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AN ALGORITHM TO ISOLATE POLES IN FEYNMAN DIAGRAMS
PRESERVING BACKGROUND GAUGE INVARIANCE AND SUPERSYMMETRY

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ABSTRACT

We regularize the heat-kernel expansion of propagators in the presence of background fields. This leads to a renormalized perturbation series which is gauge-invariant and supersymmetric with respect to background variations. This procedure permits isolation of singularities in the computation of the effective action, obeying the above symmetries.

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1. - INTRODUCTION

Besides the natural technical difficulties involved in the process of regularization and renormalization of field theories, the problem of how to preserve certain naive symmetries, such as gauge invariance and supersymmetry, is a particularly hard one. Consider, for example, a subtractive scheme such as BPHZ¹⁾. It is known that this scheme can be modified to naturally include supersymmetric theories²⁾. However, to establish gauge invariance, one has to mix different orders of the perturbation expansion, and this symmetry is violated by the renormalization scheme in intermediate steps. Thus, to ensure that physical quantities are gauge-independent, Slavnov-Taylor identities and independence with respect to gauge parameters must be proved³⁾. On the other hand, dimensional regularization⁴⁾, which explicitly preserves gauge symmetry, is incompatible with supersymmetry. Moreover, dimensional reduction⁵⁾, which has been proposed as a supersymmetric version of dimensional regularization, is inconsistent⁶⁾.

In the framework of the BPHZ machinery, one can prove that for supersymmetric Yang-Mills theory a renormalized perturbation expansion does exist, and that physical quantities are indeed gauge-invariant⁷⁾. Therefore, we take the point of view that the aforementioned difficulties are not a matter of principle, and that the main task is to build gauge invariance explicitly into the renormalized expansion. We shall achieve this with the help of the heat-kernel expansion⁸⁾ for propagators of quantum fields in the presence of a classical background⁹⁾. This amounts to a partial summation of Feynman diagrams into gauge-invariant combinations, since the Seeley coefficients are gauge independent.

In order not to break supersymmetry, the regularization can be performed either with a cut-off in the integration over the proper time τ ^{9),10)} or by inserting convergence factors τ^ϵ into the integrals, which then become analytic functions of ϵ in suitable domains. We shall use this second approach, proving later on that it is equivalent to a subtractive scheme. As it turns out, the use of a single convergence factor τ^ϵ (always with the same ϵ) does not lead to a suitable renormalization scheme (new counterterms are needed); in fact, each propagator $\underline{l}$ must come with its own factor τ^{ϵ_l} . We should point out that dimensional reduction is very similar to the scheme with a single ϵ , and this may be the reason why the scheme is inconsistent. More details will be discussed later on.

To a great extent, we shall follow the ideas and methods used in the process of analytic regularization^{11),12)}. In particular, we use some of the notations

and proofs contained in Ref. 12). The heat-kernel technique, in connection with the background field method and regularized according to either the first technique⁹⁾ or a simplified version of the second one¹³⁾, has already been considered in the literature, up to one-loop order. Our approach, however, works in all orders. Using dimensional regularization and non-supersymmetric theories, there is a proposal worked out at least up to two-loop order¹⁴⁾.

2. - THE REGULARIZED HEAT-KERNEL EXPANSION

The background field method consists in a split of the fields into a classical and a quantum piece. Integration of the quantum part defines an effective action for the classical fields. In order to perform this integration, we have to compute the propagators of the quantum field in the presence of the background, and compute vacuum diagrams.

Following de Witt and Schwinger⁸⁾ [in the supersymmetric cases see Ref. 15)] we write the propagator function as

$$Q((x, \theta_x), (y, \theta_y)) = \int_0^\infty d\tau Q((x, \theta_x), (y, \theta_y); \tau) \quad (2.1)$$

where the integrand of (2.1) satisfies

$$\frac{\partial}{\partial \tau} Q((x, \theta_x), (y, \theta_y); \tau) = -\square_B Q((x, \theta_x), (y, \theta_y); \tau) \quad (2.2)$$

with the boundary condition

$$Q((x, \theta_x), (y, \theta_y); 0) = \delta(x-y) \delta(\theta_x - \theta_y) \quad (2.3)$$

In the non-supersymmetric case, we just drop the Grassmann variable dependence above. We can solve the Eqs. (2.2) and (2.3) by the ansatz^{*}

^{*}) In our notation the Seeley coefficients may contain derivatives with respect to θ acting on the Grassmann delta functions.

$$Q((x, \theta_x), (y, \theta_y); \tau) = \frac{e^{-\frac{(x-y)^2}{4\tau} - m^2\tau}}{(4\pi\tau)^2} \sum_{n=0}^{\infty} \tau^n a_n((x, \theta_x), (y, \theta_y)) \delta(\theta_x - \theta_y) \quad (2.4)$$

The initial condition (2.3) fixes $a_0(z_x, z_y) = 1$. We can also determine $a_n(z_x, z_y)$ recursively, once we know the background field dependent Laplacian. For each theory, we promote the background quantum splitting, and calculate the necessary Seeley coefficients. For the computation of counterterms, the full knowledge of $a_n(z_x, z_y)$ is not necessary, since counterterms must be local. In renormalizable theories it is enough to have a_1 and a_2 at coinciding points. These coefficients for the (non-supersymmetric) ϕ^4 theory, for the Wess-Zumino model and $N = 1$ Yang-Mills theory were computed by several authors^{14), 15)}, and are shown in Ref. 10).

Divergences in the perturbative expansion arise from the fact that the functions

$$Q_n(x) = \int_0^{\infty} \frac{d\tau}{(4\pi\tau)^2} \tau^n e^{-\frac{x^2}{4\tau} - m^2\tau} \quad (2.5)$$

have singularities at $x = 0$. They can be explicitly computed:

$$Q_n(x) = \frac{1}{32\pi^2} \left[\frac{16m^2}{x^2} \right]^{\frac{1-n}{2}} I_{1-n}(m\sqrt{x^2}) \quad (2.6)$$

The singularity at $x = 0$ is readily seen for $n < 2$. For larger n , g_n is $n-1$ times differentiable at the origin. In four dimensions, for a renormalizable theory, no more than two derivatives may appear (by power counting), and in general we have to deal with $n < 2$, as will be seen in detail.

We also note that the g_n 's satisfy the rule

$$g_{m+n+1}(x-y) = \int d^4z g_m(x-z) g_n(z-y) \quad (2.7)$$

that we would like to preserve, in some sense, in the regularized theory.

In order to regularize the theory, we can cut the proper time integration at high frequencies (τ small). We discussed this procedure previously¹⁰⁾, but it

has the disadvantage of not providing a general algorithm valid to all orders. However, it is known that if we have a renormalized theory, cut off at high frequencies, with propagators of the form $(p^2+m^2)^{-1-\epsilon}$, or equivalently with

$$g_0^\epsilon(x) = \int_0^\infty \frac{d\tau}{(4\pi\tau)^2} \tau^\epsilon e^{-\frac{x^2}{4\tau} - m^2\tau} \quad (2.8)$$

Speer¹¹⁾ introduced a modification of Hepp's proof¹⁶⁾ to show that the limit of the ultra-violet cut-off k going to zero was a well-defined analytic function of ϵ . In our case, with higher g_n 's, it is true since

$$g_{m+n+1}^{\epsilon_1+\epsilon_2}(x-y) = \mathcal{B}(1+\epsilon_1, 1+\epsilon_2)^{-1} \int d^4z g_m^{\epsilon_1}(x-z) g_n^{\epsilon_2}(z-y) \quad (2.9)$$

where $g_n^\epsilon(x) = \int_0^\infty (d\tau/(4\pi\tau)^2) \tau^{n+\epsilon} e^{-x^2/4\tau}$. We have Speer's case again for propagators g_n , and the resulting function is also analytic in the variable $\epsilon = \epsilon_1 + \epsilon_2$. Due to this analogy, we shall take for the analytic regularized heat-kernel expansion

$$g^\epsilon((x, \theta_x), (y, \theta_y)) = \int_0^\infty \frac{d\tau}{(4\pi\tau)^2} \tau^\epsilon e^{-\frac{x^2}{4\tau} - m^2\tau} \sum_{n=0}^\infty \tau^n a_n((x, \theta_x), (y, \theta_y)) \delta(\theta_x - \theta_y) \quad (2.10)$$

Unfortunately, for several reasons, the procedure with a single ϵ does not work. One can see explicitly that new counterterms have to be defined and the theory is non-renormalizable. The fact is that the procedure is not equivalent to a subtraction of counterterms (which we will prove) and fails to provide the correct finite part beyond the one-loop level. [See also Speer's discussions in Refs. 11) and 12).]

The problem is circumvented by introducing several parameters ϵ_λ , one for each propagator. Using these regularized amplitudes, we shall now renormalize the terms of the effective action.

3. - GENERALIZED AMPLITUDES AND RENORMALIZATION

Feynman amplitudes in the presence of background fields are defined by means of the expression

$$T_{FB}((x_1, \theta_1), \dots, (x_n, \theta_n)) = \prod_L \sum_{\delta=0}^{\infty} q_{\delta}^{\varepsilon_e} (e_i^{\ell} x_i) a_{\delta}((x_{i_e}, \theta_{i_e}), (x_{f_e}, \theta_{f_e})) \delta(e_i^{\ell} \theta_i) \quad (3.1)$$

where the product is over the set of lines L of a given Feynman diagram $G(V, L)$ (see Appendix A for the graphical terminology).

In order to obtain the effective action in terms of the background fields, we must eventually multiply the above expression by the convenient fields, as dictated by the interaction Lagrangian. However, we shall not bother about this integration. In order to avoid infra-red problems, we keep the parameter m^2 in (2.5) non-zero. The Seeley coefficients also constitute an unnecessary complication in (3.1), so we renormalize instead:

$$T(\{\delta_e\}, \{\varepsilon_e\}; x_1, \dots, x_n) = \prod_L q_{\delta_e}^{\varepsilon_e} (e_i^{\ell} x_i) \quad (3.2)$$

The strategy is as follows. We look at the structure of the divergences of (3.2) by going to momentum space, and scaling the proper time variables, to explicitly show the divergences. We give then the algorithm to separate the finite part, and thereafter prove that it is equivalent to a subtractive scheme. We Fourier-transform the amplitude (3.2), which is an integral over proper time parameters of a product of exponential functions

$$T(x_1, \dots, x_n) = \prod_L \exp - \frac{1}{4\tau_e} e_i^{\ell} x_i = \exp - \frac{1}{4} \sum_{i,j} x_i A_{ij} x_j \quad (3.3)$$

where

$$A_{ij} = \sum_{\ell} \frac{e_i^{\ell} e_j^{\ell}}{\tau_{\ell}}$$

Since $\sum_i A_{ij} = 0$, we can write

$$\sum_{i,j} x_i A_{ij} x_j = \sum_{i,j \neq k} \int_i^{\epsilon_i} A_{ij} \int_j^{\epsilon_j},$$

$$x_j - x_k = \int_j^{\epsilon_j}$$

and we have, for the Fourier-transformed $\tau(\{\delta_\lambda\}, \{\epsilon_\lambda\}, k)$,

$$\tau(\{\delta_e\}, \{\epsilon_e\}, k_1, \dots, k_n) = \delta^4(P) \int_0^\infty \prod_{e \in L} \tau_e^{\epsilon_e + \delta_e - 2} \frac{d\tau_e}{(4\pi)^2} \times$$

$$\times \frac{1}{d(\tau)^2} \exp - \left(\frac{\sum_i D_{ij} k_j}{d(\tau)} + m^2 \sum \tau_e \right) \quad (3.4)$$

where δ_λ is associated to the respective g_{δ_λ} term of the propagator in the heat-kernel expansion, P is the total momentum, and the functions $D_{ij}(\tau)$ and $d(\tau)$ are¹²⁾

$$d(\tau) = \sum_T \prod_{e \notin T} \tau_e \quad (3.5a)$$

$$D_{ij}(\tau) = \sum_{T_2} \prod_{e \notin T_2} \tau_e \quad (3.5b)$$

where T are the one-trees and T_2 the two-trees (see Appendix A). The τ integrals in (3.4) are convergent for complex ϵ_λ , with $\text{Re} \epsilon_\lambda > 1$ (in fact, they converge for $\text{Re} \epsilon_\lambda > 1 - \delta_\lambda$). We seek for an analytic continuation to meromorphic functions in C^L , with poles on the non-positive real axis.

It is not difficult to convince oneself that if all τ from a loop vanish, we have a potential divergence, due to a zero in $d(\tau)$. We must isolate this divergence, making a change of variables. As shown in Appendix A, we can associate a variable t_H to each member (H) of a singularity family F , such that in the domain $D(F)$ (see Appendix A)

$$\tau_e = \prod_{H \in e} t_H \quad (3.6a)$$

$$d(\tau) = \prod_{H \in F} t_H^{L(H) - n(H) + 1} E(t) \quad (3.6b)$$

$$D_{ij}(\tau) = t_G \prod_{H \in F} t_H^{L(H) - n(H) + 1} F_{ij}(t) \quad (3.6c)$$

where $E(t)$ does not vanish.

The Jacobian of (3.6a) is

$$J = \prod_H t_H^{L(H) - 1} \quad (3.7)$$

We obtain for the Green function

$$\tilde{T}(\{\delta_e\}, \{\varepsilon_e\}, k_1, \dots, k_n) = \delta^4(p) \sum_F \int_0^\infty dt_G \int_0^1 \dots \int_0^1 \prod_{H \neq G} dt_H \prod_{H \in F} t_H^{\varepsilon(H) - \frac{1}{2} \mu(H) + \Delta(H) - 1} \exp - t_G \left[\frac{F_{ij} k_i k_j}{E(t)} + m^2 \sum_e \beta_e \right] \quad (3.8)$$

where

$$\varepsilon(H) = \sum_{e \in H} \varepsilon_e,$$

$$\Delta(H) = \sum_{e \in H} \delta_e,$$

$$\mu(H) = 2L(H) - 4(v(H) - 1)$$

and $\beta_e = t_G^{-1} \tau_e$ (β is independent of t_G).

The t_G integration can be explicitly performed, resulting in

$$\tilde{T}(\{\delta_e\}, \{\varepsilon_e\}, k_1, \dots, k_n) = \delta^4(p) \Gamma(\varepsilon(G) - \frac{1}{2} \mu(G) + \Delta(G)) \times$$

$$\sum_{\mathbb{F}} \int_0^1 \dots \int_0^1 \prod_{H \neq G} t_H^{\varepsilon(H) + \Delta(H) - \frac{1}{2} \mu(H) - 1} dt_H E(t)^{-2} \cdot$$

$$\cdot \left[m^2 \sum_e \beta_e + \frac{\kappa_i F_{ij}(t) \kappa_j}{E(t)} \right]^{\frac{1}{2} \mu(G) - \varepsilon(G) - \Delta(G)} \quad (3.9)$$

The integration in (3.9) contains the functions t^{s-1} which is a distribution that can be extended for $\text{Re } s < 0$, with poles for $s = 0, -1, -2, \dots$ [see (B.4)]. Then the generalized amplitudes can be extended to meromorphic functions on $\mathbb{C}^L$, with poles on the manifolds $\varepsilon(H) + \Delta(H) - \frac{1}{2} \mu(H) = 0, -1, -2, \dots$

Singularities appear also in overlapping diagrams, and the pole structure is quite complicated. The idea is to expand (3.9) in a Laurent series for each ε_j , and take the constant term averaged over all possible orders as the definition of the finite part. This is formalized by an operator W , called the evaluator, and satisfying the following conditions:

1. - Linearity
2. - Continuity
3. - If f is analytic at ε° , then $Wf = f(\varepsilon^\circ)$
4. - $Wf(\varepsilon_i, \varepsilon_j) = Wf(\varepsilon_j, \varepsilon_i)$
5. - If $f \in A$, $f^* \in A$, $f^*(\varepsilon) = f(\varepsilon^*)$, then $Wf^* = (Wf)^*$
6. - If f_1 and f_2 depend respectively on the disjoint sets $\{\varepsilon_j^1\}$ and $\{\varepsilon_j^2\}$, then $W(f_1 f_2) = Wf_1 Wf_2$.

We shall take the evaluator as

$$Wf = \frac{1}{(2\pi i)^L L!} \sum_{\pi(L)} \oint_{C(\pi(1))} d\varepsilon_1 \dots \oint_{C(\pi(L))} d\varepsilon_L \frac{f(\varepsilon)}{\varepsilon_1 \dots \varepsilon_L} \quad (3.10)$$

where $C(j)$ are contours $|\varepsilon_j| < R_j$, $R_1 > \frac{1}{j-1} R_j$, and $\pi(i)$ is a permutation.

Note that if we put all ε 's equal, and then take the constant coefficient as the definition of the finite part, property 6 fails to be obeyed. This happens

because of a conspiracy between the infinite part of one term and an irrelevant piece (of the order of ε) of the other part. This property is essential to prove the equivalence to other subtractive schemes. In fact, this is the question we shall treat now, namely, although the procedure is well defined to all orders of perturbation theory, we must prove that it is equivalent to a redefinition of the parameters of the Lagrangian. Only then can we call it a renormalization in the usual sense. We have first to prove that the evaluation procedure (3.10) is equivalent to the subtraction of polynomials in the external momenta. In that case, as a consequence of (3.1), the evaluation will be equivalent to the addition of derivatives of Seeley coefficients and external fields at coinciding points to the original Lagrangian.

4. - EQUIVALENCE TO SUBTRACTIVE SCHEMES

The definition of a subtraction scheme requires some detailed knowledge of the Feynman graph and of the type of counterterm required to render it finite. We shall say that a Feynman graph $G = G(V, L)$ is additively renormalized if there exist vertex parts $X(\{\varepsilon\}, g_i)$ associated to each partition P of the vertices V into generalized vertices $g_1, \dots, g_{p(P)}$, such that

$$\tau^{\text{fin}}(\{\delta_e\}, x_1, \dots, x_m) = \lim_{\varepsilon_e \rightarrow 0} \sum_P \tau_{P, X}(\{\delta_e\}, \{\varepsilon_e\}, x_1, \dots, x_m) \quad (4.1)$$

defines a tempered distribution, where the sum is over all partitions P , and $\tau_{P, X}(\{\delta_e\}, \{\varepsilon_e\}, X)$ is defined by

$$\tau_{P, X}(\{\delta_e\}, \{\varepsilon_e\}, x_1, \dots, x_m) = \prod_{i=1}^{p(P)} X(\{\varepsilon_e\}, g_i) \prod_{\text{connec}} Q_{\delta_e}^{\varepsilon_e}(e_i^e x_i) \quad (4.2)$$

with the second product being taken over all lines connecting the generalized graphs. If the partition is trivial, $P = \{\{V_1\}, \{V_2\}, \dots, \{V_n\}\}$, then $\tau_{P, X} = \tau$. The remaining terms are singularities of the diagram which have to be removed. What we have to show is that the finite amplitude obtained by the application of the evaluator (3.10) has the structure (4.1).

Consider now an analytic evaluator V (Appendix B). If H is a subgraph of G , V_H is the operator V applied to a function, but keeping the regulators ε_λ , for

$\lambda \notin H$ fixed. It follows that $V_H f$ is non-singular for $\varepsilon_\lambda \rightarrow 0$, $\lambda \in H$, but in general has singularities on $\varepsilon(H') = 0$ for H' not contained in H .

We state some lemmas. A sketch of the proofs is contained in Appendix B; for more details, see Ref. 12).

Lemma 1: if $f \in A$ has all its singularities in H' , $H' \subset H$, then

$$V_H f = V_{H'} f \quad (4.3)$$

Note that the factorization property was essential in this lemma. We defined the singular part of a diagram by the following procedure. If H is a subgraph of G , the operator $S(H)$ acts as follows:

$$S(H) = \sum_{H' \subset H} (-1)^{|H-H'|} V_{H'} \quad (4.4)$$

where the sum includes $H' = \emptyset$, and the difference $|H-H'|$ is the number of lines in H which are not in H' . The fact that $S(H)$ deserves to be called a singular part is motivated by the following lemma.

Lemma 2: a) If $S(H)f$ is non-zero, f is singular on $\varepsilon(H_1) = 0$, where $\bigcup_{i=1}^k H_i = H$.
b) If $f = f_1(\varepsilon_\lambda^1) f_2(\varepsilon_\lambda^2)$, where H_1 and H_2 are disjoint, then $S(H)f$ is non-zero only if $H \subset H_1 \cup H_2$.

In this case

$$S(H)f = [S(H \cap H_1) f_1] \cdot [S(H \cap H_2) f_2] \quad (4.5)$$

Again, the factorization property was used to prove 2b. We have:

Theorem 4.1: the analytic evaluators V can be decomposed as a sum of singular parts

$$V = \sum_{H \subset G} S(H) \quad (4.6)$$

The proof goes as follows. Substitute (4.4) into the right-hand side of (4.6). The terms V_H and $V_{H+\{\lambda\}}$ always cancel, using (4.3) and the difference in sign.

We shall relate the decomposition (4.6) to the partition (4.2). Consider the expression

$$\chi(\{\delta_e\}, \{\epsilon_e\}, \mathcal{G}) = \sum_{H \leq G'} S(H) \tau_{G'}(\{\delta_e\}, \{\epsilon_e\}) \quad (4.7)$$

where $\tau_{G'}$ is the generalized amplitude for G' , and the sum over all graphs H subordinate to $G' = G/\mathcal{G}$, where $\mathcal{G}$ is a set of generalized vertices and $G/\mathcal{G}$ is the graph obtained from G which has $\mathcal{G}_i$ as vertices, and the lines of G connecting different $\mathcal{G}_i$'s. The sum (4.7) is zero unless G' is 1PI. The expression (4.7) is a building block of the construction, namely:

Lemma 3: The distribution (4.7) is a vertex part for $\mathcal{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_n\}$.

This is a very important result, since the singular part of the amplitudes is shown to be a polynomial in the external momenta, and we have to subtract these singularities to define a renormalized theory. Locality of the Grassmann integration, in supersymmetric cases, is also immediate, since there are $L(H)$ propagators with $\delta(e_i^\lambda \theta_i)$ (or its derivatives) and $n(H)$ integrations. Since $L(H) - n(H) + 1 \geq 1$ for a non-tree graph, there are enough δ functions to perform the θ integrations.

Different ϵ -regulators were essential in practically all steps. In particular, if Lemma 1 were not true, renormalization in internal graphs would not be enough to remove the singularities, and we would have counterterms coming from higher terms in the heat-kernel expansion, as is shown by an explicit calculation, implying non-renormalizability in the usual sense. Moreover, we could not prove equivalence to a subtraction scheme.

We are very near the final result, but we need

Lemma 4: The vertex part $\tau_{P,X}$ defined in (4.2) is a sum of the singular parts of the diagram

$$\tau_{P,X} = \sum_{H \leq P} S(H) \tau_H \quad (4.8)$$

The main result now follows. Theorem 4.2: analytic renormalization is additive

$$\begin{aligned}
 W \tau_G(\{\delta_e\}, \{\epsilon_e\}, k_1, \dots, k_n) &= \sqrt{\tau_G(\{\delta_e\}, \{\epsilon_e\}, k_1, \dots, k_n)} \Big|_{\epsilon=0} \\
 &= \lim_{\epsilon_e \rightarrow 0} \sum_H S(H) \tau_G(\{\delta_e\}, \{\epsilon_e\}, k_1, \dots, k_n) \\
 &= \lim_{\epsilon_e \rightarrow 0} \sum_P \sum_{H \leq P} S(H) \tau_G(\{\delta_e\}, \{\epsilon_e\}, k_1, \dots, k_n) = \\
 &= \lim_{\epsilon_e \rightarrow 0} \sum_P \tau_{P, \chi}(\{\delta_e\}, \{\epsilon_e\}, k_1, \dots, k_n).
 \end{aligned}
 \tag{4.9}$$

5. - RENORMALIZABILITY AND GAUGE INVARIANCE IN SUPERSYMMETRIC YANG-MILLS THEORY

So far it has been proved that a theory can be rendered finite by counterterms which are locally dependent on the external fields and have a finite number of derivatives. External fields come from vertices (of Feynman rules), and also from the Seeley coefficients. Derivatives come also from the divergent products of Green functions (vertex parts). However, it is not yet clear whether these counterterms are finite in number and of the same form as the original Lagrangian. Clearly, this information can only be obtained from the specific model.

We must count the power associated to a diagram and see the type of vertex part that it produces. If the number $\mu(\gamma) - 2\Delta(\gamma)$ is positive, we have a contribution with a power corresponding in momentum to the vertex part, since the singular part has a $\delta^{(\mu-2\Delta)}(t)$ term. Then we have to count the number of derivatives coming from vertices and Seeley coefficients. Taking all this into account, we can proceed by computing the degree of divergence $\delta(\gamma)$ of some theories.

The theories to be exemplified are the ϕ^4 , the Wess-Zumino model, and the supersymmetric Yang-Mills theory. The Seeley coefficients that we need are known. In the ϕ^4 theory, the kinetic term in the presence of a background field ϕ presents the Laplacian

$$\square_B = \square + \frac{1}{2} g \phi^2$$

and we have

$$a_1(x, x) = -\frac{1}{2} g \phi(x)^2 \quad (5.1)$$

$$a_2(x, x) = \frac{1}{8} g^2 \phi(x)^4 + \text{total divergence} \quad (5.2)$$

In the Wess-Zumino model, the kinetic Lagrangian for the quantum fields is

$$\mathcal{L}^{(2)} = \int d^4\theta \quad \overbrace{\left(\begin{array}{c} \phi \\ \bar{\phi} \end{array} \right)} \left[\begin{array}{cc} -\frac{D^2}{\square} \frac{\lambda}{2} \phi & 1 \\ 1 & -\frac{\bar{D}^2}{\square} \frac{\lambda}{2} \bar{\phi} \end{array} \right] \left(\begin{array}{c} \phi \\ \bar{\phi} \end{array} \right) \quad (5.3)$$

giving rise to

$$a_1((x, \theta), (x, \theta)) = -\lambda \left[\begin{array}{cc} \bar{D}^2 \bar{\phi} & 0 \\ 0 & D^2 \phi \end{array} \right] \quad (5.4)$$

and

$$a_2((x, \theta), (x, \theta)) = \frac{\lambda^2}{2} \left[\begin{array}{cc} 0 & \bar{D}^2 D^2 \bar{\phi} \phi \\ D^2 \bar{D}^2 \bar{\phi} \phi & 0 \end{array} \right] \quad (5.5)$$

where the derivatives D^2 and $\bar{D}^2$ act on everything to the right [i.e., also on the Grassmann δ function in (2.4)].

Finally, in the Yang-Mills case, we have

$$\mathcal{L}^{(2)} = \sqrt{\square_B} \sqrt{\quad} \quad (5.6)$$

with

$$\square_B = D^a D_a - \bar{W}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} - \bar{W}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} \quad (5.7)$$

We can expand $\bar{W}^\alpha \partial_\alpha + \bar{W}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}}$ in the heat-kernel, and also the background field in $\mathfrak{D}^a$. Working it out, we have ⁽⁵⁾

$$a_1((x, \theta), (x, \theta)) = -\bar{W}^\alpha \partial_\alpha - \bar{W}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} \quad (5.8)$$

$$a_2((x, \theta), (x, \theta)) = \frac{1}{2} (\bar{W}^\alpha \partial_\alpha + \bar{W}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}}) (\bar{W}^\beta \partial_\beta + \bar{W}^{\dot{\beta}} \bar{\partial}_{\dot{\beta}}) + \frac{1}{12} F_{ab}^2 \quad (5.9)$$

where again the derivatives act on everything to the right. The last term in a_2 will not contribute to the infinite part, since it does not have enough derivatives to avoid zeros coming from products of Grassmann delta functions, as we shall see.

As a simple example, consider the ϕ^4 theory. In a given diagram with V_3 vertices with one background field, V_4 vertices with none and d_1 propagators of type G_1 , the degree of divergence in the $\underline{t}$ variables is (there is no derivative coupling):

$$\delta(\gamma) = \mu(\gamma) - 2\Delta(\gamma) = 4\Lambda - 2L - 2 \sum i d_i \quad (5.10)$$

where Λ is the number of loops. The counting identities

$$3\sigma_3 + 4\sigma_4 = 2L,$$

and

$$\Lambda = L - \sigma_3 - \sigma_4 + 1$$

imply

$$\delta(\gamma) = 4 - \sigma_3 - 2\delta_1 - 4\delta_2 \quad (5.11)$$

where we did not include higher δ_i , since they would make $\delta(\gamma)$ automatically negative. We have the following counterterms

$$1) \quad \nu_3 = \delta_1 = 0, \quad \delta_2 = 1$$

The divergence comes from the first term in the expansion of $\chi^{\varepsilon(H)-\delta(H)-1}$ [term $\delta(t)$], and there is no momentum dependence. The counterterm is a c-number times $a_2(x,x)$.

$$2) \quad \nu_3 = \delta_2 = 0, \quad \delta_1 = 2$$

Counterterm $c(a_1(x,x))^2$.

$$3) \quad \nu_3 = 2, \quad \delta_1 = 1, \quad \delta_2 = 0$$

The same divergence, and a counterterm $\phi^2 a_1(x,x)$

$$4) \quad \nu_3 = 2, \quad \delta_1 = \delta_2 = 0$$

Divergence comes from $\delta''(t)$, with momentum dependence. Counterterm $(\partial_\mu \phi)^2$.

$$5) \quad \nu_3 = 4, \quad \delta_1 = \delta_2 = 0$$

gives a counterterm $\phi(x)^4$.

The terms 1), 2), 3) and 5) renormalize the coupling term of the Lagrangian, and 4) the kinetic term. The β function can be computed¹⁰⁾.

A supersymmetric example is the Wess-Zumino model¹⁷⁾. There is only one type of vertex and the lines are all internal. The usual power counting in the t -variables gives $4-v$. Because of chirality, there are $2v$ factors D^2 and $\bar{D}^2$, corresponding to two factors for each vertex, and using 2Δ of them to avoid zeros in the product of Grassmann delta functions (we need one factor $D^2 \bar{D}^2$ for each loop), we have

$$\begin{aligned} \delta(\gamma) &= 4-v + (2v - 2(L-v+1)) - \delta_1 - 2\delta_2 = \\ &= 2 - \delta_1 - 2\delta_2 \end{aligned} \quad (5.12)$$

Again, if we have any propagator of the type G_δ , for $\delta > 3$, $\delta(\gamma)$ is automatically negative. From the explicit expression of a_1 and a_2 we see that there are counterterms of the type $\bar{\phi}\phi$ as is well known¹⁷⁾. Explicit two-loop computations have also been done¹⁰⁾.

The discussion for Yang-Mills theory goes along the same lines. We have:

From t-power counting : $4\Lambda - 2L = 2L - 4v + 4$

From vertices contributing $D^2\bar{D}^2$: $v - \Lambda = 2(2v - L - 1)$

(where 2 comes from the $\dim D^2\bar{D}^2 = 2$)

Propagator g_1 contributes $-2(W_\alpha)$ and $+\frac{1}{2}$ (derivative)

Propagator g_2 contributes -3 (-4 from power counting τ^2 , and $+1$ from two supersymmetric derivatives).

Because the final integrand is chiral, we have one extra in the counting. Thus the counting reads

$$\delta(H) = 3 - \frac{3}{2}\delta_1 - 3\delta_2 \quad (5.13)$$

The possible divergences are $\delta_1 = 0$, $\delta_2 = 1$, or $\delta_1 = 2$, $\delta_2 = 0$, always with $\delta(H) = 0$, consequently without momentum dependence. From this counting and the explicit a_1 and a_2 coefficients, we have

$$\delta\mathcal{L} = c \int w^2 d^2\theta \quad (5.14)$$

We have yet to include ghosts and matter fields. The discussion is the same as in the Wess-Zumino case and the previous one, so that only a_1 and a_2 coefficients contribute, bringing nothing new.

6. - CONCLUSIONS

We have implemented a method for isolating poles in the computation of the effective action, which is explicitly supersymmetric and background gauge-invariant. We did not consider the problem of the dependence on the α parameter (coefficient of the gauge fixing term) which at lowest order is 1, and at higher order renormalized in the same way as the other kinetic term¹⁸⁾, order by order. There is, in our opinion, an advantage over the usual dimensional reduction since no ambiguity arises, and we work always in the physical dimension.

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APPENDIX A : GRAPHICAL TERMINOLOGY

We give here the notions of graph theory used in this work.

A Feynman diagram is denoted by $G(V, L)$, where $V = \{V_1, \dots, V_n\}$ are vertices of G at the points $\{x_1, \dots, x_n\}$, and L is the set of lines (L is also the number of lines). To each line we associate an initial vertex V_{i_ℓ} and a final vertex V_{f_ℓ} . The matrix e_i^ℓ takes values ± 1 when $i = f_\ell$ (resp. i_ℓ), and zero otherwise. The heat kernel expansion functions of the propagators are denoted by $G_N(e_i^\ell x_i)$, and multiply the Seeley coefficient $a_N(x_{i_\ell}, x_{f_\ell})$.

A graph is said to be 1PI (one-particle irreducible) if it cannot be disconnected by removing a single line. It is said to be two-connected if it cannot be disconnected by removing a single vertex.

A generalized vertex in a graph $G = G(V, L)$ with vertices $V = \{V_1, \dots, V_n\}$ is a subset σ of V . G/σ is the subgraph of G with all lines $\ell \in L$, such that i_ℓ, f_ℓ are in σ .

Disjoint graphs have no common element, and overlapping graphs, having at least one line in common, are not contained in one another.

A K -tree (T_K) in a graph with n vertices is a set of $n-K$ lines not forming a loop.

A singularity family F is a maximal set of non-overlapping two-connected subgraphs of G (single lines also included), satisfying the condition that no union of elements of a family is two-connected. Moreover, G belongs to F (by maximality), and for each graph H of the family there is exactly one line, $\sigma(H)$, which is in no subgraph of H in F (the latter properly follows from the definition).

Consider a graph $G(V, L)$, a generalized vertex $\sigma = \{V_1', \dots, V_p'\}$, and $G' = G/(V_1', \dots, V_p')$. The 1PI graph H is subordinate to G' (we denote $H < G'$) if H has $V_1', \dots, V_p'$ as vertices. Given a partition P of the vertices $V_1, \dots, V_n$ of $G(V, L)$ into n generalized vertices $\{V_j^1\}_{j=1, \dots, n}$, a graph $H \subset G$ is said to be

subordinate to $P(H < P)$, if H has the connected components $H_1, \dots, H_n$, and each of them is subordinate to $G_i = G/(V_1^1, \dots, V_{n(i)}^1)$.

For each singularity family F we can find a domain $D(F)$ in the integration region of the τ parameters, such that in $D(F)$ $\tau_\ell > 0$, and $\tau_{\sigma(H)} > \tau_\ell$ if $\ell \in H$, where $\tau_{\sigma(H)}$ is the τ -parameter associated with the only line of H which is in no subgraph of H in F . The union of all above domains is R_+^L .

The scaling variables can be introduced in $D(F)$, assigning a variable t_H for each $H \in F$, $H \neq G$, $0 < t_H < 1$, and t_G for G , $0 < t_G < \infty$. The following change of variables is made in $D(F)$:

$$\tau_\ell = \prod_{H \in H} t_H$$

A vertex part for the generalized vertex $\sigma_j = \{V_1', \dots, V_n'\}$ is a distribution $X(\{\epsilon\}; \sigma_j)$ having the form

$$X(\{\epsilon\}; \sigma_j) = \begin{cases} D[\delta(x_1' - x_2') \dots \delta(x_{n-1}' - x_n')] , & n > 1 \\ 1 , & n = 1 \\ 0 & \text{otherwise} \end{cases} \quad \begin{matrix} G/\sigma_j \text{ 1PI} \end{matrix}$$

where D is any differential operator. In momentum space it is a δ of momentum conservation times a polynomial in the momenta.

APPENDIX B : SOME DEFINITIONS, AND PROOFS OF THE LEMMAS

An analytic evaluator $V: A \rightarrow A_0$, where A_0 is the subspace of functions analytic at ε_0 is defined by the following properties:

1. Linearity.
2. If $f_n \in A$, and $f_n \rightarrow 0$ uniformly in the neighbourhood of $\varepsilon = 0$, then $Vf_n \rightarrow 0$.
3. If $f \in A$ is analytic at ε^0 , $Vf = f(\varepsilon^0)$.
4. $Vf(\varepsilon_i, \varepsilon_j) = Vf(\varepsilon_j, \varepsilon_i)$.
5. If $f \in A$, $f^* \in A$, $f^*(\tau) = f(\bar{\tau})$ then $Vf^* = (Vf)^*$.
6. If f_1 and f_2 depend on $\{\tau_i^1\}$ and $\{\tau_i^2\}$ respectively, and these sets are disjoint, then $V(f_1 f_2) = (V(f_1))(Vf_2)$.
7. If $f \in A$ is independent of ε_λ , Vf will also be.

The difference between the analytic evaluator V and the evaluator W is in the defining properties 2 and 7.

An analytic evaluator is given by

$$Vf(\varepsilon) = \frac{1}{(2\pi i)^L L!} \sum_{\pi(L)} \oint_{C(\pi(1))} dv_1 \dots \oint_{C(\pi(L))} dv_L \frac{f(v)}{(v_1 - \varepsilon_1) \dots (v_L - \varepsilon_L)} \quad (B.1)$$

and $|\varepsilon|$ is smaller than the radius R_1 (smallest contour).

Proof of Lemma 1: One introduces the function

$$g(\varepsilon) = \sqrt{\tau} \varepsilon(H'') f(\varepsilon),$$

which is non-singular at $\{\varepsilon_\lambda = 0, \lambda \in H\}$ and expands it around $\varepsilon = 0$ in a power series, which is thus a sum of terms f_H, f_{H-H}, f_{G-H} . The structure of the terms is the same for g , and the lemma holds as a consequence of properties 3 and 6.

Proof of Lemma 2: 2a) Taking f singular in $\varepsilon(H')$, with $H' \subset H$ but not in $\lambda \notin H'$, the two terms cancel exactly in $S(H)f$, as a consequence of Lemma 1, namely $V_{H'+\{\lambda\}} = V_{H'}$.

2b) If $S(H)f \neq 0$, then $H \subset H_1 \cup H_2$ by 2a), and the result follows from the factorization property 6.

Proof of Lemma 3: The expression for $\tau(\{\delta\}, \{\varepsilon\}, k)$, (3.9) can be written as

$$\tau(\{\delta\}, \{\varepsilon\}, k_1, \dots, k_n) = \Gamma(\varepsilon(G') - \frac{1}{2}\mu(G') + \Delta(G')) \cdot \sum_F \tau_F(\{\delta\}, \{\varepsilon\}, k_1, \dots, k_n) \quad (B.2)$$

We apply $S(H)$ to the right-hand side of (B.2). If $H = G$, τ_F has singularities $\varepsilon(H') = 0$, where H' does not contain $\sigma(G)$; therefore, if $\mu(G) - 2\Delta(G) < 0$, the expression is zero (τ_F has no singularity). The singular part is just obtained from the explicit gamma function

$$\begin{aligned} \text{sing } \Gamma(\varepsilon(G') - \frac{1}{2}\mu(G') + \Delta(G')) \tau_F &= \\ &= \frac{(-1)^{\mu(G') - 2\Delta(G')}}{[\frac{1}{2}\mu(G') - \Delta(G')]! \varepsilon(G')} \tau_F \Big|_{\varepsilon(\sigma(G')) = -\sum_{\ell \neq \sigma(G')} \varepsilon_\ell} \end{aligned} \quad (B.3)$$

We take subgraphs H , $H \subset G'$. $S(H)$ gives zero, unless it applies to terms defined in the manifolds $\varepsilon(H_1), \dots, \varepsilon(H_n)$, $UH_1 = H$ with singularities associated to each of them. We separate the divergent part associated to the H_1 , considering the distribution t^{s-1} which is smeared with a test function $f(t)$:

$$\int_0^1 t^{s-1} f(t) dt = \sum_{i=0}^n \frac{f^{(i)}(0)}{i! (s+i)} + \int_0^1 t^{s-1} g(t) dt \quad (B.4)$$

where

$$g(t) = f(t) - \sum_{i=0}^n f^{(i)}(0) \frac{t^i}{i!}$$

The expression (B.4) provides an analytic continuation of t^{s-1} for $\text{Re } s < 0$, and at the same time gives the poles at $s = 0$. We obtain

$$\text{sing } t^{\varepsilon(H) - \frac{1}{2}\mu(H) + \Delta(H) - 1} = \frac{\delta^{\left(\frac{1}{2}\mu(H) - \Delta(H)\right)}(t)}{\left[\frac{1}{2}\mu(H) - \Delta(H)\right]! \varepsilon(H)} \quad (\text{B.5})$$

In order to have the required polynomial form as a function of the momenta, it is enough to show that $F_{ij}(t)|_{t=0} = 0$, which follows by inspection.

Proof of Lemma 4: P is a partition of the graph into generalized vertices $G_1, \dots, G_n$, and $H < P$. Then

$$\tau_G(\{\delta\}, \{\varepsilon\}, k_1, \dots, k_n) = \prod_{i=1}^n \tau_{G_i}(\{\delta_i\}, \{\varepsilon_i\}) \times \prod_{L_N} g_{\delta_{L_N}}(\varepsilon_{L_N}) \quad (\text{B.6})$$

where L_N are lines not contained in any G_i , but connecting them. It follows that

$$\begin{aligned} \sum_{H < P} S(H) \tau_G(\{\delta\}, \{\varepsilon\}) &= \\ &= \sum_{H < P} \prod_{i=1}^n S(H_i) \tau_{G_i}(\{\delta_i\}, \{\varepsilon_i\}) \prod_{L_N} g_{\delta_{L_N}}^{\varepsilon_{L_N}} = \\ &= \prod_{i=1}^n \sum_{H_i < G_i} S(H_i) \tau_{G_i}(\{\delta_i\}, \{\varepsilon_i\}) \prod_{L_N} g_{\delta_{L_N}}^{\varepsilon_{L_N}} = \\ &= \tau_{P, \times}(\{\delta\}, \{\varepsilon\}). \end{aligned} \quad (\text{B.7})$$

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