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QUANTUM MECHANICAL SYMMETRY BREAKING

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I. INTRODUCTION

My talk will be concerned with anomalies, and I shall suggest various research topics that arise from some contemporary ideas about them. But first let me take note of the occasion being marked by this Conference and place the subject in a historical setting which, as it happens, includes Niels Bohr, the originator of much of our quantum mechanical thinking.

When physicists build theories to explain fundamental interactions, we want a high degree of symmetry characterizing the correct model. This is because, first, it is aesthetically satisfying to suppose that the ultimate laws of Nature are very symmetric, and second, as a practical matter, it is important that theories with symmetry are less afflicted with divergences than those without. However, observed natural phenomena do not exhibit a large amount of symmetry, and this gap between theory and experiment must be bridged.

Explicit symmetry breaking – the oldest and most straightforward way of reducing a symmetry – need not concern us, since obviously it is of no fundamental significance. A better idea is due to Heisenberg. Inspired by examples drawn from many-body physics, he suggested that energetic stability considerations can lead to *spontaneous symmetry breaking*, also in particle physics, so that the lowest energy, equilibrium state – the vacuum – need not possess the full symmetries of the dynamical equations.¹ His idea proved to be most successful, and came to fruition in the modern theory of pseudoscalar mesons, in the unification of electromagnetism with weak interactions, and is believed to be an ingredient for the unified theory which we are seeking. Still so far, we have not understood the dynamical reason

why a symmetric state is energetically unstable.

There is, however, another more subtle mechanism for removing from the solutions of a quantized theory a symmetry which is present on the Lagrangian level. This is the *anomaly phenomenon*, which is now widely appreciated in physics, but remains poorly understood. Here, the symmetry is not violated by explicit terms in the Lagrangian, nor do energetic or stability considerations select a non-symmetric equilibrium state. Rather, the very process of quantizing, specifically second quantizing, the theory destroys the symmetry. There is no evidence for this in the formal, classical equations of the model, and the mechanism was a surprising discovery; hence, it has been called *anomalous*, but *quantum mechanical symmetry breaking* is a better name. Anomalous or quantum mechanical symmetry breaking afflicts symmetries associated with masslessness: both scale and conformal invariance as well as chiral invariance of massless fermions are broken by quantum effects. Although the former is important for the description of phase transitions in condensed matter physics and of high-energy processes in particle physics, I shall not dwell on it here. My emphasis is on chiral anomalies.

That effects of quantization may interfere with classical symmetries and conservation laws is an idea from the beginnings of modern physics, when Bohr, resisting Einstein's light quantum, proposed that energy is not conserved in electromagnetic processes. Presumably, time-translation invariance would be absent also. Later, before the neutrino became established, Bohr reiterated and extended this suggestion to include momentum non-conservation in β -decay. Of course, experiment has shown that this position is untenable; space-time translations *are* symmetries of

quantal Nature so that energy and momentum *are* conserved.² Nevertheless, we may say that our present point of view towards scale and conformal symmetry breaking was prefigured by Bohr's intuition concerning effects of quantization on space-time symmetries.

Closer to my emphasis today on anomalously broken chiral symmetries is the story of gauge invariance in quantum electrodynamics. Before correct computational methods were fixed, it was not clear that the quantized theory maintains gauge invariance. Indeed, Wentzel claimed that the theory predicts a photon mass and presumably a corresponding violation of charge conservation for Dirac fermions.³ Again, subsequent developments established gauge invariance. Therefore, the problem was removed, but its descendant forms our present understanding that massless Weyl fermions cannot in general possess gauge invariant interactions, nor can their charge be conserved, owing to quantum mechanical violation of symmetry. Moreover, for massless Dirac fermions, the charges in each of the two separate chiralities cannot be conserved, although their sum is conserved.

Nature requires that ungauged chiral symmetries of massless fermions be broken. This is deduced from details in the spectrum of pseudoscalar mesons: chiral symmetry for the constituent quarks would suppress meson decay modes and produce mass degeneracies, neither of which are seen experimentally. Since current models contain no mechanism for breaking all the relevant symmetries, the occurrence of anomalies is a phenomenologically welcome and useful result. On the other hand, models with anomalously broken gauge symmetries are inconsistent, as far

as we know. I shall speak further about these so called *anomalous gauge theories*, which thus far have no role in physics.

II ORIGIN OF CHIRAL ANOMALIES

Let us recall the various ways we have of establishing that a chiral anomaly occurs when massless fermions interact with a gauge field. First, there are the original perturbative calculations of Feynman diagrams which show that chiral Ward identities cannot be maintained. These calculations can be performed in momentum space, where the effect arises from momentum routing ambiguities,⁴ or in position space, where singularities in the product of fermion bilinears provide the operative mechanism.⁵ A second derivation, also known from the beginning, observes that the algebra of chiral generators, when calculated in perturbation theory, fails to close.⁶ Later, I shall say more about modern developments in this approach. Third, in the functional integral formulation, the anomaly arises because the fermion measure is not chirally invariant.⁷ Fourth, we now have mathematical, specifically topological and cohomological, reasons for understanding that the functional determinant of non-Abelian massless Weyl [chiral] fermions in interaction with gauge fields cannot be defined gauge invariantly.⁸

While all arguments are convincing and striking in the variety of routes they offer to the same goal, they all lack direct physical immediacy; one is still left with the central physical puzzle of the chiral anomaly. This puzzle may be stated in the following way. Consider massless Dirac fermion fields ψ , in even-dimensional space-time, interacting with an external electromagnetic gauge field A_μ . Dynamics

is governed by the Lagrangian,

$$\mathcal{L} = \bar{\psi}(i\not{D} - e\not{A})\psi \quad (1a)$$

which may be decomposed into left and right Weyl fermion pieces.

$$\begin{aligned} \psi &= \psi_L + \psi_R \\ \mathcal{L} &= \bar{\psi}_L(i\not{D} - e\not{A})\psi_L + \bar{\psi}_R(i\not{D} - e\not{A})\psi_R = \mathcal{L}_L + \mathcal{L}_R \end{aligned} \quad (1b)$$

Why is it in the second quantized theory that the separate left and right charges are not conserved, even though there is no apparent interaction between the left and right worlds, so that in the first quantized theory the left and right probability currents are conserved?

A detailed analysis of the second quantized theory gives the answer, but again in terms of a formal rather than physical concept: the Dirac negative energy sea cannot be defined in a gauge invariant way, separately for the left and right portions of the model.⁹ This is a consequence of gauge-field configurations that give rise to zero-eigenvalue modes in the Dirac equation in two dimensions lower.¹⁰ For the four-dimensional theory, we observe that the two-dimensional Dirac operator in a constant magnetic field possesses zero modes. When the full four-dimensional background gauge field in (1) includes such components, the energy spectrum in the first quantized theory cannot be divided into positive [particle] and negative [anti-particle] states in a gauge invariant manner, separately for the left and right components. It is the insistence on gauge invariance in the second quantized Dirac theory that produces a quantum mechanical coupling between the left and right worlds.

More specifically, the Lagrangian (1) in four dimensional space-time leads to a three-dimensional Hamiltonian problem for determining the modes to be second quantized.

$$H u_E = [\alpha \cdot (\mathbf{p} - e\mathbf{A}) + eA^0] u_E = E(A) u_E \quad (2)$$

With a background gauge potential chosen so that A_x and A_y produce the constant magnetic field B in the z -direction, A_z constant, and vanishing A^0 , the two-dimensional zero modes give rise to an energy-momentum dispersion law in (2) of

$$E(A) = p_z - eA_z \quad (3a)$$

for the right-handed fermions, and

$$E(A) = -p_z + eA_z \quad (3b)$$

for the left-handed ones. The zero of each branch cannot be defined in a gauge invariant manner. Moreover, when both branches are included, the ground state [at fixed A_z] of the second quantized theory is defined by "filling" the negative energy levels [at fixed A_z], and leaving the positive levels [at fixed A_z] "empty". However, if A_z is varied adiabatically, empty levels move to positive energies and filled levels move to negative energy [or *vice versa*]. This creates or destroys an amount of charge proportional to $B\delta A_z$ for each chirality, but leaves total charge conserved.

Similarly in a six-dimensional model, the four dimensional instanton produces the zero mode. The two-dimensional [Schwinger] model realizes this anomaly-producing mechanism trivially, since in two dimensions fewer there is nothing there,

and the eigenvalue is obviously zero, while the Hamiltonian problem with vanishing A^0 and constant A^1 obviously possesses eigenvalues of the form (3).¹¹

In two dimensions, moreover, we can present the coupling between the left and right worlds very explicitly, because the model is solved.¹² Owing to the two-dimensional identity $i\gamma^\mu\gamma_3 = \epsilon^{\mu\nu}\gamma_\nu$, one verifies that only one light-cone component of A_μ couples to the right fermions, and the other to the left.

$$\mathcal{L}_{(2)} = \bar{\psi}_L \gamma^- (i\partial_- - eA_-) \psi_L + \bar{\psi}_R \gamma^+ (i\partial_+ - eA_+) \psi_R \quad (4)$$

The “plus” and “minus” components are constructed from the space and time components by the rule $\pm \equiv \frac{1}{\sqrt{2}}(0 \pm 1)$. The gauge invariant, effective quantum action is known.¹²

$$\begin{aligned} -i \ln \det(i\not{\partial} - e\not{A}) &= \frac{e^2}{2\pi} \int A_\mu \left(g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\square} \right) A_\nu \\ &= -\frac{e^2}{4\pi} \int A_- \frac{\partial_+}{\partial_-} A_- - \frac{e^2}{4\pi} \int A_+ \frac{\partial_-}{\partial_+} A_+ + \frac{e^2}{2\pi} \int A_+ A_- \end{aligned} \quad (5)$$

The last contribution, a contact term unambiguously dictated by gauge invariance, puts into evidence the quantum mechanical left-right coupling. Note also that the properly gauge invariant determinant of Dirac fermions is not merely the product of left- and right-handed determinants – gauge invariance can force contact terms that spoil the factorization.

The two-dimensional model may also be viewed as providing the essence for the higher-dimensional anomaly. We begin in $2d$ dimensional space-time, and argue as follows: in 2 dimensions lower, *i.e.*, in $2d - 2$ dimensions, a zero mode may be established with the help of the index theorem. Existence of the zero mode is assured, provided there is a $2d-2$ -dimensional anomaly, which in turn requires a zero

mode in $2d - 4$ dimensions, established by the $2d - 4$ -dimensional index theorem and anomaly, *etc.* Thus, in a very precise way, the two-dimensional Abelian anomaly¹³ is at the center of the entire anomaly phenomenon.

The above is the most “physical” description of chiral anomalies known to me; but still it uses the unphysical, formal construct of a Dirac sea, and negative energy states. However, it should be recalled that charge fractionization, another unexpected effect of second quantized fermions, is also understood in terms of distortions in the negative energy Dirac sea.¹⁴

We have learned much from mathematicians about the topological and cohomological necessity of anomalies,⁸ but perhaps physics can, in its turn, advance mathematical concepts by insisting on the fact that the essence of the anomaly lies beyond present topological/cohomological ideas. The latter involve integrated, global quantities, like the Chern-Pontryagin number, yet the anomaly is local. Moreover, anomalies are present even in the absence of obstructions, like in Abelian $[U(1)]$ theories, as in the discussed example which, being two-dimensional, hardly possesses any structure, save the anomaly. The $U(1)$ anomaly, on the other hand, appears to be the heart of the matter, not only for the non-Abelian anomalies, but also for the non-perturbative ones.¹⁵

Thus, it seems to me that we are not yet at the end of the physics nor of the mathematics that can emerge from understanding anomalies. I expect that in this framework we shall find answers to questions about the precise nature of the vacuum or ground state.

III ANOMALOUS GAUGE THEORIES

Let us now consider an anomalous theory: right-handed Weyl fermions interacting with a gauge field. Apparently, gauge invariance cannot be maintained owing to the anomaly; second quantization of the coupled gauge field-matter system is problematical. In terms of our earlier discussion, the negative energy chiral anti-fermions cannot be separated gauge invariantly from the positive energy chiral fermions.

The nature of the problem has recently been couched in mathematical terms. We consider first the fermion sector, and view the gauge potential, with $A^0 = 0$, as an externally prescribed field – to be quantized later, if possible. Within the fermionic theory, we may construct the unitary operator $U(g) = \exp G_\theta$ that implements the [topologically trivial] gauge transformation $g = e^\theta$. The infinitesimal generator G_θ is $\int d\mathbf{r} \theta_a(\mathbf{r})[\delta_a(\mathbf{r}) - i\rho_a(\mathbf{r})]$, where $\delta_a(\mathbf{r}) \equiv -\mathcal{D}_{ab} \cdot \frac{\delta}{\delta \mathbf{A}_b(\mathbf{r})}$ effects an infinitesimal gauge transformation on $\mathbf{A}_a$ and ρ_a , the fermion charge density, performs the same job on the fermion degrees of freedom; $\mathcal{D}$ is the covariant derivative. The occurrence of anomalies in the generators' commutator algebra⁶ has led to the suggestion that the operators U give a projective representation for the gauge group,

$$g_1 g_2 = g_{12} \tag{6}$$

$$U(g_1)U(g_2) = e^{i2\pi\omega_2(g_1, g_2)}U(g_{12})$$

where the 2-cocycle, $2\pi\omega_2$, can be determined by *a priori* arguments when it is cohomologically non-trivial.¹⁶

While such projective representations of transformation groups are familiar in quantum mechanics, this is the first time they make an appearance in quantized

gauge theory. Moreover, since gauge invariance is conventionally enforced as the first-class constraint that $U(g)$ leave physical states invariant [for topologically trivial gauge transformations], we see that owing to the projective composition law the constraint cannot be satisfied, and the anomalous gauge theory loses gauge invariance. Note, however, that the cocycle occurs even in the cohomologically trivial Abelian theory, for which the original calculation⁶ of anomalous commutators was performed. Yet no *a priori* mathematical argument is available to establish existence of the cocycle in the cohomologically trivial case. Here again we have further evidence that the anomaly goes beyond present mathematical ideas.

What to do with an anomalous gauge theory, other than simply abandoning it? The conventional remedy is to adjust fermion content so that the anomaly is absent — i.e., the 2-cocycle vanishes. This principle leads to a physical prediction which is thus far satisfied: the number of leptons must match the number of quarks.¹⁷ Moreover, the same principle has focused attention on the most recent version of the string.¹⁸

Nevertheless, one may still inquire whether an anomalous theory can yield consistent physics. One suggestion is implicit in Faddeev's¹⁶ work: perhaps gauge invariance in the quantum theory should not be implemented as a first-class constraint, but as a higher order one. Thus far, nothing has come from this idea, and it seems to me that further problems abound: the constraints do not properly commute with the electric field, nor do they commute with the Hamiltonian. [However, since an anomalous current divergence can be expressed as a derivative of a gauge non-covariant quantity, one *can* construct a conserved, gauge non-invariant

object – but it is unclear that this may be used as a symmetry generator in a Lorentz-invariant theory.]

An alternative position, recently taken by Rajaraman and me, is to give up the constraints altogether, i.e., abandon gauge invariance and view anomalies as a mechanism for gauge symmetry breaking.¹⁹ The original arguments¹⁷ for cancelling anomalies called attention to two possible pitfalls in our adventurous approach: (1) renormalizability may be lost, (2) unitarity may be lost. We have nothing to say about renormalizability; indeed, we study a finite, two-dimensional model – the chiral Schwinger model with gauge coupling of strength e – which is anomalous. Owing to the model's simplicity, its spectrum may be analyzed. We find that unitarity is preserved; the vector particle acquires a mass, which, however, is not calculable but lies between $\frac{e}{\sqrt{\pi}}$ and infinity; also, there are massless excitations, which appear to be deconfined fermions. Of course, this two-dimensional model is far from reality, and accepting anomalies runs counter to current practice, which has led to the determination of a unique pair of string models. Nevertheless, I find it attractive to speculate that weak interactions, with their deconfined fermions and massive vector mesons, are a remnant of an anomalous chiral gauge theory.

IV. 3-COCYCLES

I have mentioned 2-cocycles – they arise in a projective representation of a transformation group and introduce a phase in the composition law. Familiar also are 1-cocycles: they are phases that represent the action on vectors. For example, the action of Galilean transformations on quantum mechanical wave functions

includes a phase, as does the action of gauge transformations on quantum field theoretical wave functionals in gauge theories with topological Chern-Simons terms.²⁰

How about 3-cocycles? These have not arisen in the mathematical discussions of representation theory, but several physicists²¹ have pointed out that a non-vanishing 3-cocycle, $2\pi\omega_3$, is a measure of non-associativity in the operator algebra of a representation.

$$\left(U(g_1)U(g_2) \right) U(g_3) = e^{i2\pi\omega_3(g_1, g_2, g_3)} U(g_1) \left(U(g_2)U(g_3) \right) \quad (7)$$

An example can be found in the quantum mechanics of a point particle with charge e , moving at $\mathbf{r}$ in an external magnetic field $\mathbf{B}(\mathbf{r})$, which is not divergence free: $\nabla \cdot \mathbf{B} \neq 0$. The Hamiltonian for this dynamics does not see the magnetic field, because the magnetic field does no work on the charged particle.

$$H = \frac{1}{2} v^2 \quad (8)$$

[We set the mass of the particle to unity.] In order that the Lorentz force law be obtained by commutation with the Hamiltonian, we define

$$[r^i, r^j] = 0, \quad [r^i, v^j] = i\delta^{ij}, \quad [v^i, v^j] = ie\epsilon^{ijk} B^k \quad (9)$$

and the Lorentz law follows.

$$\begin{aligned} \dot{r}^i &= i[H, r^i] = v^i \\ \dot{v}_i &= i[H, v^i] = e\epsilon^{ijk} v^j B^k \end{aligned} \quad (10)$$

Finite translations of $\mathbf{r}$ are represented by

$$U(\mathbf{a}) = e^{i\mathbf{a} \cdot \mathbf{v}} \quad (11)$$

since

$$U(\mathbf{a})\mathbf{r}U^{-1}(\mathbf{a}) = \mathbf{r} + \mathbf{a} \quad (12)$$

However, these do not represent the translation group faithfully since one finds from

(9)

$$U(\mathbf{a}_1)U(\mathbf{a}_2) = e^{-ie\Phi}U(\mathbf{a}_1 + \mathbf{a}_2) \quad (13)$$

where Φ is the flux through the triangle at $\mathbf{r}$ formed from $\mathbf{a}_1$ and $\mathbf{a}_2$; see Figure 1.

Fig. 1 The triangle at $\mathbf{r}$ through which the flux Φ is calculated.

Moreover, by considering the triple product $U(\mathbf{a}_1)U(\mathbf{a}_2)U(\mathbf{a}_3)$, associated in the two different ways as in (7), one finds a 3-cocycle, given by $-e$ times the total flux out of the tetrahedron formed at $\mathbf{r}$ from $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$; see Figure 2.

Of course, when $\nabla \cdot \mathbf{B}$ vanishes, so does the total flux through any closed surface; there is no 3-cocycle and $\mathbf{v}$ may be realized by linear operators on a Hilbert space: $\mathbf{v} = -i\nabla - e\mathbf{A}(\mathbf{r})$, $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$.

Fig. 2 The tetrahedron at r through which the flux determining the 3-cocycle is calculated.

When there are sources, the flux is non-zero, but we may still achieve associativity provided ω_3 is an integer, since then $e^{i2\pi\omega_3} = 1$. This requirement forces: (1) $\nabla \cdot \mathbf{B}$ to consist of localized delta functions, so that the total flux not vary continuously when the $\mathbf{a}_i$'s change; *i.e.*, the sources must be monopoles; (2) since a monopole of strength g produces the cocycle $-4\pi eg$, eg must satisfy the Dirac quantization condition. In this way, removal of the 3-cocycle, which is necessary for conventional quantum mechanics with associative operators on Hilbert space, limits magnetic sources to quantized Dirac monopoles. Other magnetic sources lead to a non-associative algebra.

Finally note that for infinitesimal generators, a non-vanishing 3-cocycle, *i.e.* non-associativity, implies failure of the Jacobi identity. Indeed, from (9) one deduces

that²²

$$\left[v^1, [v^2, v^3] \right] + \left[v^2, [v^3, v^1] \right] + \left[v^3, [v^1, v^2] \right] = e \nabla \cdot \mathbf{B} \quad (14)$$

[The triple products are associated in the obvious way.]

V. EXTERNAL U(1) CONNECTIONS IN FIELD THEORY

Is there anything similar in field theory? Violations of the Jacobi identity had been found years ago, but not in gauge group generators, rather in the old $U(6) \times U(6)$ current algebra.²³ While it may be interesting to frame this into a coherent mathematical picture, it seems far removed from current interest. However, you may be surprised to hear that an “external magnetic field” of particle mechanics has a sensible analog in gauge field theory.

The aspect of particle dynamics in an external magnetic field that is of interest for the field theoretic generalization, is that the magnetic field is invisible in the Hamiltonian, but reappears in the velocity commutator, as in (8) and (9) *i.e.*, the canonical momentum does not coincide with the velocity.

We have grown accustomed to topological terms contributing to gauge field actions – the θ parameter multiplying the Chern-Pontryagin density in even dimensions, the Chern-Simons term with quantized coefficient in odd dimensions.²⁰ Since both are topological world scalars, not involving the metric tensor, they do not contribute to the energy-momentum tensor, and the Hamiltonian retains its conventional form.

$$H = \int d\mathbf{r} \left(\frac{1}{2} E_a^i E_a^i + \frac{1}{4} F_a^{ij} F_a^{ij} \right) \quad (15)$$

However, the canonical field momenta differ from the field “velocities” – $\mathbf{E}_a$, and

the difference may be ascribed to an external $U(1)$ field connection. The $U(1)$ field curvature – the analog of the external magnetic field – is determined by the equal-time commutator of the field velocities.

For theories in even-dimensional space-time with a Chern-Pontryagin density, the commutator vanishes – the external field curvature is zero, the connection is flat. This is to be expected, since the topological term does not affect equations of motion, and the connection is a pure functional gauge. It can be gauged away, and this is the familiar procedure which shifts the vacuum angle from the Lagrangian to the state.²⁰ In this way, the situation is similar to point vortices on a plane.

The Chern-Simons term in odd-dimensional space-time gives rise to a non-vanishing external field curvature, since it does modify equations of motion. In the three-dimensional theory for example, the equal-time commutator [which lives in two-space] reads

$$i[E_a^i(\mathbf{r}), E_b^j(\mathbf{r}')] = \frac{m}{4\pi} \delta_{ab} \epsilon^{ij} \delta(\mathbf{r} - \mathbf{r}') \quad (16)$$

$$m = 0, \pm 1, \dots$$

The curvature may be described as a constant, functional external $U(1)$ “magnetic” field. That the coefficient is quantized brings out the analogy, once again, between the quantization of Chern-Simons parameters in field theory and the point Dirac monopole.

While this viewpoint towards topological contributions to the gauge field action²⁴ does not produce new results, it suggests a direction for further investigation. Are there other forms of external $U(1)$ field connections and curvatures that may be added to conventional gauge dynamics?

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