CHIRAL FERMIONS IN STOCHASTIC QUANTIZATION

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A B S T R A C T

The simultaneous conservation of chiral and gauge currents in the framework of stochastic quantization is discussed. By means of the stochastic regularization procedure we explicitly compute the axial anomaly for fermions with mass $m \neq 0$ and the fictitious time $t \rightarrow \infty$. However, when $m \equiv 0$, an ambiguity appears: it turns out that the two limits ($m \rightarrow 0$, $t \rightarrow \infty$) do not commute. In this case non-perturbative methods show that the difference between left-handed and right-handed zero modes cancels; therefore no anomaly is present and stochastic regularization is unable to describe chiral theories at finite fictitious time. It is in any case unclear how stochastic quantization can describe a massless fermion at finite t .

In this letter we study problems related to the presence of chiral fermions in the stochastic quantization scheme¹⁾. The stochastic regularization procedure in connection with anomalies will be developed first, while at the end we consider problems associated to the poles of the massless fermion propagator in stochastic quantization in general.

Breit, Gupta and Zaks²⁾ have pointed out the possibility of making use of the ideas inherent to stochastic quantization in order to regularize field theories. Their proposal aims at the production of a new scheme that will respect, if possible, all the symmetries in the theory. For instance, such a stochastic regularization scheme has opened great expectations as to the possibility of preserving supersymmetry in a non-perturbative regularization scheme. Nevertheless, independent of the presence or absence of supersymmetry, the assumed simultaneous conservation of chiral and gauge symmetries in that scheme presents problems by itself. This last question is the one to be discussed here.

There is no compelling reason which forces the Langevin equations to exhibit explicitly all the symmetries of the theory, as long as the correct symmetric Green functions are recovered when taking the equilibrium limit $t \rightarrow \infty$ (where t is the fictitious time) which leads to the usual field theory. Nevertheless it is highly desirable to keep those symmetries explicit all along in order to control at any moment the relevant Ward identities as well as to be able to identify any eventual source of symmetry breaking. The requirement of chiral and gauge symmetry conservation leads us in the simplest case to the consideration of two independent stochastic noises in the field evolution. For instance, QED with dynamical gauge bosons and massless fermions could be described by

$$\frac{\partial \Psi(x,t)}{\partial t} = \not{D}_x^2 \Psi(x,t) + i \not{D}_x \eta_1(x,t) + \frac{1}{2} \eta_2(x,t) \quad (1a)$$

$$\frac{\partial \bar{\Psi}(x,t)}{\partial t} = \bar{\Psi}(x,t) \not{D}_x^2 + \bar{\eta}_1(x,t) i \not{D}_x + \frac{1}{2} \bar{\eta}_2(x,t) \quad (1b)$$

$$\frac{\partial A_\mu(x,t)}{\partial t} = (\square \delta_{\mu\nu} - \partial_\mu \partial_\nu) A_\nu(x,t) + e \bar{\Psi}(x,t) \gamma_\mu \Psi(x,t) + \eta_\mu(x,t) \quad (1c)$$

$$\langle \eta_1^a(x,t) \bar{\eta}_2^b(x',t') \rangle = \delta(x-x') \Delta(t-t') \delta_{ab} \quad (1d)$$

$$\langle \eta_\mu(x, t) \eta_\nu(x', t') \rangle = \delta(x-x') \Delta'(t-t') \delta_{\mu\nu} \quad (1e)$$

where a and b are Dirac indices, $\Delta(t-t')$ and $\Delta'(t-t')$ are stochastic regulators and $\not{D} = (\not{D} - ieA)$. Gauge covariance is explicit as well as chiral invariance under $\Psi \rightarrow e^{i\beta\gamma_5} \Psi$, $\eta_2 \rightarrow e^{i\beta\gamma_5} \eta_2$, $\eta_1 \rightarrow e^{-i\beta\gamma_5} \eta_1$, etc.

Except for Ishikawa's³⁾ supersymmetric analysis which already includes two noises in the non-supersymmetric limit of QED, no published work^{2), 4)} considers more than one fermionic stochastic noise for the fermion evolution equation but neither of those explicitly conserves both chiral and gauge invariances. The integrated anomalies being a low-energy phenomenon, the restriction to gauge background fields is certainly correct for their study, in which case perfectly symmetric equations with just one fermionic noise are easy to write. We will choose to work here with our more general type¹⁾ equations even for background gauge fields [omitting (1c) and (1e)], as the calculations are very simple anyway.*)

Existing non-perturbative methods⁵⁾ for testing the presence of proper axial anomalies have been recently extended by Ninomiya and Chung-I-Tan⁶⁾ to the study of lattice gauge theories. We will do the same for stochastic regularization. Let us consider fermions moving in a classical non-Abelian background field; the needed equations are (1a), (1b) and (1d), where now $A_\mu = A_\mu^a T^a$ in an obvious notation. But we will introduce from now on a fermionic mass in the Lagrangian equations in order to exhibit the limit $m \rightarrow 0$. A possible choice is

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= (\not{D}^2 - m^2) \Psi + i(\not{D} + im) \eta_1 + \frac{1}{2} \eta_2 \\ \frac{\partial \bar{\Psi}}{\partial t} &= (\not{D}^{2T} - m^2) \bar{\Psi} + \bar{\eta}_1 i(\not{D} + im) + \frac{1}{2} \bar{\eta}_2 \\ \langle \eta_1^a \bar{\eta}_2^b \rangle &= \delta(x-x') \Delta(t-t') \delta_{ab} \end{aligned} \quad (2)$$

Let Ψ_E denote the eigenvalues of the massless Dirac equation $\not{D} \Psi_E(x) = E \Psi_E(x)$ and consider the Green function

*) When this work was practically finished, we received a preprint by Egorian et al.⁸⁾ discussing anomalies in stochastic quantization, using equations with just one stochastic noise. Certainly our method of study differs from theirs, and the results do not coincide in the massless case.

$$\left[\frac{\partial}{\partial t} - \mathcal{D}_x^2 \right]^{ab} G^{ba'}(x, x'; t, t') = \delta(x-x') \delta(t-t') \delta_{aa'} \quad (3)$$

It is then possible to write

$$G^{ba'}(x, x'; t, t') = \sum_E \theta(t-t') \Psi_E^{*a'}(x') \Psi_E^b(x) e^{-(E^2+m^2)(t-t')} \quad (4)$$

We now define the axial current by $j_\mu^5(x) = \bar{\Psi}(x) \gamma_5 \gamma_\mu \Psi(x)$ and compute⁵⁾

$$\langle j_\mu^5(x) \rangle \equiv \lim_{x \rightarrow y} \langle \bar{\Psi}^a(x) (\gamma_5 \gamma_\mu)_{ab} \Psi^b(y) \rangle = \lim_{x \rightarrow y} \text{tr} \{ \gamma_5 \gamma_\mu R(y, x) \} \quad (5)$$

where $R^{ba}(y, x) \equiv \langle \Psi^b(y) \bar{\Psi}^a(x) \rangle$. In other words

$$\langle j_\mu^5(x) \rangle \equiv \text{tr} \left\{ \gamma_5 \gamma_\mu R_{\text{regulated}}(x, x) \right\} \quad (6)$$

and we will use the stochastic regularization scheme for $R_{\text{regulated}}(x, x)$. A straight use of (2) and (4) leads to the spectral resolution

$$R^{ba}(y, x) = i \int_0^t d\tau \int_0^t d\bar{\tau} \Delta(\tau - \bar{\tau}) \cdot \sum_E \Psi_E^b(y) (E - im) e^{-(E^2+m^2)(2t-\tau-\bar{\tau})} \Psi_E^{a*}(x) \quad (7)$$

The identification at this point $\Delta(t-t') = \delta(t-t')$ allows to check that the unregulated result $R^{ba}(y, x) = \sum_E \Psi_E^b(y) \Psi_E^{a*}(x) / (E + im)$ is recovered.

Similarly

$$\partial_\mu \langle j_\mu^5(x) \rangle = \int d\tau d\bar{\tau} \Delta(\tau - \bar{\tau}) \cdot \sum_E \left\{ \Psi_E^+(x) \gamma_5 \Psi_E(x) E (E - im) \cdot e^{-(E^2+m^2)(2t-\tau-\bar{\tau})} \theta(t-\tau) \theta(t-\bar{\tau}) \right\} \quad (8)$$

Introducing the stochastic regulator⁷⁾ $\Delta(\tau) \equiv \sigma/2 |\tau|^{\sigma-1}$, $\sigma \rightarrow 0$, we get for large t

$$\partial_\mu j_\mu^5 = -im \sum_E \frac{\bar{\Psi}_E^+(x) \gamma_5 \Psi_E(x)}{E+im} \frac{1}{(E^2+m^2)^\sigma} +$$

$$+ \sum_E \frac{\bar{\Psi}_E^+(x) \gamma_5 \Psi_E(x)}{(E^2+m^2)^\sigma} \quad (9)$$

and the anomaly reads

$$A(x) = \sum_E \frac{\bar{\Psi}_E(x) \gamma_5 \Psi_E(x)}{(E^2+m^2)^\sigma} \quad (10)$$

It is then possible to evaluate $A(x)$ as follows:

$$A(x) = \lim_{x \rightarrow y} \text{tr} \left\{ \gamma_5 (D^2 + m^2)^{-\sigma} \delta(x-y) \right\} =$$

$$= \lim_{x \rightarrow y} \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left\{ \gamma_5 (D^2 + \frac{1}{2} \sigma_{\mu\nu} F^{\mu\nu} + m^2)^{-\sigma} e^{ik(x-y)} \right\} \approx$$

$$\approx \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left\{ \gamma_5 \left(\frac{1}{(k^2+m^2)^\sigma} - \frac{\sigma}{2} \frac{\sigma_{\mu\nu} F^{\mu\nu}}{(k^2+m^2)^{\sigma+1}} + \right. \right. \quad (11)$$

$$\left. \left. + \frac{\sigma(\sigma+1)}{2} \frac{\sigma_{\mu\nu} F^{\mu\nu} \sigma_{\alpha\beta} F^{\alpha\beta}}{(k^2+m^2)^{\sigma+2}} + \dots \right) \right\} =$$

$$= \frac{\sigma(\sigma+1)}{2} \text{tr} \left\{ \gamma_5 \sigma_{\mu\nu} \sigma_{\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \right\} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2+m^2)^{\sigma+2}}$$

The last integral has a pole at $\sigma = 0$ which conspires with the coefficient in front yielding the expected finite answer at $\sigma = 0$.

$m = 0$ versus $\underline{m} = 0$. Let us consider the anomaly in the massless case, as t plays anyway the role of an infrared regulator. The ultraviolet unregulated anomaly $\langle q \rangle$ reads

$$\partial_\mu \langle j_\mu^5(x) \rangle = \sum_E \bar{\Psi}_E^+(x) \gamma_5 \Psi_E(x) (1 - e^{-2(E^2+m^2)t}) \quad (12)$$

and the difference in the number of left-handed and right-handed zero modes is

$$\begin{aligned} n_+ - n_- &= \int dx \sum_E \Psi_E^\dagger(x) \gamma_5 \Psi_E(x) (1 - e^{-2(E^2 + m^2)t}) = \\ &= \int dx \sum_{E=0} \Psi_E^\dagger(x) \gamma_5 \Psi_E(x) (1 - e^{-2m^2t}) \end{aligned} \quad (13)$$

where the integration goes over the compact four-dimensional space-time. The last step stems from the fact that $\int dx \Psi_E^\dagger(x) \gamma_5 \Psi_E(x) = 0$ for $E \neq 0$.

If in expression (13) the $m \rightarrow 0$ limit is taken, we see that $n_+ - n_- = 0$, and no chiral theory can be written at finite fictitious time. In order to recover the standard anomaly, it would be necessary to take first the infrared regulator $t \rightarrow \infty$. In the latter case the anomaly should be seen as beautifully developing from 0 at $t = 0$ to its regular value at $t = \infty$ (imposing only then $m = 0$). In any case, it is evident that the double limiting procedure ($t \rightarrow \infty$ and $m \rightarrow 0$) does not exist: namely, the two limits do not commute and the behaviour of the product $m^2 t$ determines a unique direction in which the continuum should be reached.

An intuitive interpretation of this result is the following: the zero anomaly result for massless fermions at finite t can be considered by rewriting the massless Langevin equations (1) in terms of explicit left-handed (L) and right-handed (R) objects. For instance (1a) gives

$$\frac{\partial \Psi_L}{\partial t} = \not{D}^2 \Psi_L + i \not{D} \not{\eta}_1 R + \frac{1}{2} \not{\eta}_2 L \quad (14)$$

The evolution equation tells us that the propagation of an initially left-handed field always develops a right-handed component in the form of one stochastic noise η_1 , as the fictitious time develops.

The origin of the problems we encounter with the axial anomaly can be traced to a peculiarity of the fermion propagator which we now discuss. Consider the unregulated fermion propagator (8)

$$R^{ab}(x, y) = \sum_E \frac{\Psi_E(x) \Psi_E^\dagger(y)}{E + im} (1 - e^{-2(E^2 + m^2)t}) \quad (15)$$

An amazing fact is transparent in this equation: the massless fermion propagator does not present a pole at $E = 0$. This behaviour is not inherent to our

regularization procedure, but already appears in any previous proposals^{3),4)} for fermionic propagators with a correct converging behaviour when $t \rightarrow \infty$. Although t is a fictitious time, intuitively it can be looked at as a finite "volume" in which chiral long-range correlations ($m = 0$) are ambiguously defined. For the same reason, the physically relevant part of massless boson propagators does not present poles at zero momentum, but of course this has not such far-reaching consequences as for the chiral modes.

We have explicitly shown how the stochastic regulator recovers the axial anomaly when $m \neq 0$ (m being the mass of the fermion in the theory). In this case the Langevin equation is not chiral-invariant but it is gauge-invariant and accordingly the regulator exhibits the anomaly for large t . The $m = 0$ case, where the Langevin equation is chiral-invariant, does not exhibit the anomaly and thus stochastic regularization cannot implement chiral theories at finite fictitious time: the right theory is not recovered when $t \rightarrow \infty$. The massless fermions are absent from the spectrum in this limit, in stochastic quantization in general. This comes about because the equilibrium is reached only for $t \gg m^{-1}$.

Formally it is still possible to use the stochastic regulator for large t but keeping in mind the limiting recipe $\lim_{m \rightarrow 0} \{ \lim_{t \rightarrow \infty} \}$. With this artifact, the regularization procedure is expected to be suited for calculations, preserving the right anomaly factors.

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