THE QUANTUM MECHANICS OF THE SCALAR FIELD
IN THE NEW INFLATIONARY UNIVERSE

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ABSTRACT

An attempt is made to clarify the quantum theory of the "slow-rollover" phase transition which characterizes the new inflationary universe model. We discuss the theory of the upside-down harmonic oscillator as a toy model, with particular emphasis on the fact that the system can be described at late times by a classical probability distribution. An approximate but exactly soluble model for the scalar field is then constructed, based on three principal assumptions: 1) exact de Sitter expansion for all time; 2) a quadratic potential function which changes from stable to unstable as a function of time; and 3) an initial state which is thermal in the asymptotic past. It is proposed that this model would be the proper starting point for a perturbative calculation in more realistic models. The scalar field can also be described at late times by a classical probability distribution, and numerical calculations are carried out to illustrate how this distribution depends on the parameters of the model. For a suitable choice of these parameters, a sufficient period of inflation can be easily obtained. Density fluctuations can be calculated exactly in this model, and the results agree very well with those previously obtained using approximate methods.

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To determine the modulation for the new material, we begin by summarizing what information is available about its properties. The goal is to understand how the material interacts with light and how this interaction can be controlled and explored. The key aspects to consider include:

1. The refractive index of the material, which determines the speed of light in the material compared to vacuum.
2. The absorption coefficient, which indicates how much light is absorbed by the material.
3. The extinction coefficient, which relates to the scattering of light by the material.

These properties are essential for designing optical devices and understanding the behavior of light in the material. The equations below summarize the relationships involved:

\[
\begin{align*}
\alpha &= \frac{1}{\lambda} \ln \left( \frac{1}{R} \right) \quad \text{(refractive index)} \\
\kappa &= \frac{1}{\lambda} \ln \left( \frac{1}{T} \right) \quad \text{(absorption coefficient)} \\
\end{align*}
\]

where \( \alpha \) is the absorption coefficient, \( \kappa \) is the extinction coefficient, \( R \) is the reflectance, and \( T \) is the transmittance.

Additionally, the interaction of light with the material can be described by the following equation:

\[
\frac{I}{I_0} = 1 - 2R - T
\]

where \( I \) is the intensity of the light after interaction, \( I_0 \) is the initial intensity, and \( R \) and \( T \) are as defined above.

Understanding these relationships is crucial for the development of novel optical materials with specific applications in various fields.
the linearized equation

$$\ddot{\phi} + 3\dot{\phi} \dot{\phi} - e^{-2\phi} \nabla^2 \phi = - \frac{\partial^2 V}{\partial \phi^2} (\phi_0) \delta \phi.$$  \hspace{1cm} (1.5)

Using these ideas, it is then possible to calculate the resulting spectrum of mass density
perturbations. (For more details, see Section VIII.)

This standard picture relies on a number of assumptions, and various questions have
been raised about its validity:

1. Is the picture of a classical slow rollover valid? It has been pointed out
by Mazenko, Unruh, and Wald\textsuperscript{11} that at high temperatures, when one says that
$\phi \approx 0$, one really means that the spatial or time average of $\phi$ is about equal to
zero— the field itself is undergoing large fluctuations. As the system cools, they
argue, it is possible that these fluctuations, which extend initially out to $\phi = \pm \phi_*$
or more, will cause the scalar field to settle quickly into small regions with $\phi \approx
\pm \phi_*$ in each of these regions. When one looks at the spatial average one might see
what appears to be a rolling motion, but the actual local dynamics could be quite
different. In particular, these authors claim that the picture of a classical slow
rollover is invalid for Coleman-Weinberg models\textsuperscript{12,9,3} with $g^2 \approx 1$, and also for
all models in which the scalar field which drives the inflation interacts only with
itself. We find, however, that self-interacting scalar field models appear perfectly
acceptable provided that the coupling constant is sufficiently small. Presumably
a similar statement could be made for the Coleman-Weinberg case, although we
do not study the case in this paper. On the other hand, our results agree with
Mazenko, Unruh, and Wald for cases with couplings of order unity.

2. What is the physical significance of the classical function $\phi_0(t)$? Hawking
and Moss\textsuperscript{13} point out that the system begins in a thermal ensemble which
possesses an exact symmetry, $\phi \rightarrow -\phi$. The dynamics is also consistent with this
symmetry, and it therefore follows that $\langle \phi(\vec{x}, t) \rangle$ remains zero for all time—
the field presumably does roll down the hill, but since it is equally likely to roll in
any direction, the expectation value remains zero. Hawking and Moss developed
an alternative method of calculation in which only the operator $\phi^2(\vec{x}, t)$ appears.
Using this method, they calculated\textsuperscript{13} a value for the fluctuation amplitude which
was in significant disagreement with calculations done using the standard picture.\textsuperscript{5} These authors have since retracted their calculation,\textsuperscript{14} but the question
which they raise still requires an answer. In our method of calculation, $\phi_0(t)$ will
be given a very explicit meaning.

3. The standard picture described above gives no prediction for how long
the scalar field will hover about $\phi = 0$ before it begins rolling down the hill
of the effective potential. This question is of course crucial, since it determines
whether or not sufficient inflation is obtained. The question has been studied by
Linde\textsuperscript{15} and by Vilenkin and Ford\textsuperscript{16}, who calculated the behavior of $\langle \phi^2(t) \rangle$.
This operator is infinite and requires a subtraction, and it was never completely
clear to us how this expectation value is related to the behavior of $\phi$ itself. Our
methods will address this question in what we feel is a more precise way, but we
should say at the beginning that our results are in qualitative agreement with
the results of these previous authors.

4. It is well known\textsuperscript{17} that the effective potential shown in Figure 1 is actually
ill-defined. In particular, the behavior shown for the zero temperature potential
at $\phi < \phi_*$ violates rigorous convexity theorems. The shape of the true effective
potential depends on choices made in its definition. One choice leads to a Maxwell
construction, $V(\phi) = 0$ for $|\phi| < \phi_*$. Another choice, based on analytic coninuation in the parameters of the theory, leads to an effective potential which is
complex when $|\phi| < \phi_*$. In the light of these statements, it is not clear that
the standard picture is reliable. The problem is essentially that the standard picture
attempts to describe a time-dependent situation using the effective potential,
which is a quantity defined to describe static, equilibrium situations. Our method
will avoid this question, since it will involve a dynamical calculation which makes
no use of the effective potential.

The goal of the present work is to provide answers to these questions. In order to
Section III

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it is probed at length scales large compared to a. In terms of these parameters, one finds
\[
H = \frac{1}{2a^2} \tan(\phi - iw_0t) \\
= \frac{1}{2a^2} \sin(2\phi - i\sinh 2w_0t) \\
= \frac{1}{2a^2} \cos 2\phi + \cosh 2w_0t,
\]
where \(\phi\) is a real constant of integration which is related to the width of the wave packet at \(t = 0\). The differential equation is satisfied for complex values of \(\phi\), but we choose to absorb the imaginary part of \(\phi\) into a redefinition of the origin of \(t\). The wave packet is then at its minimum width when \(t = 0\). For a properly normalized wave function one finds
\[
A = (2\pi)^{-1/4} \delta(\cos(\phi - iw_0t))^{-1/2},
\]
where
\[
\delta = a(\sin 2\phi)^{-1/2}.
\]

We are particularly interested in the behavior of \(\psi(x,t)\) for large \(t\), which is given by
\[
\psi(x,t) \underset{t \to \infty}{\approx} (2/\pi)^{1/4} e^{-x^2/4a^2} \exp \left\{ -e^{-2at^2} x^2 + \frac{ix^2}{2a^2} \right\}.
\]
The probability distribution for \(x\) is then Gaussian, with
\[
< x^2 > \approx \frac{1}{4} \frac{1}{a^2} e^{2at^2}.
\]

Note that the \(\phi\)-dependence of \(< x^2 >\) is easily understood. As \(\phi\) is varied, \(\delta^2\) is minimized when \(\phi = \pi/4\), with \(\delta = a\). For \(\phi < \pi/4\), the initial probability distribution for \(x\) is broad, and it therefore spreads quickly. For \(\phi > \pi/4\), the initial wave function is narrowly peaked in \(x\), and hence the spread at momentum is large; this spread of momentum results in a rapid spreading of the \(x\)-distribution.

Our main point in introducing this model is to show that the quantum mechanical wave function at large times is accurately described by classical physics. Let us first consider the commutator of the operators \(x\) and \(p\), where \(p = -i\hbar \frac{\partial}{\partial x}\). Note that
\[
p\psi = \hbar \frac{\partial}{\partial x} \psi + O(e^{-2at^2})
\]
\[
= \sqrt{mk} \frac{\partial}{\partial x} \psi + O(e^{-2at^2}),
\]
where \(\sqrt{mk}\) is in fact the momentum which would be attained by a classical particle which rolled from rest at the peak of the potential to the point \(z\). Let us now examine the noncommutativity of \(z\) and \(p\):
\[
zp\psi = \sqrt{mk} x^2 \psi + O(e^{-2at^2})
\]
\[
pz\psi = \sqrt{mk} x^2 \psi - i \hbar \psi + O(e^{-2at^2}).
\]
The commutator contribution \(-i\hbar \psi\) will be insignificant compared to the other term if \(\hbar \ll \sqrt{mk} x^2\), or equivalently if \(a^2 \ll x^2\). Thus at large times, in the region \(x^2 \gg a^2\), the commutator \([x,p]\) becomes negligible and thereby poses no barrier to a classical description of the system. The general rule is that whenever distance over which the phase of the wave function changes by \(2\pi\) (i.e., the de Broglie wavelength) is much shorter than any other relevant length, then classical physics applies. Alternatively, one can say that classical physics applies whenever it is possible to ignore uncertainties \(\Delta x\) and \(\Delta p\) satisfying
\[
\Delta x \Delta p \approx \hbar.
\]

Note however that the wave function is definitely not sharply peaked about one particular classical trajectory. Thus, the large \(t\) behavior of the system must be described not by a classical trajectory, but instead by a classical probability distribution:
\[
f(x,p,t) = |\psi(x,t)|^2 \delta(p - \sqrt{mk} x)
\]
\[
= \left( \frac{2}{\pi \hbar} \right)^{1/4} e^{-x^2/4a^2} \exp \left\{ -e^{-2at^2} x^2 / \hbar^2 \right\} \delta(p - \sqrt{mk} x).
\]
To see that \(f(x,p,t)\) provides an accurate classical description of the system at late times, one must verify the following two statements:

a) \(f(x,p,t)\) describes classical physics — that is, it obeys the classical equations of motion:
\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} = 0.
\]

b) For any dynamical variable (i.e., any function \(Q(x,p,t)\)), the expectation value of \(Q\) can be computed by using either the quantum mechanical wave function \(\psi\) or the classical probability distribution \(f\). More precisely, it can be
a problem which can many important features in common with the problem in which they
are both measurable. The novelty of this approach is that we consider the problem as a
problem of a probability distribution of complex interactions.

The approach to a problem of this type is due to the work of (1) the work of (2) in
reconnection (3) and their ideas are extended to the problem of (4) in this
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approach is that we consider the problem as a problem of a probability distribution of complex interactions.

The probability distribution is defined by the following equation:

\[ \mathbb{P}(\mathbf{x}) = \frac{1}{Z} \exp \left( -\frac{1}{2} \mathbf{x}^T \mathbf{K} \mathbf{x} \right) \]

where \( \mathbf{K} \) is the covariance matrix and \( Z \) is the normalization constant.

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This approach is similar to the one in (1) where the probability distribution is
defined by the following equation:

\[ \mathbb{P}(\mathbf{x}) = \frac{1}{Z} \exp \left( -\frac{1}{2} \mathbf{x}^T \mathbf{K} \mathbf{x} \right) \]

where \( \mathbf{K} \) is the covariance matrix and \( Z \) is the normalization constant.
concerns us. In this section we will describe an idealized quantum field theory model which approximates the behavior of the scalar field in the new inflationary universe scenario. The model is a free field theory, and is therefore exactly soluble. It is of course not totally realistic, but we believe that it qualitatively describes the correct physics. Furthermore, we hope that it will serve as a valid zero order approximation to a more complicated calculational scheme in which the interactions are taken into account perturbatively.

The idealized model is based on three assumptions, which we will discuss one at a time.

**Assumption 1:** We assume that the space-time is described exactly by the de Sitter metric:

$$ds^2 = -dt^2 + e^{2\chi}d\mathbf{x}^2,$$  
(3.1)

where the expansion rate \( \chi \) is related to the mass density \( \rho_0 \) of the false vacuum by

$$\chi = \sqrt{\frac{8\pi}{3} G \rho_0},$$  
(3.2)

where \( G \) denotes Newton’s constant.

Thus, we are paying no attention to the approach to de Sitter space, but simply assuming that the space is de Sitter for all time. For comoving wavelengths which are short compared to the Hubble length before the onset of inflation, this inaccuracy in the treatment of the metric should have very little effect—wavelengths which are very short compared to the Hubble length evolve in any case as if they were in Minkowski space. However, we must remember that the longest wavelengths—those which are comparable to or larger than the Hubble length at the onset of inflation—are not being treated accurately.

By using Assumption 1, we are also ignoring the fluctuations of the metric which result from the fluctuations of the matter fields.

We will consider a single scalar field evolving in this idealized de Sitter space. Such a field can be described by an action

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right),$$  
(3.3)

where the potential \( V(\phi) \) is given by

$$V(\phi) = \frac{1}{4}\lambda \left( \phi^2 - \frac{\mu^2}{\lambda} \right)^2.$$  
(3.4)

For pedagogical purposes we are setting \( c = 1 \) and \( k \) (Boltzmann constant) equal to one, but in Sections III-V we will keep factors of \( \kappa \). It is then necessary to specify, for example, whether \( \mu \) denotes a mass or an inverse length. We make this choice according to the usual prescription that \( S \) should represent the action of a classical field theory, and the quantity \( \kappa \) should not appear until the classical theory is quantized. \( S \) must therefore have the units of \( \text{mass} \times \text{length} \), from which it follows that \( \phi \) has units of \( \text{mass}^{1/2} \times \text{length}^{-1/2} \), \( \mu \) has units \( \text{length}^{-1} \), and \( \lambda \) has units \( \text{mass}^{-1} \times \text{length}^{-1} \). Note that \( \kappa \lambda \) is then dimensionless.

A plot of this potential is shown as Figure 4. The potential has a minimum at

$$\phi_0 = \frac{\mu}{\sqrt{\lambda}}.$$  
(3.5)

The false vacuum energy density is given by

$$\rho_0 = \frac{\mu^4}{4\lambda},$$  
(3.6)

leading to a de Sitter expansion rate

$$\chi = \sqrt{\frac{2\pi G}{3\lambda}} \mu^2.$$  
(3.7)

Using the metric (3.1), the action can be written out more explicitly as

$$S = \int d^4x L,$$  
(3.8a)

where the Lagrangian density is given by

$$L = e^{2\chi} \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} e^{-2\chi} (\nabla \phi)^2 - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4 - \frac{\mu^4}{4\lambda} \right).$$  
(3.8b)
We consider the continuous problem of the motion of a mass point under the Lorentz force described by the equation of motion:

\[ m \frac{d^2 \mathbf{r}}{dt^2} = q \mathbf{E} \times \mathbf{B} + F(\mathbf{r}, \dot{\mathbf{r}}, t) \]

where \( m \) is the mass of the point, \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic fields, respectively, and \( F(\mathbf{r}, \dot{\mathbf{r}}, t) \) is an additional force term.

The Lorentz force is given by:

\[ F(\mathbf{r}, \dot{\mathbf{r}}, t) = q(\mathbf{E} \cdot \dot{\mathbf{r}}) \mathbf{E} - q \mathbf{E} \times (\mathbf{E} \times \mathbf{r}) - q \mathbf{B} \times (\mathbf{B} \times \dot{\mathbf{r}}) \]

where \( q \) is the charge of the point.

In the limit of small \( \mathbf{r} \), the Lorentz force can be approximated by:

\[ F(\mathbf{r}, \dot{\mathbf{r}}, t) \approx q(\mathbf{E} \cdot \dot{\mathbf{r}}) \mathbf{E} \]

The motion of the point is then described by the equation:

\[ \mathbf{r}(t) = \mathbf{r}(0) + \frac{1}{m} \int_0^t \dot{\mathbf{r}}(\tau) \, d\tau \]

where \( \dot{\mathbf{r}}(\tau) \) is the velocity at time \( \tau \).

The conservation of momentum is given by:

\[ \dot{p}_x = \frac{d}{dt} (p_x + qA_x) = \frac{d}{dt} (\dot{p}_x + qB_x) \]

where \( p_x = m \dot{x} \) is the momentum in the x-direction.

The conservation of angular momentum is given by:

\[ \omega(t) = \frac{d}{dt} \mathbf{L} = \frac{d}{dt} (m \mathbf{r} \times \mathbf{v}) = q \mathbf{E} \times \mathbf{v} \]

where \( \omega(t) \) is the angular velocity and \( \mathbf{L} \) is the angular momentum.
Weinberg\cite{12} type. However, by studying the case \( \mu^2 = 0 \) in our model one can at least learn about the initial behavior of fluctuations in Coleman-Weinberg models. We hope that in the future we will be able to treat this case in more detail.

The dynamics of our model system has now been completely defined, and our one remaining assumption will specify the nature of the initial state. It is here that questions about the nature of the de Sitter space "vacuum" will arise. However, before we can describe this last assumption, it is necessary to develop the description of the dynamics in more detail.

Since the problem has been reduced to a free field theory, we can use a Fourier decomposition in order to obtain decoupled degrees of freedom. Order to keep the analogy with the upside-down harmonic oscillator of Section II as close as possible, we will expand in sine and cosine functions rather than in complex exponentials, so that the expansion coefficients will be real. Furthermore, we will begin by imagining that our system is confined to a cubic box with periodic boundary conditions, with the coordinate length of each side fixed at a value \( \delta \)-- the Fourier decomposition is then a discrete sum. The limit \( \delta \to \infty \) can be taken at the end. Thus we write

\[
\phi(x, t) = \frac{1}{(2\pi)^{3/2}} \left( \frac{2\pi}{\delta} \right)^3 \left\{ \sqrt{2}\sigma(0, t) + \sum_{\vec{k} \neq 0} \sigma_+(\vec{k}, t) \cos \vec{k} \cdot \vec{x} + \sigma_-(\vec{k}, t) \sin \vec{k} \cdot \vec{x} \right\},
\]  

(3.15)

where

\[
\sigma_+(-\vec{k}, t) \equiv \sigma_+(-\vec{k}, t)
\]

and

\[
\sigma_-(\vec{k}, t) \equiv -\sigma_-(\vec{k}, t).
\]

Note that the zero-momentum mode lacks the \( \vec{k} \to -\vec{k} \) redundancy exhibited by all other modes, and it therefore requires special treatment. The normalisation used for the zero-momentum mode has been chosen to simplify the subsequent equations.

It can be seen already that the expansion in sines and cosines is both unfamiliar and a bit awkward. We will proceed with it nonetheless, because we think that the analogy with the upside-down harmonic oscillator is conceptually very important. However, in Appendix A we will translate our results into the language of the usual expansion in complex exponentials.

To simplify our notation, we denote the mode variables \( \sigma(0, t) \) and \( \sigma_\pm(\vec{k}, t) \) generically by \( \sigma_\alpha(t) \). Whenever \( \alpha \) is summed, we adopt the convention that the sum includes the zero momentum mode, and also one entry of \( \sigma_+ \) and one entry of \( \sigma_- \) for each pair \( (\vec{k}, -\vec{k}) \).

Since we have reduced the problem to a free field theory, the mode variables \( \sigma_\alpha(t) \) will completely couple from each other. Thus we can study the quantum mechanics of these mode variables one at a time, and each variable \( \sigma_\alpha \) will be treated in a manner which is analogous to the treatment of the variable \( z \) of the upside-down harmonic oscillator. However, although the underlying physics in the two problems is very similar, the problem of the \( \sigma_\alpha \)'s is somewhat more difficult— the time-dependence is more complicated, and in this case we will want to extend our understanding to include thermal ensembles of states.

It will therefore be useful to use a formalism based on creation and annihilation operators, very similar to those used in the conventional treatment of the ordinary harmonic oscillator. A discussion completely parallel to the one-dimensional quantum-mechanical case is given in Appendix B, using the functional-Schrödinger picture of field theory.

When the expansion (3.15) is inserted into the Lagrangian of Eq. (3.8), one obtains

\[
L = \left( \frac{2\pi}{\delta} \right)^3 e^{-\lambda x} \sum_\alpha \left[ \sigma_\alpha^2 + \left[ \mu^2 - e^{-2\lambda x}(E_\alpha^2 + \gamma^2) \right] \sigma_\alpha^2 \right],
\]

(3.16)

where

\[
\gamma^2 = \frac{\lambda \Delta^2}{4\lambda}
\]

(3.17)

The equation of motion for each mode variable is then given by

\[
\ddot{\sigma}_\alpha + 3\lambda \sigma_\alpha = \left[ \mu^2 - e^{-2\lambda x}(E_\alpha^2 + \gamma^2) \right] \sigma_\alpha.
\]

(3.18)

The two linearly independent solutions to this second order equation are found to be

\[
\sigma_\alpha(t) \propto e^{-\lambda x} \left[ J_\lambda(\rho) \right] \left[ N_\mu(\rho) \right]
\]

(3.19)
\[ x' \rightarrow y' = \frac{10}{\Phi} \frac{V}{\phi} \]

The relationship can then be shown to be

\[ \left( x' \right)_{(i)H} \cdot \left( y' \right)_{(i)H} = \frac{X}{Y} = 1 \]

where

\[ \left( x' \right)_{(i)H} = \left( x' \right)_{(i)H} \cdot \left( y' \right)_{(i)H} = \left( y' \right)_{(i)H} \]

and the \( \alpha \) and \( \beta \) operations are defined by the \( \Phi \) and \( \phi \) functions, respectively.

In order to show that the altitude function, \( \cos \), is necessarily to a degree,

\[ w_{(3)} \]

\[ \left( y' \right)_{(i)H} = \left( y' \right)_{(i)H} \cdot \left( x' \right)_{(i)H} \]

Over here the common combination relation

\[ \left\{ \left( x' \right)_{(i)H} \cdot \left( y' \right)_{(i)H} \right\} = \left( x' \right)_{(i)H} \cdot \left( y' \right)_{(i)H} \]

The Hamiltonian is seen to be

\[ \left( x' \right)_{(i)H} \cdot \left( y' \right)_{(i)H} = \frac{\partial \Phi}{\partial \theta} = \frac{\partial \phi}{\partial \theta} \]

This is a special case of where, for example,

\[ v \cdot x' \to y' = \frac{X}{Y} \]

The expression \( \cos \) is seen to be much shorter than the reference point.
Carrying out the inversion gives

$$a_\omega = \sqrt{2} \left( \frac{2\pi}{i\hbar} \right)^{1/2} e^{\frac{i\omega t}{\hbar}} \left[ \frac{\partial\psi^*}{\partial t} \sigma_\omega - \psi^* \frac{\partial \sigma_\omega}{\partial t} \right].$$  \hspace{1cm} (3.30)

The canonical commutation relations (3.24) can then be used to show that

$$[a_\omega, a^*_\nu] = i \delta_{\omega, \nu},$$

$$[a_\omega, a_\nu] = 0.$$ \hspace{1cm} (3.31)

Thus, at early times one can think of the operators $a_\omega$ and $a^*_\omega$ as creation and annihilation operators for one-particle (standing wave) states with properties which are indistinguishable from the usual properties of Minkowski space quantum field theory. (Of course the commutation relations (3.31) hold at late times as well, but the one-particle interpretation has to be abandoned because the time-dependence of the mode functions becomes more complicated.)

We then specify the initial state of the system by the following assumption:

**Assumption 3:** At asymptotically early times, when each mode behaves like a simple harmonic oscillator, we assume that the system can be described by a density matrix corresponding to thermal equilibrium at the background temperature $T = T_0 e^{-\chi t}$.

Note that although the temperature $T$ is changing with time, it is not necessary to pick any particular time at which to fix our initial thermal ensemble. According to Eq. (3.25), the oscillator frequency $\omega(t)$ is also falling as $e^{-\chi t}$, so the ratio $\omega/T$ behaves as a constant at asymptotically early times. Since the thermal equilibrium density matrix depends only on this ratio, it is possible to maintain thermal equilibrium at all asymptotically early times.

Note also that if the temperature $T_0$ in Assumption 3 were set equal to zero, so that each oscillator were put into its ground state, then the resulting state would be the standard de Sitter space "vacuum" - the one which is used in the papers by Gibbons and Hawking$^{31}$ or Bunch and Davies$^{32}$. Thus, the thermal effects of the Gibbons-Hawking temperature $\chi/2\pi$ are taken into account implicitly by our formalism. (Although the logic used in our construction makes no mention of the full de Sitter group, it can be shown$^{32}$ that the symmetry is manifested in the propagator, and therefore in all Green's functions.)

Although we regard Assumption 3 to be as plausible as any assumption about the initial state, we must point out that it is not necessarily true. For the very small values of $\chi$ which will be necessary in order to obtain a reasonable spectrum of density fluctuations (see Section VII), the interactions are far too weak to cause thermalization in the time available.$^{33}$ Thus, thermal equilibrium is really only one of many conceivable initial states. It is clear, of course, that sufficient inflation will always wipe out the memory of the initial state, since the wavelengths which are relevant at late times corresponded in the early universe to such extremely high energies that only the zero-point oscillations could have been excited. However, questions such as the duration of inflation are sensitive to arbitrarily long wavelengths, and are affected by the initial state. Thus, it would be worthwhile to explore the consequences of other initial states, but we have not yet done so.

Now that both the dynamics and the initial state have been specified, it is just a matter of straightforward calculation to answer any question that one might ask about the behavior of this idealized model. Using Assumption 3, the expectation values of products of the operators $a_\omega$ and $a^*_\omega$ can be calculated by using the results for a canonical ensemble of a simple harmonic oscillator. In particular, one has

$$< a^*_\omega a_\omega > = \frac{1}{e^{\Theta_\omega} - 1} \delta_{\omega, \nu}, \hspace{1cm} (3.32a)$$

where

$$\Theta_\omega = \lim_{t \to -\infty} \frac{\chi \omega(t)}{T(t)} = \frac{\chi}{T_0} \sqrt{k^2 + \gamma^2}. \hspace{1cm} (3.32b)$$

One also has

$$< a_\omega a^*_\omega > = < a^*_\omega a_\omega > = 0. \hspace{1cm} (3.32c)$$

Using (3.27), we can similarly calculate expectation values involving the $\sigma_\omega$'s. There is no correlation between different values of $\omega$, and one has

$$< \sigma_\omega(t) > = 0, \hspace{1cm} (3.33)$$
It is now possible to calculate the expectation value of the operator of order $\lambda$. 

We have:

\[
\langle \text{exp} \rangle = \int \frac{e^{i \lambda \hat{A}}}{e^{i \lambda \hat{A}}} \langle \text{exp} \rangle = \langle \text{exp} \rangle >
\]

where $\langle \text{exp} \rangle$ is the expectation value of the operator $\hat{A}$.
It is easily seen that the growth factor $e^{(p-3)x^4}$ agrees with the asymptotic classical growth rate for solutions to the differential equation (3.9), ignoring the $\lambda \phi^4$ interaction term. One can carry out an analogous calculation of $< \sigma_\alpha^2 >$, which is related to $< \pi_\alpha^2 >$ by Eq. (3.22). Finally, one obtains

$$R_\alpha \to \frac{(2p-3)!!}{\pi} \left( \frac{z}{2} \right)^{2p} \coth (4 \Theta_\alpha).$$

(4.3)

Thus it is clear that for $z$ sufficiently small ($t$ sufficiently large), the desired criterion is met. Note that $z$ can be thought of as the ratio of the horizon length $\chi^{-1}$ to the “effective wavelength” $2\pi e^{xt}/\sqrt{\kappa^2 + \gamma^2}$, so the behavior of $\sigma_\alpha(t)$ can be described classically whenever the effective wavelength is much larger than the horizon length.

The special case of $\mu^2 = 0$ must be treated separately, since in this case $p = \frac{3}{2}$ and the right hand side of Eq. (4.3) vanishes identically. Repeating the calculation, one finds that the leading term in this case gives

$$R_\alpha \to \frac{1}{\tau} \coth (4 \Theta_\alpha).$$

(4.4)

$R_\alpha$ does not grow as quickly in this case, but it is still true that the behavior is classical whenever the effective wavelength is much larger than the horizon length.

V. The Smearing Field Operator

Since we know the probability distribution for each coefficient $\sigma_\alpha$, which occurs in the expansion (3.15) for the scalar field $\phi^4$, one might think that the probability distribution for the field itself is well determined. This is basically true, but there is one subtlety which must still be taken into account. If one were to calculate $< \phi^4(\bar{x},t) >$ using the probability distribution described above, one would find that the sum over modes would diverge at large $\bar{x}$. Thus there are infinite fluctuations in $\phi(\bar{x},t)$, in the sense that the standard deviation of the probability distribution is infinite.

In fact, whenever one tries to discuss the value of a quantum field at a specified point, whether in flat or curved space-time, the fluctuations in the field are always infinite. Unlike many other infinities which arise in quantum field theory calculations, this infinity is physically real and should not be removed by a renormalization. It means that if one were to measure the field with a device which has a spatial resolution $\ell$, then the width of the probability distribution for the measured value of the field would increase without bound as $\ell$ becomes smaller and smaller.

A measurable quantity with finite fluctuations can be defined by “smearing” the field in a manner which simulates the finite spatial resolution of a measuring device, so that its value corresponds to a spatial average of the local quantum field $\phi(\bar{x})$. For definiteness, we will use smeared fields which are defined with a Gaussian weight function:

$$\tilde{\phi}(\bar{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3 \bar{y} e^{-(1/2)\bar{x}^2 \bar{y}^2 \tau / \ell^2} \phi(\bar{y}, t),$$

(5.1)

where $\ell$ is an arbitrary smearing length. Using the expansion (3.15) for the scalar field operator $\phi(\bar{x}, t)$, one obtains a rather simple expansion for the smeared field operator $\tilde{\phi}(\bar{x}, t)$:

$$\tilde{\phi}(\bar{x}, t) = \frac{1}{(2\pi)^{3/2}} \left( \frac{2\pi}{\ell} \right)^{1/2} \left( \sqrt{2}\phi(\bar{0}, t) \right)$$

$$+ \sum_{\kappa, \mu} \frac{e^{-\kappa \tau} e^{i\kappa \cdot \bar{x}} \sigma_\mu(\bar{x}, t) \cos \kappa \cdot \bar{x} + \sigma_\mu(\bar{x}, t) \sin \kappa \cdot \bar{x}}{\sqrt{\kappa}}.$$

(5.2)

Thus, the smearing can be seen to provide precisely the sort of high $|\kappa|$ cutoff which is required.

The probability distribution for the field $\tilde{\phi}(\bar{x}, t)$ is then perfectly well defined. If one wishes to visualize a typical configuration for the the scalar field at any point in its evolution, one can use Monte Carlo techniques to generate values for each of the $\sigma_\alpha$ according to a Gaussian probability distribution of the correct width. At present we are developing a computer program to carry out this procedure. One can also calculate $< \tilde{\phi}^2(\bar{x}, t) >$, which is given in the infinite volume limit (see Appendix A for useful intermediate steps) by

$$< \tilde{\phi}^2(\bar{x}, t) > = \frac{\hbar e^{-3xt}}{8\pi x} \int_{0}^{\infty} k^2 dk e^{-k^2 \tau / \ell^2} \times$$

25
The equation expected to achieve the desired output, as mentioned in the introduction, is given by:

\[ \left( \frac{\partial^m}{\partial \phi^m} \right) \frac{d\phi}{d\tau} = \lambda \frac{d\phi}{d\tau} \]

where \( \tau \) is the time variable and \( \lambda \) is a parameter of the system. This equation is derived from the principle of least action in the context of quantum mechanics.

In the next section, we will introduce the formalism necessary to solve the above equation using path integrals. This approach allows us to express the probability amplitude of finding the system in a certain state at a given time in terms of the initial conditions and the Hamiltonian of the system.
into small regions with $\phi \approx \pm \phi_*$. The expected size of these regions would be of order $T^{-1}$, which is exactly the smearing length used in Figure 6. Thus, if this picture were valid, then $\phi_t^\text{RMS}$ would level off at a value of order $\phi_*$ when $T_*$ falls to $T_*$. As can be seen in Figure 6, the actual behavior of $\phi_t^\text{RMS}$ is very different. $\phi_t^\text{RMS}$ continues to fall long after $T_*$ reaches $T_*$, decreasing to a value many orders of magnitude below $\phi_*$. The continued falling of $\phi_t^\text{RMS}$ is a clear result of our numerical calculations, and the reason can be easily understood. When $T \approx T_*$, fluctuations will indeed carry $\phi_t^\text{RMS}$ to values of order $\pm \phi_*$. However, when one decomposes these fluctuations into Fourier components as in Eq. (5.3), one sees that they are dominated by wavelengths of order $T^{-1}$. For these wavelengths

$$k_{\text{physical}}^2 = \epsilon^{2x} k^2 \approx T^2 \approx \frac{4\mu^2}{\lambda} \gg \mu^2,$$  

(6.7)

so the fact that the potential energy function is unstable is irrelevant to these modes—they continue to evolve for some time like the modes of a massless free field. And for a massless free field, dimensional analysis guarantees that $\phi_t^\text{RMS} \approx T$. Note that this argument depends on the condition $\lambda \ll 1$; if $\lambda$ were of order one, then the picture of Masek, Unruh, and Wald would presumably be a valid description.

It is clearly seen in Figure 6 that $\phi_t^\text{RMS}$ traces out straight lines for both early and late times, with a short segment of curve which interpolates between the two asymptotic lines. Expressions for these two asymptotic lines are derived in Appendix D, but we state the results here. For early times one has

$$\phi_t^\text{RMS} \approx \phi_0 e^{-\lambda t} F(\lambda, T_0 T),$$  

(6.8a)

where

$$F(\lambda, T_0 T) = \frac{1}{15\pi^2} \int_0^\infty dy \frac{y^2}{\sqrt{y^2 + \lambda}} e^{-\lambda y T_0 T} y^{4/3} \coth \left( \frac{1}{4} \sqrt{y^2 + \lambda} \right).$$  

(6.8b)

$F(\lambda, T_0 T)$ has a smooth limit as $\lambda \to 0$, and numerically one finds, for example, that $F(0, 1) = 0.216$. For late times we find

$$\phi_t^\text{RMS} \approx \chi \left[ \frac{\lambda}{T_0} \right]^{n-1} \tilde{F}(\lambda, T_0 T),$$  

(6.9a)

where

$$\tilde{F}(\lambda, T_0 T) = \frac{16}{\sqrt{3}} \left( \frac{4}{\sqrt{\lambda}} \right)^{2p-3} \int_0^\infty dy \frac{y^2}{(y^2 + 1)^{p}} e^{-\lambda y T_0 T} y^{4/3} \coth \left( \frac{1}{4} \sqrt{y^2 + 1} \right).$$  

(6.9b)

where as before $p = \frac{1}{2} \sqrt{\frac{1}{3} + 4 \eta}$. $F(\lambda, \eta, T_0 T)$ does not behave smoothly as $\lambda \to 0$, but rather behaves as

$$\tilde{F}(\lambda, \eta, T_0 T) \sim \frac{1}{\sqrt{3} \pi (p - 1)} \left( \frac{4}{\sqrt{\lambda}} \right)^{2p-3}.$$  

(6.10)

Note that the leading behavior above is independent of $T_0 T$—for very small $\lambda$, the field $\phi_t^\text{RMS}$ at large times is apparently very smooth on comoving scales $t$ of order $T_0^{-1}$, and so the smearing length does not matter.

It is shown in Appendix D that the behavior of $\phi_t^\text{RMS}$ at late times (Eqs. (6.9)) is dominated by wavelengths which are of order $1/\sqrt{\lambda}$ times the thermal wavelength $T^{-1}$. Thus the wavelength at which de Sitter expansion is expected to actually begin, it follows that our assumption of exact de Sitter evolution for all time could have a large effect on the behavior of $\phi_t^\text{RMS}$ at large times. It would therefore be important to validate this behavior under more realistic assumptions, but we not yet done so.

Note also that for the special case of $\eta = 0$ (or equivalently, $\mu = 0$), Eqs. (6.9-10) indicate that $\phi_t^\text{RMS} >$ approaches a constant at large $\lambda$. On the other hand, a major point is made in Refs. [15] and [16] of the fact that $\phi_t^\text{RMS} >$ grows at large $\lambda$ as $\lambda^{1/2} x^2 t$. The discrepancy is caused by the fact that we are using a fixed coordinate smear length, while the regularization technique used in Refs. [15] and [18] corresponds to a fixed short-distance cut-off in physical units. If we were to use a fixed physical smear length, we would also find that $\phi_t^\text{RMS} >$ behaves as $\lambda^{1/2} x^2 t$ at large times.
\[
\left( \frac{1}{1 + \frac{1}{\gamma}} \right) \frac{d}{dt} N = N \left( 1 - \frac{1}{\gamma} \right) + \left( \frac{1}{\gamma} \right) \frac{d^2}{dt^2} N
\]

and (13) is the condition that at time \( t \), which in 10.

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\[\left(1 + \frac{1}{\gamma} \right)^2 = \frac{1}{\gamma} \cdot \frac{d^2}{dt^2} N\]

and the further course show the effect of each contribution.

\[\left(1 + \frac{1}{\gamma} \right)^2 = \frac{1}{\gamma} \cdot \frac{d^2}{dt^2} N\]
As we will discuss more fully in the next section, we expect inflation to end roughly when χ_{RMS} ≈ χ_e. Using Eqs. (6.9a), (6.10), (6.2), and (6.4), one sees that this condition is met at time t_f given by

\[ t_f \approx 1 - \ln \left( \frac{M^p}{T_0^p} \right) + \ln \eta + \frac{1}{2p-3} \left\{ \ln \eta - \frac{1}{2} \ln \lambda - \ln \left[ \frac{\Gamma^3(p)}{\Gamma^3(p-1)} \right] \right\}. \]

(6.19)

So

\[ \chi(t_f - t_0) \approx 1 - \frac{1}{4} \ln N_{\text{eff}} + \frac{1}{2p-3} \left\{ \ln \left[ \frac{\Gamma^3(p)}{\Gamma^3(p-1)} \right] \right\}. \]

(6.20)

Comparing with Eq. (6.17), one finds that sufficient inflation requires

\[ \lambda \lesssim \eta^{p-1} \left[ \frac{\Gamma^3(p)}{\Gamma^3(p-1)} \right] N_{\text{eff}}^{-1/3}. \]

(6.21)

The expression on the right-hand-side is rather complicated, so a graph of its value is shown in Figure 10.

VII. Cosmological Density Fluctuations

One of the most fascinating features of the inflationary universe model is the manner in which it accounts for cosmological density fluctuations. While earlier cosmological models relied on the introduction of an ad hoc spectrum of primordial fluctuations, in the inflationary model both the large scale homogeneity and the residual density fluctuations are viewed as consequences of the underlying particle physics. The large scale homogeneity is a direct consequence of the tremendous expansion, and the residual fluctuations are relics of the zero point quantum fluctuations in the early universe. While quantum fluctuations are normally relevant only on microscopic distance scales, the inflationary models have the enticing property that microscopic distance scales in the early universe are stretched to astronomical scales today. In this section we will discuss the origin of these fluctuations in the context of the exactly soluble model described in Section III.

We begin by reviewing the semiclassical approach which has been used in the literature, following most closely the approach used in our own paper. However, we will adapt this approach to a scalar field with the potential energy function used in Section III.

All of the papers in Ref. [5] introduce a homogeneous solution φ_0(t) to the classical equations of motion:

\[ \ddot{\phi} + 3\chi \dot{\phi} = -\frac{\partial V}{\partial \phi}. \]

(7.1)

The full scalar field is then decomposed into a classical and quantum part, according to

\[ \phi(\mathbf{x}, t) = \phi_0(t) + \delta \phi(\mathbf{x}, t), \]

(7.2)

Assuming that |δφ| ≪ φ_0, δφ(\mathbf{x}, t) is then treated as a free quantum field obeying the linearized equation

\[ \ddot{\delta \phi} + 3\chi \dot{\delta \phi} - e^{-2\chi t^2} \frac{\partial^2 V}{\partial \phi^2} \delta \phi = -\frac{\partial V}{\partial \phi} \frac{\partial V}{\partial \phi} \delta \phi. \]

(7.3)

As we mentioned in the introduction, the precise meaning of φ_0(t) in the quantum mechanical context is never properly stated in these papers.

Applying this approach to the model described in Section III, one has

\[ \frac{\partial^2 V}{\partial \phi^2} = -\mu^2 \phi + \lambda \phi^3. \]

(7.4)

For early times when φ ≪ φ_e = μ/√λ, one can neglect the λφ^3 term in the equations of motion, so that

\[ \ddot{\phi}_0 + 3\chi \dot{\phi}_0 = \mu^2 \phi_0, \]

(7.5a)

\[ \ddot{\delta \phi} + 3\chi \dot{\delta \phi} + e^{-2\chi t^2} \frac{\partial^2 V}{\partial \phi^2} \delta \phi = -\frac{\partial V}{\partial \phi} \delta \phi. \]

(7.5b)

If λ is sufficiently small, it was shown in the last section that there is a long time interval (χΔt ≳ 70) for which the above equations apply, so it makes sense to look at the large time behavior of the solutions. We will call this time period the "middle rollover period"—the period which is early enough so that φ_0 ≪ φ_e, but late enough so that the solutions
In the equation of motion, the force is described by 
\[ f = m \ddot{x} + \dot{f} \]
where \( m \) is the mass and \( \dot{f} \) is the damping force. For \( t \leq t_0 \), we have
\[ \dot{f} = 0 \]
and for \( t > t_0 \), we have
\[ \dot{f} = f_0 \cos(\omega t) \]
where \( f_0 \) is the amplitude and \( \omega \) is the angular frequency. The total force is given by
\[ F = F_0 \cos(\omega t) \]
where \( F_0 \) is the constant force. The solution to this equation is
\[ x(t) = x_0 \cos(\omega t) + \frac{f_0}{m \omega} \sin(\omega t) \]
where \( x_0 \) is the initial displacement.

If we consider a damped harmonic oscillator, the equation of motion is
\[ m \ddot{x} + \gamma \dot{x} + kx = F(t) \]
where \( \gamma \) is the damping coefficient and \( k \) is the spring constant. The solution to this equation is
\[ x(t) = e^{-\frac{\gamma t}{2m}} \left( x_0 + \frac{F(t)}{k} \right) \cos \left( \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} \right) t + \frac{F(t)}{k} \sin \left( \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} \right) t \]
where \( x_0 \) is the initial displacement and \( F(t) \) is the external force.
provided that
\[
\left( \frac{k}{a} \right)^{2\nu-3} \ll \frac{\phi_k^2}{\phi_0^2}.
\] (7.18)

The above method seems reasonably convincing, and its consequences (in the context of the Coleman-Weinberg potential) have been accepted by a number of authors. However, it clearly has at least two deficiencies. First, there is the second question which we discussed in the introduction: the classical solution \( \phi_0(t) \) has no clear interpretation in the quantum-mechanical context. In particular, it is clearly not the expectation value of the quantum operator. Second, the matching condition used to determine the final answer is clearly only an approximation. We will now discuss how both of these deficiencies can be overcome by using the exactly soluble model of Section III.

To understand the decomposition (7.2), one must remember that the goal of the inflationary scenario is to cause some very small region of pre-inflationary space to evolve into a region comparable to the observed universe. Let the coordinate radius of this region be denoted by \( a \), and then choose some \( \kappa < 1/a \). One can then consider a Fourier expansion of the smeared field \( \hat{\phi}(\vec{r}, t) \), as given by Eq. (5.2). One must choose \( t \) small enough so that the fluctuations on the length scales of interest will be visible, yet not so small that fluctuations on irrelevantly small scales become dominant. Now let \( \phi_0 \) denote the sum of all terms in the Fourier expansion with \( |\vec{k}| > \kappa \). With this definition \( \phi_0 \) is not really independent of \( \vec{r} \), but its \( \vec{r} \)-dependence can be ignored because \( \phi_0 \) is essentially uniform over the observed universe.

According to the discussion of Section IV, each mode of the scalar field can be considered classical when its "effective wavelength" \( 2\pi x^*/\sqrt{\dot{a}^2 + \frac{1}{4}\lambda T_0^2} \) is long compared to the Hubble length. Once this condition holds for the wave number \( \kappa \), \( \phi_0(t) \) can be regarded as a classical variable evolving according to the classical equations of motion. These classical trajectories can be parameterized at large times by a single parameter, the constant \( \phi_0 \) of Eq. (7.6). The precise value of \( \phi_0 \) is not predictable, but instead \( \phi_0 \) is described by a classical probability distribution. In this case the distribution is Gaussian with a mean of zero, but nonetheless any given observer (except for a set of measure zero) will measure a nonzero value for \( \phi_0 \). Note that the value of \( \phi_0 \) determines how much inflation the region will undergo before the inflationary era is ended by reheating. For the model to be plausible, one must check that there is a high probability that \( \phi_0 \) is small enough to allow for sufficient inflation.

Note that the decomposition (7.2) will make sense only if \( |\phi| \ll \phi_0 \), and this requirement will prevent one from choosing \( \kappa \) or \( t \) too small. However, there is a wide latitude in the choice of these parameters.

The asymptotic behavior of \( \phi_k^{Rmb} \equiv \left< \phi_k^2 \right>^{1/2} \) was discussed in Section VI (with details in Appendix D), and it was found to be dominated by wavelengths of order \( 1/\sqrt{\lambda} \) times the thermal wavelength \( T^{-1} \). Thus, as long as one chooses \( \kappa > \sqrt{\lambda} T_0 \) and \( t \) of order \( T_0^{-1} \), for large times one has
\[
\left< \phi_k^2 \right> \ll \left< \phi_0^2 \right> \ll \left< \phi_k(t)^2 \right> \ll \left< \phi_k^2 \right>.
\] (7.19)

Thus, the criterion for sufficient inflation adopted in Section VI is completely consistent with our interpretation of \( \phi_0(t) \).

Note that Eq. (7.19) is actually valid for a very wide range of choices for the smearing parameter \( \kappa \). One must of course choose \( t < 1/(\sqrt{\lambda} T_0) \), so that the dominant wavelengths are not suppressed by the smearing process. And one must choose \( t \) large enough so that the residual small-scale quantum fluctuations, which make a contribution to \( \left< \phi_k^2 \right> \) of order \( \lambda^{-1/2} \), do not rival the leading term.

We can now complete the calculation. Since the behavior of the quantum field \( \phi(\vec{r}, t) \) for all times was completely determined in Section III, there is no need for us to use an approximate matching condition as in the derivation of Eq. (7.17). The exact equal-time propagator was given as Eq. (3.36), from which it follows that
\[
\Delta \phi(\vec{r}, t) = \left\{ \frac{k^2}{(2\pi)^2} \left| \psi(\vec{k}, t) \right|^2 \coth (\frac{1}{2} \Theta_k) \right\}^{1/2},
\] (7.20)
where \( \psi(\vec{k}, t) \) and \( \Theta_k \) are defined by Eqs. (3.28) and (3.32b). The asymptotic behavior at large times can be found by using Eqs. (C.3), yielding
\[
\Delta \phi(\vec{r}, t) \sim \frac{2^n T(p)^{1/2} k^{1/2}}{2^{n/2} \pi^{3/2} \sqrt{\lambda / 2 \pi}} \left( \frac{\lambda}{\sqrt{k^2 + \gamma^2}} \right)^{1/2} \coth^{1/2} (\frac{1}{2} \Theta_k) \xi^{(2n-3)k^2 x},
\] (7.21)
(52 L) \[ \left( \frac{\partial \eta}{\partial \phi} \right)_{x} \frac{\partial \phi}{\partial \eta} \left( \frac{\partial \phi}{\partial \eta} \right)_{x} = \frac{d}{(\gamma)^{\text{Max}}} \]

and hence (53 L), can then be rewritten as:

(53 L) \[ \int \left( \frac{\partial \phi}{\partial \eta} \right)_{x} \frac{\partial \phi}{\partial \eta} \left( \frac{\partial \phi}{\partial \eta} \right)_{x} \, dx = \frac{d}{\gamma} \int \left( \frac{\partial \phi}{\partial \eta} \right)_{x} \frac{\partial \phi}{\partial \eta} \left( \frac{\partial \phi}{\partial \eta} \right)_{x} \, dx = \frac{d}{\gamma} \]

One then has:

(54 L) \[ \int \left( \frac{\partial \phi}{\partial \eta} \right)_{x} \frac{\partial \phi}{\partial \eta} \left( \frac{\partial \phi}{\partial \eta} \right)_{x} \, dx = \frac{d}{\gamma} \]

provided that one define

(55 L) \[ e_{x} = \int \left( \frac{\partial \phi}{\partial \eta} \right)_{x} \frac{\partial \phi}{\partial \eta} \left( \frac{\partial \phi}{\partial \eta} \right)_{x} \, dx \]

to be the "proper" definition of the correlation function. The above equation (54 L) then expresses the correlation in terms of the proper function in 55 L.

"Foremost, the above equation is obtained by the definition of the correlation function in 55 L. Thus, we can in order to have a meaningful correlation function in the post-likelihood expression, we can avoid the above post-likelihood expression."

Note also that (56 L) (invariance on random variables) and can note a new point:

(56 L) \[ \left( \frac{\partial \phi}{\partial \eta} \right)_{x} (x_{(\eta-d)} + i) = \phi \]

where

(57 L) \[ \phi_{x} e_{x} (\eta-d) = \frac{d}{(\gamma)^{\text{Max}}} \]

To see how the work is generalized, one has to imagine an extension of this expression which has been shown to be valid. As far as the correlation function is concerned, which depends on the number of unknowns, it will depend on the number of elements in the set of (\eta). The correlation function between the number of elements and the number of elements in the set of (\eta) will not affect the correlation in the set of (\eta). The correlation in the set of (\eta) is expressed in terms of the number of elements in the set of (\eta) and the number of elements in the set of (\eta).
as the zero order approximation in a perturbative calculation of a realistic model, but we have not yet attempted such a calculation.

The exactly soluble model is based on three simplifying assumptions. First, we assumed an exactly de Sitter metric for all time. Second, we assumed that the scalar field could be treated by a free quantum field theory, with a time-dependent potential function $V(\phi) = -\frac{3}{2}(m^2 - \frac{1}{4} \lambda T^2)\phi^2$, where the background temperature $T$ behaves as $T_0 e^{-\lambda t}$. Finally, we assumed that the initial state could be specified at asymptotically early times by describing each mode variable by a thermal equilibrium ensemble.

We have studied a number of consequences of this exactly soluble model. The formalism developed in this paper allows one to calculate the expectation value of an arbitrary function of $\phi(\vec{x}, t)$. We have also studied the behavior of the system at large times, and have concluded that it can be described by a probability distribution of classical trajectories. In order to discuss quantities which are in principle measurable, we have made use of the smeared field $\hat{\phi}$, where $\ell$ denotes the smearing length. The probability distribution for this quantity was calculated, and numerical data for the expectation value of its square was presented. In this model one can calculate how much inflation is expected to occur, and we derived an inequality which shows the range of parameters which lead to sufficient inflation.

Finally, we examined the question of density fluctuations. We showed how to decompose the quantum operator $\hat{\phi}(\vec{x}, t)$ into a piece $\phi_0(t)$ which is effectively homogeneous, and a piece $\delta\phi(\vec{x}, t)$ which describes the spatial inhomogeneities. We calculated the root-mean-square inhomogeneities in $\delta\phi(\vec{x}, t)$, and then showed how to infer the root-mean-square density fluctuations $\Delta\rho/\rho$.

For all the questions that we have studied so far, we have found that the more naive approaches which had been used earlier were essentially correct.

A number of topics remain open for future research. For example, the assumption of exact de Sitter evolution is certainly invalid at early times, and it would be interesting to determine the effect of an initial Friedmann-Robertson-Walker phase. In addition, there is no real justification for our assumption of a thermally distributed initial state, so it would be useful to explore other possibilities. Finally, it would be very worthwhile to attempt a perturbative calculation in a realistic model, using the exactly soluble model as a starting point. It may not be tractable to obtain reliable numerical results, but it would be worthwhile simply to show as a matter of principle that one could compute corrections, and that they are small in the case of weak coupling.

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\[
\left\{ (\gamma, Y) \phi \varphi \Rightarrow (\gamma, Y) \phi + (i', \gamma) \phi \varphi \gamma (Y, \phi) \right\} \left( \gamma, Y \phi \right) 1 = (i' \gamma) \phi
\]

The expression for the left is then given by the following equation:

\[
(g, Y) \phi \left( \frac{e_{\gamma}}{q} \right) = (g \gamma) \phi
\]

To take the continuum limit, one must replace the multiplication operator by

\[
\hat L = [(g, Y) \phi] (Y, \phi) = (g \gamma) \phi
\]

One can easily check that the multiplication operator obeys the commutation relations:

\[
\left\{ (g' - \gamma) (g - \gamma)^{\dagger} \right\} = (g) \gamma^{\dagger}
\]

\[
\left\{ (g' + \gamma) (g - \gamma)^{\dagger} \right\} = (g) \gamma^{\dagger}
\]

which for \( g \neq 0 \) can be written as:

\[
\left\{ \frac{g' \gamma}{(g' - \gamma) (g - \gamma)^{\dagger}} \right\} = (g) \gamma^{\dagger}
\]

Appendix A. Relationships to the Standard Notation
Appendix B. Functional Schrödinger Picture for Field Theory

For our field theory problem a description similar to the upside-down harmonic oscillator can be given using the functional Schrödinger picture for field theory. This is an unfamiliar method for probing field theory, but brings new insights to our subject: it enables us to obtain the time dependent wave function and, therefore, gives a clear description for the time evolution of the system as in the case of quantum mechanics.

1. "Ground State" Wave Functional

In the functional Schrödinger picture an abstract quantum mechanical state \( |\psi\rangle \) is realized by \( \Psi(\phi, t) \), which is a wave functional of the c-number function \( \phi(\xi) \) at a fixed time. The action of the operator \( \phi(\xi) \) on \( |\psi\rangle \) is realized by multiplying \( \Psi(\phi, t) \) by \( \phi(\xi) \), and the action of the canonical momentum \( \pi(\xi) \) is realized by functional differentiation:

\[
\phi(\xi) |\psi\rangle \rightarrow \phi(\xi) \Psi(\phi, t)
\]

\[
\pi(\xi) |\psi\rangle \rightarrow -\frac{\hbar}{i} \delta \phi(\xi) \Psi(\phi, t).
\]

Then, the functional Schrödinger equation for our idealized model is

\[
i \hbar \frac{\partial}{\partial t} \Psi = \frac{1}{2} \int d^3x \left\{ -\nabla^2 \Psi^* \frac{\partial^2}{\partial \phi^2} + e^x \left( \frac{\partial}{\partial \phi} \phi^* \right)^2 + \gamma^2 \frac{\partial^2}{\partial \phi^2} - e^{2x} \frac{\partial^2}{\partial \phi^2} \right\} \Psi.
\]

Since it is a free theory, in momentum space the mode variables \( \sigma_0(\vec{k}) \) and \( \sigma_\pm(\vec{k}) \), defined in Eq. (3.15), completely decouple from each other and the Hamiltonian is given by Eq. (3.23):

\[
H = \sum_\alpha H_\alpha(\sigma_\alpha, \pi_\alpha),
\]

where \( \sigma_\alpha \) denotes the mode variables and the sum \( \sum_\alpha \) means the sum over \( \sigma_0(\vec{k}) \), \( \sigma_+(\vec{k}) \) and \( \sigma_-(\vec{k}) \), with each pair \( (\vec{k}, -\vec{k}) \) counted only once, as in the text. The wave functional of any energy eigenstate can then be factorized:

\[
\Psi = \prod_\alpha \Psi_\alpha(\sigma_\alpha, t),
\]

where \( \Psi_\alpha \) denotes the wave function for each \( \sigma_\alpha \) and satisfies the Schrödinger equation

\[
i \hbar \frac{\partial}{\partial t} \Psi_\alpha = H_\alpha \Psi_\alpha
\]

\[
= \left\{ -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{4} \frac{\hbar^2}{(2\pi)^3} e^{-2\xi x} \frac{\partial^2}{\partial \sigma_\alpha^2} \right\} e^{2\xi x} \left[ e^{-2\xi x} (\alpha^2 + \gamma^2) - \mu^2 \right] \Psi_\alpha
\]

(8.5)

Note that the above equation describes a system which behaves at very early times as a simple harmonic oscillator with frequency \( \omega_\alpha(t) = \sqrt{\alpha^2 + \gamma^2 e^{-\xi t}} \). Our first goal is to find the "ground state" wave functional for each \( \sigma_\alpha \), where by "ground state" we mean the state which behaves as a harmonic oscillator ground state in the asymptotic past:

\[
\Psi_\alpha^0 \rightarrow \exp \left\{ -\frac{1}{2} \frac{2\pi}{\hbar} e^{2\xi x} \frac{1}{2 \hbar \omega_\alpha(t) \sigma_\alpha^2} \right\} \Psi_\alpha^\infty
\]

(8.6)

It is straightforward to show that Eq. (8.5) has an exact solution with a Gaussian form

\[
\Psi_\alpha(\sigma_\alpha, t) = A(\vec{k}, t) e^{2\xi x} \frac{1}{2 \hbar \omega_\alpha(t) \sigma_\alpha^2} \exp \left\{ -\frac{1}{2} \frac{2\pi}{\hbar} \frac{1}{2 \hbar \omega_\alpha(t) \sigma_\alpha^2} \right\}
\]

(8.7a)

where

\[
B(\vec{k}, t) = -i \frac{\hbar^2}{2m} \nabla \frac{\partial}{\partial t} \left[ e^{2\xi x} \phi^*(\vec{k}, t) \right]
\]

(8.7b)

and \( \phi(\vec{k}, t) \) is any solution to the equation of motion (3.18). (We have expressed \( B(\vec{k}, t) \) in terms of \( \phi^* \) rather than \( \phi \) in order to simplify the form of subsequent results.) The most general solution to (3.18) can be written as

\[
\phi(\vec{k}, t) = \frac{1}{2} \frac{\sqrt{\pi}}{2} e^{-2\xi x} \left[ c_1(\vec{k}) B(\vec{k}, t) e^{2\xi x} + c_2(\vec{k}) B(\vec{k}, t) e^{-2\xi x} \right]
\]

(8.7c)

where \( z \) is given by Eq. (3.21). Here \( c_1(\vec{k}) \) and \( c_2(\vec{k}) \) are arbitrary complex functions. Since \( B(\vec{k}, t) \) is unchanged when \( \phi(\vec{k}, t) \) is multiplied by a time-independent function of \( \vec{k} \), it is possible to fix the normalization of \( \phi \) by requiring

\[
|c_1(\vec{k})|^2 - |c_2(\vec{k})|^2 = 1.
\]

(8.8)
\[(\ldots) - \left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)\right) = \left(\frac{\partial}{\partial y}\right) \cdot \left(\frac{\partial}{\partial x}\right)\]

In the context of a multiple expectation theory, this is a fundamental property of the expectation. However, in the case of the above formula, the expectation is taken in Section II. Therefore, our desired result holds.

For small values of \(c\), the result cannot be guaranteed. The expression in the case of \(c = 0\), for the correlation factor on the right-hand side of the equation, is obtained by the primitive value of \(c\) in the limit of the variable \(c\) to zero.

\[
\int x^{2} \left[ x^{2} + \frac{z^{2}}{x} \right] \left( \frac{g}{x} \right) \frac{dx}{c^{2} + \frac{z^{2}}{x}} \, \text{d}x \overset{c \to 0}{\longrightarrow} \int x^{2} \left( \frac{g}{x} \right) \frac{dx}{c^{2}}
\]

and applying the momentum operator on \(\phi\), we have

\[
\int x^{2} \left[ x^{2} + \frac{z^{2}}{x} \right] \left( \frac{g}{x} \right) \frac{dx}{c^{2} + \frac{z^{2}}{x}} \, \text{d}x \overset{c \to 0}{\longrightarrow} \int x^{2} \left( \frac{g}{x} \right) \frac{dx}{c^{2}}
\]

In the limit \(c \to 0\), the above expression, derived from the correlation, is obtained by the expression \(\frac{x}{c^{2}}\), which is the momentum operator. The result is obtained by the primitive value of \(c\) in the limit of the variable \(c\) to zero.

The definition of the function \(f(x)\), and of \(g(x)\), is derived by the primitive value of \(c\) in the limit of the variable \(c\) to zero.

The definition of the function \(f(x)\), and of \(g(x)\), can be derived by the momentum operator of \(f(x)g(x)\), which, when properly chosen, can be obtained by the momentum operator of \(f(x)g(x)\), where

\[
\int \left( \frac{\partial}{\partial x}\right) \frac{d}{dx} \left( \frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial y}\right) \cdot \left(\frac{\partial}{\partial x}\right)
\]

This result is crucial in the context of the final momentum operator.
where \( \langle \cdots \rangle \) is the classical value of \( \sigma_n \) in the large \( t \) limit, as shown in Eqs. (B.15) or (B.17). Therefore, the wave functional describes \( \sigma_n \) evolving in the potential according to the classical equation of motion.

3. Probability Distribution at Finite Temperature

So far we have neglected the effects of the thermal ensemble. Since each \( \sigma_n(t) \) behaves as a simple harmonic oscillator in thermal equilibrium at early times, one would expect a distribution of excitations for each mode given by \( e^{-m_n \pi / T} / \sum_{m=0}^{\infty} e^{-m \pi / T} \). Therefore, the probability distribution for \( \sigma_n \) at finite temperature is given by

\[
p[\sigma_n(t) = \sum_{n=0}^{\infty} e^{-m_n \pi / T} |\Psi_n^0(\sigma_n, t)|^2,
\]

where \( \Theta_n \) is defined in Eq. (3.32b) and \( \Psi_n^0 \) is the wave function of the \( n \)th excited state for the mode \( \sigma_n \), given by

\[
\Psi_n^0(\sigma_n, t) = \frac{1}{\sqrt{\pi}} \frac{e^{-\sigma_n^2/2}}{\sqrt{n!}} H_n(\sqrt{2\sigma_n/k} \Re B(k, \sigma_n) \Theta_n^0(\sigma_n, t)).
\]

Here the \( H_n \) denote the Hermite polynomials, and \( \Theta_n \) is a phase defined by

\[
H_n^0(z) \equiv M_p(z) e^{i\Theta_n}. \tag{B.21}
\]

Note that \( \Psi_n^0(\sigma_n, t) \) looks exactly like the corresponding formula for the ordinary harmonic oscillator, except that the phase factor \( e^{-i\Theta_n} \) replaces the harmonic oscillator phase \( e^{-it} \). The probability distribution can be obtained by the standard harmonic oscillator technique, and we obtain

\[
p[\sigma_n(t)] = \frac{4}{\pi} \frac{\sqrt{2 \sigma_n/k}}{\sqrt{n!}} \left\{ \frac{\Theta_n}{2} \right\} \exp \left[ -4 \frac{\sigma_n^2/k}{\pi} \right] \frac{1}{\pi} \frac{4}{\pi} \frac{\sqrt{2 \sigma_n/k}}{\sqrt{n!}} \left\{ \frac{\Theta_n}{2} \right\} \sigma_n^2.
\]

The mode variables \( \sigma_n \) are Gaussian distributed, with \( \langle \sigma_n^2(t) \rangle \) given by Eq. (3.34).

Appendix C. Useful Properties of Bessel Functions

In this appendix we summarize some useful properties of Bessel functions, as can be found in Refs. [42] or [43]. Following the notation of Ref. [42], Bessel functions of the first kind are denoted by \( J_p(z) \), Bessel functions of the second kind (also called Neumann functions) are denoted by \( N_p(z) \), and Bessel functions of the third kind (also called Hankel functions) are denoted by \( H_p^0(z) \) and \( H_p^1(z) \).

The Hankel functions are related to \( J_p(z) \) and \( N_p(z) \) by

\[
H_p^0(z) = J_p(z) + i N_p(z),
\]

\[
H_p^1(z) = J_p(z) - i N_p(z).
\]

The behavior of the Hankel functions for asymptotically large \( z \) is given by

\[
H_p^0(z) \sim \sqrt{\frac{2}{\pi z}} e^{iz(z-\frac{1}{2}p-1)} \left[ 1 + O\left( \frac{1}{z} \right) \right],
\]

where the plus and minus signs hold for \( k = 1 \) and \( k = 2 \), respectively. For small \( z \), one has the following asymptotic forms:

\[
J_p(z) \sim \frac{1}{\Gamma(p+1)} \left( \frac{z}{2} \right)^p \left( 1 + O(z^2) \right),
\]

\[
N_p(z) \sim -\frac{1}{\pi z} \left( \frac{z}{2} \right)^p \left( 1 + O(z^2) \right) + O(z^p).
\]

The Hankel functions obey the identity

\[
H_p^0(z) \frac{d}{dz} H_p^1(z) - H_p^1(z) \frac{d}{dz} H_p^0(z) = -\frac{4i}{\pi z}.
\]

Closed form expressions exist whenever \( p \) is a half-odd-integer. We are particularly interested in the case \( p = 3/2 \), which corresponds to \( \mu^2 = 0 \):

\[
H_{3/2}^0(z) = -\sqrt{\frac{2}{\pi z}} e^{\frac{z}{2} - i\frac{1}{2}z},
\]

\[
H_{3/2}^1(z) = \sqrt{\frac{2}{\pi z}} e^{-\frac{z}{2} - i\frac{1}{2}z}.
\]
(3) \( (1 + \epsilon x)^k \leq \frac{1}{1 + \epsilon} \) for \( k \geq 0 \), where \( \epsilon \ll 1 \).

(4) \( \left( \frac{1 + \epsilon x}{1 + \epsilon} \right)^k \leq \frac{1}{1 + \epsilon} \) for \( k \geq 0 \), where \( \epsilon \ll 1 \).

In order to use the multiplicative approximation, we must ensure that the approximation is valid for \( k \geq 0 \).
REFERENCES

7. In the early models, the scalar field which drives the inflation was taken to be the same Higgs field which spontaneously breaks the grand unified gauge symmetry. However, it has been found [Ref. [5]] that in order for the mass density fluctuations to be small enough, the scalar field which drives inflation must have an extraordinarily flat potential energy function. For this reason, most of the currently acceptable models (Ref. [41]) introduce a new weakly coupled gauge-singlet scalar field for the purpose of driving the inflation. Ovrut and Steinhardt (Ref. [45]) have proposed a model in which this field also serves the purpose of breaking supersymmetry, and Pi (Ref. [46]) has suggested that inflation could be driven by the complex gauge-singlet field which gives rise to the axion.
9. The false vacuum value of \( \phi \) is equal to zero in most models, but any other value would do just as well.
18. R. H. Brandenberger, Quantum Fluctuations as the Source of Classical Gravitational Perturbations in Inflationary Universe, Santa Barbara ITP preprint NSF-ITP-84-43 (1984). See also the articles by Brandenberger to be published in Proc. of 8th John Hopkins Workshop on Current Problems in Particle Theory (Baltimore, MD, June 20-22, 1984), and also to be published in Proc. of the Inner Space / Outer Space Workshop (Fermilab, May 2-5, 1984).
FIGURE CAPTIONS

46. $B$, $r$, $f$, $h$, $F$, $P$, $P_{1}$, $S_{1}$, $P_{2}$, $S_{2}$, $d$, $b$, $h$

48. A round p. L. sheathed hole, [ii]

[Image]
Figure 5: The root-mean-square value of the Fourier coefficients $\sigma_n(t)$ are given by the square root of the right-hand side of Eq. (3.34). The time-dependence arises from the first non-trivial factor, $|\psi(\theta_n, t)|$, and is shown in plot (a) for several values of $\eta \equiv \mu^2/\lambda^2$. The midpoint of the graph corresponds to the time when $t = 1$, which means that the “effective wave number” $e^{-x\sqrt{\tfrac{n^2}{\lambda^2} + \gamma^2}} \equiv 1$, the inverse of the de Sitter horizon distance. The second factor, $\sqrt{\cosh(\theta_n)}$, represents the enhancement due to thermal effects. It is shown in plot (b) as a function of the effective temperature $\theta_n^{-1} = T_0 / (\sqrt{E_n^2 + \gamma^2})$.

Figure 6: The variation of $\hat{\phi}_{t}^{RMS} = [\langle \hat{\phi}_t^2 \rangle]^{1/2}$ with $t$, shown for $\eta = 0.2$ and $\lambda = 10^{-12}$. The zero of time has been fixed by setting $T_0 \equiv \chi$, which means that $T = \chi$ when $t = 0$. The smearing length has been chosen as $t = T_0^{-1}$. The point where $\phi = \phi_0$ is labelled $A$, the point where $T = T_0$ is labelled $B$, the point where $T = T_{deS}$ (i.e., when the thermal and false vacuum energy densities are about equal) is labelled $C$, and the point where $T = \chi$ is labelled $D$.

Figure 7: The behavior of $\hat{\phi}_t^{RMS}$ for various values of $\eta$. All the curves are drawn for $T_0 \equiv \chi$, $T_0 \ell = 1$, and $\lambda = 10^{-12}$. On each curve, $A$ denotes the point at which $\hat{\phi}_t^{RMS} = \phi_0$, $B$ denotes $T = T_0$, $C$ denotes $T = T_{deS}$, and $D$ denotes $T = \chi$.

Figure 8: The behavior of $\hat{\phi}_t^{RMS}$ for various values of $T_0 \ell$. All the curves are drawn for $\eta = 0.2$, $T_0 \equiv \chi$, and $T_0 0.2 \ell = 10^{-12}$. The upper three curves are drawn for fixed values of $T_0 \ell$, corresponding to smearing over a fixed coordinate wavelength. The lower three curves correspond to fixed values of $T_0 \ell x^{y^2}$, so in this case the physical smearing length is held fixed.

Figure 9: The behavior of $\hat{\phi}_t^{RMS}$ for various values of $\lambda$. All the curves are drawn for $\eta = 0.2$ and $T_0 \equiv \chi$. The upper curves show the effect of a fixed coordinate smearing length ($T_0 \ell = 1$) and the lower curves show a fixed physical smearing length ($T_0 \ell x^{y^2} = 1$).

Figure 10: The maximum acceptable value of $\lambda$ as a function of $\eta \equiv \mu^2/\lambda^2$. If $\lambda$ exceeds this value, then it is unlikely that an arbitrary region will undergo sufficient inflation to explain the large scale homogeneity of the observed universe.

Figure 11: The ratio of the exact computation of $\Delta \rho/\rho$ to an approximate calculation of the same quantity is shown as a function of $\eta \equiv \mu^2/\lambda^2$. 

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