20)$$

We have now succeeded in expressing the action for an arbitrary configuration in terms of the gauge invariant quantities ρ , f , $\{g^i\}$ which directly describe the physical excitations of the theory. It remains to express $\partial\bar{\phi}\partial\phi$ in terms of these variables. To begin with consider a field configuration containing only a single vortex. Suppose that the bare vortex world line cuts each constant Euclidean time surface only once, at the point $x_\mu(s)$. With suitable

continuity assumptions on ϕ and its derivatives we can expand the field near the bare vortex, where it must vanish, for fixed x_0

$$\varphi(z, z^*) = \alpha z + \beta^* z^* + O(|z|^2)$$

$$z = x_1 - x_1(s) + i [x_2 - x_2(s)] \quad (5.21)$$

Of course α and β may vanish but initially we will limit ourselves to the simple case when they do not. In terms of the complex variables z, z^* the charge of the vortex is

$$n = \frac{1}{4\pi i} \oint dz \frac{\bar{\varphi} \partial \varphi}{\bar{\varphi} \varphi} + \frac{1}{4\pi i} \oint dz^* \frac{\bar{\varphi} \partial^* \varphi}{\bar{\varphi} \varphi}$$

$$\partial = \frac{\partial}{\partial z}, \quad \partial^* = \frac{\partial}{\partial z^*} \quad (5.22)$$

For the configuration (5.21) this formula gives the charge as

$$n = \begin{cases} 1 & |\beta| < |\alpha| \\ 0 & |\beta| = |\alpha| \\ -1 & |\beta| > |\alpha| \end{cases} \quad (5.23)$$

where the winding number is calculated taking the direction of increasing Euclidean time as that of increasing s .

It is extremely useful to have a function which registers the presence of the vortex. In order to construct one we choose the function J such that

$$\int dx_1 dx_2 \delta(\rho^2) J = 1 \quad (5.24)$$

for the configuration (5.21) when n is non-zero. Clearly the left-hand side vanishes when there is no vortex present. In the neighbourhood of the zero of ρ^2 we have

$$\rho^2 = (|\alpha|^2 + |\beta|^2) |z|^2 + \alpha \beta z^2 + \alpha^* \beta^* z^{*2} \quad (5.25)$$

but under a rotation of x_1, x_2 co-ordinates through an angle ξ

$$z \rightarrow z e^{i\xi} \quad (5.26)$$

and we can use this freedom to cancel the phase of $\alpha\beta$ so that

$$\begin{aligned}\alpha\beta &= \alpha^*\beta^* \\ \rho^2 &= |\alpha^* + \beta|^2 x^2 + |\alpha^* - \beta|^2 y^2 \\ x &= x_1 - x_1(s) \quad , \quad y = x_2 - x_2(s) .\end{aligned}\tag{5.27}$$

If we scale the co-ordinates

$$x \rightarrow |\alpha^* + \beta| x \quad , \quad y \rightarrow |\alpha^* - \beta| y \tag{5.28}$$

and introduce polar variables (r, θ) we obtain

$$\begin{aligned}\int d^2x \delta(\rho^2) J &= \int d\theta r dr \delta(r^2) J / [|\alpha|^2 - |\beta|^2] , \\ \text{for } [|\alpha|^2 - |\beta|^2] &\neq 0 ,\end{aligned}\tag{5.29}$$

so by choosing

$$J = 2 [|\alpha|^2 - |\beta|^2] / \pi$$

expression (5.24) is satisfied. Also from (5.27) we see that

$$\det(\partial_i \partial_j \rho^2) = 4 [|\alpha|^2 - |\beta|^2]^2 \quad ; \quad i, j = 1, 2 \tag{5.30}$$

so we conclude that

$$\frac{1}{\pi} \int dx_1 dx_2 \delta(\rho^2) \sqrt{\det \partial_i \partial_j \rho^2} \tag{5.31}$$

vanishes in the absence of a vortex and equals one for a single vortex configuration of the kind considered. This expression does not look invariant under $SO(3)$ rotations in the Euclidean space, although it must be. In the Appendix we show that it may be written in the manifestly invariant form

$$\begin{aligned}\int d\Sigma_\mu K_\mu \\ K_\mu = \frac{1}{\pi} \frac{dx_\mu(s)}{ds} \sqrt{\det' \partial_\alpha \partial_\beta \rho^2} \delta(\rho^2) .\end{aligned}\tag{5.32}$$

The prime on the determinant is meant to denote that it is the product of non-zero eigenvalues only.

So far we have kept x_0 fixed. If we now multiply together the expressions (5.31) for all the different values of x_0 we arrive at the functional integral

$$\int \mathcal{D}x_1 \mathcal{D}x_2 \delta[\rho^1] \prod_{x_0} \left(\frac{1}{\pi} \sqrt{\det \partial_i \partial_j \rho^1} \right), \quad (5.33)$$

which again vanishes in the absence of vortices and equals one for the single vortex configuration we have considered. This may be put into the more compact form

$$\int \widetilde{\mathcal{D}x} \delta[\rho^1], \quad (5.34)$$

where the measure $\widetilde{\mathcal{D}x}$ is defined in terms of a metric on variations of x orthogonal to the bare vortex

$$\|\delta x\|^2 = \frac{1}{\pi} \int ds \delta x_\mu (\partial_\mu \partial_\nu \rho^1) \delta x_\nu. \quad (5.35)$$

Now consider a slightly more general configuration consisting of ν vortices for each of which the zero of ϕ is of first order as in (5.21). The functional integral which counts the number of vortices and generalizes (5.34) is

$$\lim_{\alpha \rightarrow 0} \int \widetilde{\mathcal{D}x} \delta[\rho^1(x)] e^{-\alpha \int ds \kappa^2} = \nu \quad (5.36)$$

where κ is the curvature of the path $x(s)$

$$\kappa^2 = \frac{d^2 x_\mu}{ds^2} \frac{d^2 x_\mu}{ds^2} \quad (5.37)$$

and the metric on variations of $x(s)$ is the same as before, (5.35). Implicit in this expression is a summation over all possible topologies of bare vortex world line, for example the Figure contains lines which are loops and double loops as well as simple curves of the kind we have considered up until now, and these should be treated separately. We assume that the bare vortex world lines are smooth and the curvature is included in (5.36) to ensure that only those integration paths which are close to the world lines contribute. Jagged paths which go from one world line to another are suppressed as these would have infinite curvature. The choice of $\int \kappa^2 ds$ as a damping term is somewhat arbitrary, and there is a strong temptation to use $\alpha \int ds$ instead because this is the action of a particle of mass α which would therefore act as an infra-red regulator but this would fail to suppress jagged paths of finite length. With these

qualifications, (5.36) may be verified by expanding in the neighbourhood of each bare vortex world line in turn.

In general the bare vortices will not correspond only to first-order zeroes of ϕ . Given suitable assumptions on the continuity of ϕ it may be expanded to leading order in the neighbourhood of a bare vortex as

$$\varphi(z, z^*) = \sum_{p=0}^m \alpha_p (z^*)^p z^{m-p}, \quad (5.38)$$

with p and m integers. The charge, n , of the vortex can be any integer between m and $(-m)$ depending on the relative magnitudes of the coefficients α_p . We will assume that such a configuration can be approximated by one for which the behaviour near the vortex differs by a kind of point-splitting, i.e.,

$$\varphi(z, z^*) = \sum_{p=0}^m \alpha_p \prod_{j=1}^p (z^* - \beta_j^*) \prod_{j=p+1}^m (z - \beta_j), \quad (5.39)$$

where $\{\beta_j, j=1 \dots m\}$ is a set of small complex numbers, no two of them the same. These two configurations have the same charge because if the right-hand sides are inserted into (5.22) we can integrate around a large circle on which the β_j are negligible in comparison with z . The advantage of (5.39) is that it is a configuration of the kind we have already considered, each zero is only first order. Using this point-splitting trick we can regard any vortex as being composed from elementary vortices each with unit charge, so that (5.36) is valid for any configuration entering the partition function (5.11). If we divide both sides of (5.36) by v and raise it to the power v we can write it as

$$1 = \sum_{\bar{v}=0}^{\infty} \frac{\mathcal{L}t}{\alpha_i \rightarrow 0} \left(\frac{1}{\bar{v}}\right)^{\bar{v}} \delta_{\bar{v}v} \int \prod_{i=1}^{\bar{v}} \mathcal{D}x^i \delta[f^i(x^i)] e^{-\alpha_i \int ds_i \kappa^2}, \quad (5.40)$$

which may insert into the partition function to obtain, after rearranging the order of integration

$$\begin{aligned} \mathcal{Z} &= \sum_{\bar{v}} \int \prod_{i=1}^{\bar{v}} \mathcal{D}x^i \mathcal{D}x_i^i T_{\bar{v}}[\{x^j\}], \\ T_{\bar{v}} &= \left(\frac{1}{\bar{v}}\right)^{\bar{v}} \frac{\mathcal{L}t}{\alpha_i \rightarrow 0} \int \mathcal{D}f \mathcal{D}\bar{\varphi} \mathcal{D}\varphi \Gamma \delta_{\bar{v}v} \prod_{i=1}^{\bar{v}} \delta[f^i(x^i)] \\ &\quad \times \exp \left(-S + \sum_j \int ds (ig^j \frac{dx_j^i}{ds} B_{\mu} - \alpha_j \kappa^2) \right). \end{aligned} \quad (5.41)$$

Because of the delta functionals the only configurations that contribute to T are those for which ρ vanishes on the world lines $x^k(s)$ where the phase of ϕ becomes singular with winding number ± 1 . Furthermore $\delta_{\nu\nu}^-$ ensures that these are the only singularities in the phase. If n^i is the winding number associated with the singularity at $x^i(s)$ then we also have the identity

$$\sum_{\{m^j = \pm 1\}} \prod_{i=1}^{\bar{\nu}} \delta_{n^i m^i} = 1 \quad (5.42)$$

which we insert into T

$$\begin{aligned} (\bar{\nu})^{\bar{\nu}} T &= \sum_{\{m^j\}} t[\{x^j, m^j\}] \\ t &= \lim_{\alpha \rightarrow 0} \int \mathcal{D}f \mathcal{D}\bar{\varphi} \mathcal{D}\varphi \Gamma \delta_{\bar{\nu}\nu} \prod_{i=1}^{\bar{\nu}} \delta[\rho^i(x^i)] \delta_{m^i n^i} \\ &\quad \times \exp\left(-S + \sum_j \int ds \left[i g^j \frac{dx^j}{ds} B_\mu - \alpha_j k^j \right]\right) \end{aligned} \quad (5.43)$$

The configurations which contribute to t may be parametrized

$$\varphi = \rho e^{i(\theta + \theta_s)}$$

where ρ and θ are continuous functions but θ_s has singularities at $x = x^i(s)$ with winding number m^i . Only variations of ρ and θ contribute to the integral so we can write in (5.43)

$$\mathcal{D}\bar{\varphi} \mathcal{D}\varphi = \mathcal{D}\rho \mathcal{D}\theta \prod_x \rho \quad (5.44)$$

The θ integration may be done trivially leading to an infinite constant to be absorbed into the normalization. Finally we rewrite the delta functionals by introducing additional fields λ^i on the vortex world lines to obtain the partition function as

$$Z = \sum_{\vec{p}} \sum_{\{m^i\}} \frac{L^4}{\alpha \rightarrow 0} \left(\frac{1}{\vec{p}}\right)^{\vec{p}} \int \mathcal{D}f \mathcal{D}g \tilde{F} \delta_{\vec{p}, \nu} \prod_{i=1}^{\vec{p}} \widetilde{\mathcal{D}x^i} \mathcal{D}\lambda^i \delta_{m^i n^i} \\ \times \exp \left(-S + \sum_j \int ds \left[i g^j \frac{dx^j}{ds} B_\mu + i \lambda^j p^2 - \alpha_j k^2 \right] \right).$$

This expresses the partition function (once the source terms have been reinstated) entirely in terms of gauge invariant variables. The dependence on the vortices has become manifest and the amplitude has been decomposed into a sum over the different vortex number and charge sectors.

5.2. Nielsen-Olesen vortices

In four dimensions the topological charge is associated with two-dimensional surfaces on which the phase of ϕ becomes singular. These surfaces are like the world sheets of strings, and the possibility of obtaining a string model from a semiclassical expansion was discussed by Nielsen and Olesen⁶⁾. Their interest was in part motivated by the hope that by starting from a field theory it might be possible to overcome the well-known pathologies of the usual string models. However their programme met with two difficulties. Firstly the exact classical solutions are not known which makes a systematic expansion of the functional integral difficult. Secondly the classical solutions are extended objects whereas typical string Lagrangians describe strings that have no thickness at the classical level. Of course the extension of the Nielsen-Olesen vortex can be modified by changing the coupling constants but it was found that doing this resulted in a value for the action that was typically much smaller than Planck's constant so that the semiclassical approximation became inapplicable. Our method for making the vortex degrees of freedom explicit has the merit of being exact whilst dealing with surfaces of zero thickness. Although our work is somewhat formal it will be clear from the result how some of the problems of the dual string are resolved.

To generalize the discussion of the previous section is straightforward so we shall not repeat the argument but merely quote the final results. The contribution of the vortices to the action is

$$S_g = \sum_{\text{vortices}} \frac{ig}{2} \int dS_{\mu\nu} f_{\mu\nu}^*, \quad g = \frac{2\pi n}{e} \quad (5.47)$$

where dS is the area element of the vortex world sheet

$$dS_{\mu\nu} = d_1 x_\mu d_2 x_\nu - d_2 x_\mu d_1 x_\nu \quad (5.48)$$

(with $d_1 x, d_2 x$ two independent displacements within the sheet) and the dual is defined with the convention

$$*f_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} f_{\lambda\sigma} \quad , \quad \epsilon_{1234} = 1 \quad (5.49)$$

Equation (5.40) remains valid in four dimensions provided we generalize, in an obvious way, the metric (5.39) on variations of x in the space normal to the world sheet

$$2\pi \cdot \| \delta x \|^2 = \int dS \delta x_\mu \partial_\mu \partial_\nu \varphi^2(x) \delta x_\nu , \quad (5.50)$$

where dS is the scalar element of area

$$dS = d^2 \xi \sqrt{\det \left(\frac{\partial x_\mu}{\partial \xi_i} \frac{\partial x_\mu}{\partial \xi_j} \right)} \quad (5.51)$$

and ξ_i $i = 1, 2$ are two variables parametrizing the world-sheet. κ is now the curvature of the world-sheet. Again the delta functionals can be written as exponentials by introducing fields λ so that S'_g becomes

$$S'_g = S_g + \sum_{\text{vortices}} \int dS (i\lambda \varphi^2 + \alpha \kappa^2) . \quad (5.52)$$

With these modifications we again have expression (5.46) for the transition amplitude of Nielsen-Olesen vortices.

We can see immediately how this formalism might overcome some of the problems of the old dual string. These are associated with reparametrization invariance. Both the action (5.52) and metric (5.50) are independent of the way we parametrize the world-sheet as they would be for a functional integral representation of the Nambu string. However, here the quantum fluctuations of x must be orthogonal to the world-sheet because of (5.50), so there are no variations δx corresponding to changes of parameter. In the old models all variations of δx are integrated over, including those lying in the world-sheet which do not change the action and so lead to infinities which must be isolated from the functional integral with the consequent introduction of ghosts and all their attendant difficulties.

6. - CONCLUSION

We have described a gauge invariant method for quantizing the Abelian-Higgs model that in principle can be applied also to non-Abelian gauge theories. The formulation expresses the functional integral in terms of fields that directly describe the physical excitations of the theory, as a result the particle content of the Higgs mechanism is manifest and we have been able to calculate an effective potential which has a physical significance. Furthermore the contribution of vortices to the functional integral has been made explicit without resorting to any approximation, so that we have been able to see how the Nielsen-Olesen vortices may overcome some of the difficulties of the dual string.

The role of vortices in the functional integral is of physical interest in condensed matter physics¹⁴⁾ and in applications of strings to cosmology¹⁵⁾. The present method may be directly applied to non-Abelian gauge theories where instead of vortices there are other field configurations which are characterized by topological quantum numbers. Thus we can hope to make explicit the exact dependence of transition amplitudes on monopole and instanton-like configurations. This would elucidate their physical significance for values of the coupling constant beyond those for which the current approximations that are used to discuss these configurations are valid.

An interesting formal development would be to express the vortex degrees of freedom in terms of a field. Our expression (5.46) treats the vortices "quantum mechanically" in that their co-ordinates enter explicitly. If this could be rewritten using a field that creates and destroys vortices then, as usual, the notation would become more compact but also we would obtain the analogue of Mandelstam's treatment of the sine-Gordon equation in which he showed that sine-Gordon solitons are created and destroyed by the fermionic fields of the massive Thirring model¹⁶⁾. This generalization of two-dimensional bosonization would be of particular interest in the case of the magnetic monopoles of the non-Abelian Higgs model in the light of the speculations of Ref. 11).

APPENDIX

In this Appendix we rewrite expression (5.31)

$$\frac{1}{\pi} \int dx_1 dx_2 \delta(\varphi^2) \sqrt{\det \partial_i \partial_j \varphi^2} \quad (A.1)$$

$x^0 = \text{const.}$

in a form that is manifestly invariant under rotations in the three-dimensional Euclidean space. φ vanishes on the world line of the bare vortex $x = x(s)$ so

$$\varphi^2(x(s)) = 0, \quad \partial_\mu \varphi^2(x(s)) = 0. \quad (A.2)$$

Differentiating the second equation along the world line

$$\frac{dx_\nu(s)}{ds} \partial_\mu \partial_\nu \varphi^2(x(s)) = 0, \quad (A.3)$$

so $\partial_\mu \partial_\nu \varphi^2(x(s))$ has at least one vanishing eigenvalue and

$$\det \partial_\mu \partial_\nu \varphi^2(x(s)) = 0. \quad (A.4)$$

As previously we choose the orientation of the x_1, x_2 axes so that at the value of x^0 under consideration

$$\varphi^2 = |\alpha^* + \beta|^2 x^2 + |\alpha^* - \beta|^2 y^2,$$

to leading order. This diagonalizes $\partial_i \partial_j \varphi^2$ so that at this point we call the components of $\partial_\mu \partial_\nu \varphi^2$

$$(\partial_\mu \partial_\nu \varphi^2(x(s))) \equiv \begin{pmatrix} A & D & E \\ D & B & 0 \\ E & 0 & C \end{pmatrix}. \quad (A.5)$$

Dividing (A.3) by dx_0/ds on the assumption that it does not vanish leads to

$$\begin{pmatrix} A & D & E \\ D & B & 0 \\ E & 0 & C \end{pmatrix} \begin{pmatrix} 1 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = 0, \quad (A.6)$$

where the dots denote differentiation with respect to x_0 , thus

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -D/B \\ -E/C \end{pmatrix} \quad (\text{A.7})$$

Expression (A.4) may be rewritten in terms of the new variables

$$ABC - D^2C - E^2B = 0 \quad (\text{A.8})$$

Now define $\Delta(\lambda)$

$$\Delta(\lambda) \equiv \det (\lambda \delta_{\mu\nu} + \partial_\mu \partial_\nu \rho^2) \quad (\text{A.9})$$

so that if ω_1 and ω_2 are the non-zero eigenvalues of $\partial_\mu \partial_\nu \rho^2$ then

$$\Delta(\lambda) = \lambda (\lambda + \omega_1) (\lambda + \omega_2) \quad (\text{A.10})$$

We previously denoted the product of non-zero eigenvalues by $\det' \partial_\mu \partial_\nu \rho^2$

$$\det' \partial_\mu \partial_\nu \rho^2 = \omega_1 \omega_2 = \left. \frac{d\Delta}{d\lambda} \right|_{\lambda=0} \quad (\text{A.11})$$

and this is manifestly invariant under rotations of the axes. We can calculate this in terms of (A.5) as

$$\det' \partial_\mu \partial_\nu \rho^2 = [(BC)^2 + (BE)^2 + (DC)^2] / (BC) \quad (\text{A.12})$$

where we have used (A.8) to eliminate A.

The determinant in (A.1) is not $\det' \partial_\mu \partial_\nu \rho^2$ but rather $\det \partial_i \partial_j \rho^2$ which is just

$$\det \partial_i \partial_j \rho^2 = BC \quad (\text{A.13})$$

From (A.7) we obtain

$$\left(\frac{ds}{dx_0} \right)^2 = 1 + \dot{x}_1^2 + \dot{x}_2^2 = 1 + D^2/B^2 + E^2/C^2 \quad (\text{A.14})$$

so that we may express the required determinant (A.13) as

$$\det \partial_i \partial_j \varphi^2 = \left(\frac{dx_0}{ds} \right)^2 \det' \partial_\mu \partial_\nu \varphi^2 \quad (\text{A.15})$$

and write (A.1) as

$$\frac{i}{\pi} \int dx_0 dx_1 \delta(\varphi^2) \frac{dx_0}{ds} \sqrt{\det' \partial_\mu \partial_\nu \varphi^2} \quad (\text{A.16})$$

where s is assumed to be measured in the direction of increasing Euclidean time. More generally this is

$$\int d\Sigma_\mu \kappa_\mu,$$

where

$$\kappa_\mu = \frac{i}{\pi} \frac{dx_\mu}{ds} \sqrt{\det' \partial_\alpha \partial_\beta \varphi^2} \delta(\varphi^2) \quad (\text{A.17})$$

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FIGURE CAPTION

A typical configuration of world lines interesting a plane of constant Euclidean time.