CERN-TH.4045/85

DIMENSIONAL REDUCTION OF GAUGE THEORIES YIELDING UNIFIED MODELS
SPONTANEOUSLY BROKEN TO $SU_3 \times U_1$

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ABSTRACT

By dimensional reduction of pure gauge theories (with gauge group G) over a compact coset space S/R , one obtains four-dimensional theories where scalar fields and a symmetry breaking potential appear naturally. We present a complete analysis (including the fermion sector) of all unified models with simple G which are spontaneously broken to $SU_3 \times U_1$, and which can be obtained by this technique with the added restriction that $S \subset G$. Such models only exist when G is an exceptional group; however, the surviving fermions do not have the correct quantum numbers. The paper also provides an exhaustive list of SU_3 embeddings in the exceptional groups.

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CERN-TH.4045/85
March 1985

1. - INTRODUCTION

In recent years, the Kaluza-Klein idea of unification through dimensional reduction or compactification has been revived. The most ambitious attempt, namely to generate the standard model from the spontaneous compactification of eleven-dimensional supergravity, has met with serious obstructions. Severe problems arise in the fermion sector with respect to the correct lepton and quark quantum numbers and the left-right asymmetry.

A variation on the Kaluza-Klein principle is to start with a Yang-Mills theory in higher dimensions and to reduce it over some compact "internal" space to a four-dimensional theory¹⁾⁻⁴⁾. This approach is clearly less restrictive, as one gains the freedom of choosing the gauge group G , the internal space S/R and eventually the fermion representations. Models of this type can be analyzed at various levels. One way is to take the extra dimensions seriously, in which case one has to take into account all higher-dimensional excitations, both of the Yang-Mills and the gravitational type, leading to complications with respect to stability and anomalies. In this picture one furthermore has to accommodate the "extra" Yang-Mills theory with a gauge group corresponding to the isometry group of the internal manifold which appears due to the reduction of the gravitational fields. Another way to utilise the idea is to limit oneself to the reduction of the Yang-Mills fields only. In this approach, the extra dimensions serve only as a book-keeping device. It is this type of model building that we will pursue in this paper. The theories one obtains in this way have several attractive features. The parameters and couplings in the resulting four-dimensional theory are completely fixed once G and the internal space are chosen. It turns out that two symmetry breaking scales appear naturally. The first one is associated with the geometric breaking of the gauge group G to some subgroup H , at a scale which corresponds to the inverse radius of the internal space. Secondly, because through the reduction four-dimensional scalar fields and a symmetry breaking potential appear naturally, the residual group H will be broken spontaneously to some group $K \subset H$. Another appealing feature in the fermion sector of these theories is that if suitable conditions are met, it is possible to have left-right asymmetry, and hence one may hope to construct realistic models.

In this paper, we report on an exhaustive search for models of the following type. We start with a Yang-Mills theory in $(4+d)$ dimensions with a simple gauge group G and reduce it over a d -dimensional homogeneous space S/R . Here we have made the extra restriction that $S \subset G$, because in that case the symmetry breaking

pattern can be determined directly⁴⁾. The compactification with symmetric gauge fields reduces G to $H = C_G(R)$, i.e., the centralizer of R in G . This group H is then spontaneously broken to $K = C_G(S) \subset H$. We have focused on models where $K = SU_3 \times U_1$, the standard low-energy theory, and where H is such that the electromagnetic U_1 is unified in some non-Abelian group (not necessarily $SU_2^L \times U_1^Y$ though). In order to have the possibility of parity violation, we furthermore required that $\text{rank}(S) = \text{rank}(R)$ (S semisimple).

One would expect that under these mild assumptions there would be a large class of possibilities deriving from the freedom to choose G as well as S/R , etc. Our analysis leads, however, to the opposite conclusion. It is only possible to obtain the desired bosonic sector if G is an exceptional group. If one adds fermions, there are only very few cases where some of them survive. Among these models there are even less examples where parity violation occurs. Furthermore, none of these models yielded only fermions with the correct quantum numbers. In short, we did not find any realistic model in the wide class of models we considered.

The paper is organized as follows. In Section 2, a review is given of the coset space dimensional reduction (CSDR) method, both for the bosonic and fermionic sectors of the theory. In Section 3, we give a reformulation of the problem yielding an efficient group theoretical recipe to determine the bosonic sector. The procedure should be applicable to a wide variety of similar problems. In Section 4, we use these tools to show that all classical groups are eliminated because it is impossible to embed the correct bosonic structure. In Section 5, we analyze the exceptional groups, and some interesting but not realistic models with surviving fermions are described. In the Appendices we have gathered some technical details of the construction. Appendix B in particular may be useful, as it contains a complete list of all SU_3 embeddings in the exceptional groups, including all the sequences of maximal embeddings through which they can be reached.

2. - COSET SPACE DIMENSIONAL REDUCTION (CSDR)

In this section we give a short review of the dimensional reduction procedure using symmetric gauge fields. We start with a pure gauge field theory with gauge group G defined on an extended space-time manifold, M , which is assumed to be a direct product

$$M = M_4 \times S/R \quad (2.1)$$

where M_4 is a four-dimensional pseudo-Riemannian manifold (in the following, it is just the usual Minkowski space-time) and S/R is a compact coset space. In the rest of the paper, S will denote a compact, semisimple Lie group and R is a Lie subgroup of S . The gauge group G is also assumed to be compact and semisimple.

We now define S symmetric gauge fields¹⁾ by requiring that the action of S on the gauge potentials should be equivalent to a gauge transformation. This implies that the gauge potential A_μ depends on the coset space co-ordinates in a non-trivial way. The formulation of the problem of symmetric gauge fields (invariant connections over homogeneous spaces) in the context of fibre bundle theory was first studied in Ref. 2) and subsequently pursued in Refs. 3) and 4). In this paper we follow the viewpoint of Ref. 1) and work with local co-ordinate components.

Let us first clarify our use of co-ordinates. $x^{\hat{\mu}}$ denotes the co-ordinates of the manifold M . In view of (2.1), it is natural to split $x^{\hat{\mu}}$ into two parts:

$$x^{\hat{\mu}} = (x^\mu, y^\kappa) ; \mu = 0, \dots, 3, \quad \kappa = 1, \dots, \dim S/R = d$$

corresponding to the decomposition of M . The gauge potential $A_{\hat{\mu}}$ takes its value in the Lie algebra of G : $A_{\hat{\mu}} = A_{\hat{\mu}}^a T^a$ and the generators T^a satisfy $[T^a, T^b] = f^{abc} T^c$. The infinitesimal left action of S on the fields $A_{\hat{\mu}}$ is

$$\delta_N A_{\hat{\mu}} = -(\xi_N^{\hat{\rho}} \partial_{\hat{\rho}} A_{\hat{\mu}} + A_{\hat{\rho}} \partial_{\hat{\mu}} \xi_N^{\hat{\rho}}) = L_{\xi_N} A_{\hat{\mu}} \quad (2.2)$$

L_{ξ_N} is the Lie derivative with respect to $\xi_N^{\hat{\rho}}$. The Lie algebra of S is represented by the differential operators $L_{\xi_N} (\delta_N)$ [and the (left) vector fields $\xi_N^{\hat{\rho}}$ on the coset space S/R]:

$$[L_{\xi_N}, L_{\xi_M}] = g_{NMP} L_{\xi_P} \quad (2.3)$$

where g_{NMP} are the structure constants of the Lie algebra of S . The equations defining the symmetric gauge fields are

$$L_{\xi_N} A_{\hat{\mu}} = D_{\hat{\mu}} W_N \quad N=1, \dots, \dim S \quad (2.4)$$

where $D_{\hat{\mu}}$ denotes the gauge covariant differential operator

$$D_{\hat{\mu}} W_N = \partial_{\hat{\mu}} W_N - [A_{\hat{\mu}}, W_N]$$

Clearly, the right-hand side of (2.4) is just an infinitesimal gauge transformation of $A_{\hat{\mu}}$. For a detailed analysis of (2.4), we refer to Ref. 1); here we merely summarize the results. In order to obtain non-trivial solutions of (2.4), one has to embed R into G faithfully. The solution of the symmetry Eqs. (2.4) for the vector potential is the following:

$$A_{\mu} = A_{\mu}(x^{\alpha})$$

$$A_{\alpha} = \sum_{M=1}^{\dim S} \Phi_M(x) \tilde{\xi}_{M\alpha}(y^{\alpha}, y^{\tau})|_{y^{\tau}=0} \quad (2.5)$$

where $\tilde{\xi}_{M\alpha} = h_{\alpha\beta}^{-1} \tilde{\xi}_M^{\beta}$, $\tilde{\xi}_M^{\beta}(y^{\alpha}, y^{\tau})$ are the (right) vector fields generating the Lie algebra on S and $h_{\alpha\beta}^{-1}$ is the bi-invariant metric on the group space. The co-ordinates of S are split as $y^{\alpha} = (y^{\alpha}, y^{\tau})$, y^{τ} denoting the co-ordinates of R . $y^{\tau} = 0$ corresponds to the unit element of R , and the non-trivial dependence of $A_{\hat{\mu}}$ on the coset space co-ordinates is given by (2.5).

$\Phi_M(x^{\mu})$ are Minkowski space scalar fields subject to the constraint

$$[\Phi_M, \Phi_r] + g_{Mrp} \Phi_p = 0 \quad \begin{matrix} r=1, \dots, \dim R \\ M=1, \dots, \dim S \end{matrix} \quad (2.6a)$$

Also the four-dimensional components of the gauge fields, $A_{\mu}(x)$, must satisfy

$$[A_{\mu}(x) \Phi_r] = 0 \quad r=1, \dots, \dim R \quad (2.6b)$$

In the above, the Φ_r components are constant and generate an R subalgebra of G . For example, in the case of spherical symmetry ($S = SU_2$, $R = U_1$)

$$S/R = S^2, \quad y^{\alpha} = (\theta, \varphi) \quad (2.7)$$

$$A_{\theta} = -\Phi_1(x)$$

$$A_\varphi = \bar{\Phi}_2(x) \sin \theta - \bar{\Phi}_3 \cos \theta \quad (2.7) \text{ cont.}$$

the constraints (2.6a) and (2.6b) reduce to

$$\begin{aligned} [\bar{\Phi}_3, \bar{\Phi}_1(x)] &= -\bar{\Phi}_2(x) \\ [\bar{\Phi}_3, \bar{\Phi}_2(x)] &= \bar{\Phi}_1(x) \\ [\bar{\Phi}_3, A_\mu(x)] &= 0 \end{aligned} \quad (2.8)$$

The gauge group preserving the S symmetry reduces from G to the little group of R (the centralizer of R in G), which we denote by $H = C_G(R)$. If R contains U_1 factors, the same U_1 's appear in H.

We now want to reduce the pure gauge action on M to an effective four-dimensional theory. Assuming that the metric tensor of M is block diagonal [this is a very natural assumption in view of (2.1)]:

$$g^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \frac{1}{L^2} h^{\alpha\beta}(y) \end{pmatrix} \quad (2.9)$$

where L is an arbitrary length scale. If we now insert (2.5) into the Yang-Mills action on M,

$$\mathcal{L} = -\frac{1}{4} \int d^{4+N} x \sqrt{-g} F_{\hat{\mu}\hat{\nu}}^a F_{\hat{\rho}\hat{\sigma}}^a g^{\hat{\mu}\hat{\rho}} g^{\hat{\nu}\hat{\sigma}} \quad (2.10)$$

using (2.9), we get

$$\begin{aligned} \mathcal{L} = L^N \int d^4 x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2L^2} \sum_N (D_\mu \bar{\Phi}_N)^a (D^\mu \bar{\Phi}_N)^a \right. \\ \left. + \frac{1}{4L^4} (g_{MNP} \bar{\Phi}_P + [\bar{\Phi}_M, \bar{\Phi}_N])^a (g_{MNS} \bar{\Phi}_S + [\bar{\Phi}_M, \bar{\Phi}_N])^a \right\} \end{aligned} \quad (2.11)$$

where Ω is the volume of the coset space.

We see that the pure gauge theory on M yielded a Yang-Mills-Higgs theory in four dimensions. The original gauge group reduces to H, and also a quartic scalar potential appears which leads in general to a spontaneous symmetry breaking of H to some gauge group $K \subset H$. Generally speaking, the only way to determine K is to solve first the constraint (2.6a) to get the physical Higgs fields; then

one has to minimize the Higgs potential explicitly. The Higgs content of (2.11) is not hard to compute as it follows from an application of Schur's lemma¹⁾. The result is the following.

Decompose the adjoint representation of S ($\text{ad}S$) into irreducible representations (irreps) of R :

$$\text{ad}S = \text{ad}R + \sum_i \underline{r}_i \quad (2.12)$$

where $\underline{r}_1$ is an irrep of R . Then decompose $\text{ad}G$ into irreps of $R \times H$:

$$\text{ad}G = \sum_j (\underline{s}_j, \underline{h}_j) \quad (2.13)$$

where $\underline{s}_j$ is an irrep of R , and $\underline{h}_j$ is an irrep of H . Now for each pair of $(\underline{r}_1, \underline{s}_j)$ where $\underline{r}_1$ and $\underline{s}_j$ are identical irreps, there is an $\underline{h}_j$ multiplet of (physical) Higgs fields in the four-dimensional theory.

It was observed in Ref. 4) that in the special case when one embeds S in G (faithfully), the determination of the final unbroken gauge group, K , is very easy and one evades the explicit minimization of the Higgs potential. To show this, let us suppose that $\{T^{\hat{a}}\}_{\hat{a}=1}^{\dim S}$ generates an S subalgebra of G . Now it is easy to see that $\Phi_N^a = \{-\delta_N^{\hat{a}}, 0\}$ solves (2.6a) for all values of M, r (not only for $r = 1, \dots, \dim R$), so this is an allowed choice for the values of the physical Higgs field. Moreover, it obviously corresponds to the minimum of the potential since the latter vanishes for these values of Φ_N (this is the "true" vacuum). The final unbroken gauge group, K , is clearly the little group of S in G ; $K = C_G(S)$. We summarize this result in the following diagram:

$$\begin{array}{ccc} & G & \\ \vee & & \\ S & \times & K \\ \vee & & \wedge \\ R & \times & H \end{array} \quad (2.14)$$

The most interesting feature of the dimensional reduction procedure is the existence of a two-stage symmetry breaking. First the gauge group G is broken geometrically to H , then in the second stage H is spontaneously broken to K . The rather attractive features of these theories are mainly due to the fact that with only a few free parameters (the coupling constant of the $4+N$ dimensional theory and the length scale(s) of the coset space) we get the complete bosonic sector, in particular a completely fixed Higgs structure. In a sense, (2.14) gives a

group theoretical construction of Higgs potentials, and it is not inconceivable that such a group structure could emerge naturally, independently of the presently invoked extra dimensions. Indeed, in this paper, we do not take the existence of the extra dimensions seriously - we consider them as a book-keeping device only. If one takes the extra dimensions as physical, one should analyze the non-symmetric fluctuations as well. This was investigated in Ref. 5). It is interesting to note that (2.14) leads automatically to vanishing vacuum energy density.

It seems that there are at least three interesting possible applications of the coset space reduction scheme (here not necessarily restricted to $S \subset G$).

- 1) Alternative to grand unified theories (GUTs)⁶⁾⁻⁸⁾ where one requires that a simple gauge group be broken to $H = SU_3 \times SU_2 \times U_1$ geometrically, then this is spontaneously broken to $SU_3 \times U_1^{em}$.
- 2) Derivation of GUTs, where one would demand H to be a GUT group (SU_5 , SO_{10} , etc.) broken to $SU_3 \times U_1$ hierarchically with $SU_3 \times SU_2 \times U_1$ as an intermediate step. That would clearly require the introduction of at least two length scales, and the obvious way of achieving this is to choose more general S invariant (not the projection of the bi-invariant group metric) coset space metrics. An interesting example of a hierarchical breaking was given in Ref. 7).
- 3) An "other alternative" to GUTs. Here one would require $H \supset SU_3 \times SU_2 \times U_1$ while H is not a GUT group, i.e., spontaneously broken to $K = SU_3 \times SU_2 \times U_1$. K could then be further broken to $SU_3 \times U_1$ dynamically by fermionic condensates. An attempt in this direction can be found in Ref. 9).

In this paper, we concentrate on 1) and 2) in order to try to build an acceptable model with $S \subset G$, so that we have the group structure (2.14) associated to the Higgs potential. We are looking for $H \supseteq SU_3 \times SU_2 \times U_1$ which is spontaneously broken to $K = SU_3 \times U_1^{em}$. H can be either a "GUT group" or any conceivable group containing an electroweak $SU_2 \times U_1^Y$ subgroup. Surprisingly enough, we found a complete answer to this problem. Surprisingly, because there appears to be a great amount of arbitrariness; namely, G is an arbitrary simple group, S is any semisimple group and we fix only the centralizer of S , $C_G(S)$ to be $SU_3 \times U_1^{em}$.

At this stage, we have to incorporate fermions into our scheme before we can discuss realistic model building. It is rather straightforward to define symmetric fermions. The generalization of (2.4) for fermi fields is^{10),11)}:

$$L_{\xi_N} \Psi = \Omega_N \Psi + R(W_N) \Psi \quad (2.15)$$

i.e., a symmetry transformation on ψ should be equivalent to a local Lorentz transformation $(\Omega_N \psi)$, plus a gauge transformation $[R(W_N) \psi]$, where $R()$ denotes the appropriate representation of the gauge group]. The detailed analysis of Eq. (2.15) can be found in Ref. 11). There is an analogous constraint to Eq. (2.6a):

$$\Omega_r \Psi + R(\Phi_r) \Psi = 0 \quad r=1, \dots, \dim R \quad (2.16)$$

where $\{-\Omega_r\}_{r=1}^{\dim R}$ generates an R subalgebra in SO_d , to be compared with $\{-\Phi_r\}_{r=1}^{\dim R}$ also generating an R subalgebra in G . In contrast to the latter case, where one is free to choose the embedding of R into G , the embedding of R into SO_d is fixed by the action R on the tangent space of S/R . This embedding is completely characterized by

$$d = \sum_i r_i \quad (2.17)$$

where d is the vector of SO_d and $\sum r_i$ is the induced representation of R over S/R , that is, the corresponding piece on the left-hand side of Eq. (2.12).

The four-dimensional physical fermion fields (surviving fermions) solving Eq. (2.16) are determined by the application of Schur's lemma again. To compute them, we have to decompose the spinor, s , of SO_d under R :

$$s = \sum_{\alpha} r_{\alpha} \quad (2.18)$$

and compare (2.18) with the decomposition of R (the corresponding representation of G) under $R \times H$:

$$R = \sum_{\beta} (r_{\beta}, h_{\beta}) \quad (2.19)$$

where r_{β} (h_{β}) is an irrep of $R(H)$. For each pair (r_{α}, r_{β}) where r_{α} and r_{β} are identical irreps, there is an h_{β} multiplet of surviving fermions.

We incorporate fermions into our scheme by simply adding to the pure gauge action (2.10) a Dirac action

$$\frac{i}{2} \bar{\Psi} \Gamma^{\hat{\mu}} (\partial_{\hat{\mu}} \Psi - \omega_{\hat{\mu}} \Psi - R(A_{\hat{\mu}}) \Psi) + h.c. \quad (2.20)$$

(where $\Gamma^{\hat{\mu}}$ are the curved space Dirac matrices, $\omega_{\hat{\mu}}$ is the spin connection) and impose the symmetry Eqs. (2.16). Then (2.20) reduces to a four-dimensional action where the fermions are in H multiplets with Yukawa couplings to the geometrical Higgs fields and explicit mass terms of geometrical origin. These geometrical mass terms vanish identically if the coset space, S/R , is a symmetric space (i.e., $g_{mnp} = 0$ with the indices m, n, p corresponding to the coset space). In general, after spontaneous symmetry breaking from H to K, the fermions obtain masses through the Yukawa couplings. A very important property of the CSDR scheme is the possibility of parity violation, in contrast to the Kaluza-Klein type models¹²⁾. Here we indicate only roughly the mechanism that makes left-right asymmetry possible; for more details we refer to the numerous papers in this field^{11), 8), 13)}.

In the following, we shall assume that $d = \dim(S/R)$ is even, so we can define Weyl spinors on M. The Weyl condition can be written as

$$\Gamma^{5+d} \Psi_{\pm} = \pm \Psi_{\pm} \quad (2.21)$$

One can find a basis where

$$\Gamma^{5+d} = \gamma^5 \otimes \tilde{\gamma}^{d+1} \quad (2.22)$$

(where γ^5 is the four-dimensional chirality operator and $\tilde{\gamma}^{d+1}$ is the chirality operator in the extra dimensions. Therefore a Weyl spinor, $\Psi_{\pm}$, on M can be written as

$$\Psi_{\pm} = (\psi_L, \psi_+) + (\psi_R, \psi_-) \quad (2.23)$$

A Weyl spinor on M contains both left- and right-handed spinors, and the handedness in four dimensions is linked to the $\tilde{\gamma}^{d+1}$ chirality in the extra dimensions.

Now the symmetry Eqs. (2.15) do not involve the Minkowski space structure, but their solutions may be asymmetric with respect to the $\tilde{\gamma}^{d+1}$ chirality. This asymmetry then becomes a left-right asymmetry in four dimensions.

It was argued in Ref. 11) that it is necessary but not sufficient to put the fermions into a complex representation of G , and that $\text{rank } S = \text{rank } R$ in order to have parity violation. However, as was shown in Ref. 8), if the total number of dimensions is $D = 4n+2$, there is a way to get left-right asymmetry, starting with real (if $D = 2+8n$) or pseudoreal (if $D = 6+8n$) representations of G . Imposing both Weyl and Majorana conditions on the fermions can halve the number of states surviving. Roughly speaking, this possibility arises because the spinor of $SO(1, D-1)$ is real (resp. pseudoreal) if $D = 2+8n$ (resp. if $D = 6+8n$), and if Ψ is in a real (resp. pseudoreal) representation of G , then Ψ is overall real, in which case all the right-handed partners in (2.23) can be interpreted as the corresponding antiparticles. This is a very interesting observation since, starting with vectorlike theories (where the fermions are assigned to a self-conjugate representation of G), one can end up with models where the fermions appear in a non-self-conjugate representation of H .

We look for models where we have zero mass fermions, even after spontaneous symmetry breaking with correct electromagnetic charges. In a sense, the CSDR scheme predicts the H group and the number of generations.

3. - REFORMULATION OF THE PROBLEM

In this section, we explore further consequences of the group structure (2.14) which we discussed in the previous section. We deduce a number of properties of (2.14) which can then be used to reformulate the problem of finding all possible models so that it becomes tractable.

Let us first settle our notations. If $A \subset G$ (i.e., A is a proper subgroup of G), then we denote the centralizer (or the little group) of A in G as follows:

$$A \Rightarrow C_1$$

i.e., $C_1 = C_G(A)$. Let us now look at the centralizer of the centralizer:

$$A \Rightarrow C_1 \Rightarrow C_2$$

Then we have the following properties for the above sequence:

Property 1 $A \subseteq C_2$

Property 2 $C_{C_2}(A) = C_1 \cap C_2$, and is therefore Abelian.

To prove Property 2, let us suppose that the semisimple factor in $C_{C_2}(A)$ is M . Then, by definition, $M \subseteq C_2$, but $M \cap C_1 = 0$. Since by assumption $[A, M] = 0$, we have $M \subseteq C_2$. Hence $M = 0$. If C_1 contains Abelian factors, then these carry over to C_2 and form $C_1 \cap C_2$. These obviously commute with A .

Note that $C_{C_2}(A) = 0$ does not imply that A is a maximal subgroup of C_2 (for example, consider $SO_3 \subset SU_3 \subset SO_8$ characterized by $3 + 5 = 8 = 8$).

Property 3 $C_1 \leftrightarrow C_2$

Indeed, from Property 1 we have $C_3 = C_G(C_2) \supseteq C_1$, but on the other hand

$$C_2 \supseteq A \quad \text{so} \quad C_G(C_2) \subseteq C_G(A) \quad \text{i.e.} \quad C_3 \subseteq C_1$$

therefore $C_3 = C_1$.

Let us now recall that the final unbroken gauge group $K [= C_G(S)]$ is of the form $K = K_0 \times U_1$, with K_0 being semisimple (in the case we are studying, $K_0 = SU_3$ but other K_0 groups may also be interesting, e.g., $K_0 = SU_2 \times SU_3$). A very important step in our reformulation of the original problem is to introduce the centralizer of K :

$$C_G(K) = U_1 \times P \quad (3.1)$$

where the U_1 factor on the right-hand side is the same U_1 as in $K = K_0 \times U_1$. Now, using Property 2, we immediately obtain the relation:

$$C_P(S) = 0 \quad (3.2)$$

which implies that P is a semisimple group. We now define the centralizer of K_0 , the semisimple factor of K :

$$C_G(K_0) = P_0 \quad (3.3)$$

Then, since $C_G(P) = K$, we have

$$C_{P_0}(P) = P_0 \cap K = U_1 \quad (3.4a)$$

and in view of (3.1) and (3.3), we also have

$$C_{P_0}(U_1) = U_1 \times P \quad (3.4b)$$

Furthermore, since

$$C_{P_0}(S) = C_G(S) \cap P_0 = (K_0 \times U_1) \cap P_0 = U_1$$

we have

$$C_{P_0}(S) = U_1 \quad (3.5)$$

In view of the fact that we want to have a model where there is a unification of electroweak forces, it is important to know the centralizer of K_0 (in our case, SU_3^{colour}) in the gauge group H , $C_H(K_0)$, which we require to be non-Abelian. There is an important relation between $C_H(K_0)$ and the centralizer of R in P_0 ($R < S$):

$$C_H(K_0) = C_{P_0}(R) \quad (3.6)$$

Indeed, since

$$H \cap P_0 = H \cap C_G(K_0) = C_H(K_0)$$

and also

$$H \cap P_0 = C_G(R) \cap P_0 = C_{P_0}(R)$$

(3.6) immediately follows. In a very similar way, one can deduce that

$$C_H(K) = C_P(R) \times U_1 \quad (3.7)$$

Finally, let us consider the centralizer of P_0 :

$$C_G(P_0) = Q$$

Since $K_0 \subset K$ it follows that $P_0 \geq P \times U_1$, therefore

$$C_G(P_0) = Q \subseteq C_G(P \times U_1) = K$$

From Property 1, we obtain $K_0 \times U_1 \geq Q \geq K_0$. This implies that either $Q = K_0$, which means that

$$P_0 \longleftrightarrow K_0$$

(3.8)

and P_0 is semisimple, or $Q = K$. In this latter case,

$$C_H(K_0) = H \cap P_0 = H \cap (P \times U_1) = (H \cap P) \times U_1$$

so that $C_H(K_0)$ contains the U_1 factor separately. All these results can be summarized in the diagram

$$\begin{array}{ccc}
 G & & G \\
 \downarrow & & \downarrow \\
 U & & U \\
 P_0 & \longleftrightarrow & K_0 \\
 \downarrow & & \downarrow \\
 U_1 \times P & \longleftrightarrow & K = K_0 \times U_1 \\
 \downarrow & & \downarrow \\
 U & \nearrow & \cap \\
 S & & \\
 \downarrow & & \\
 R & \longrightarrow & H \\
 & & \downarrow \\
 & & G
 \end{array}
 \tag{3.9}$$

where the following comments are in order:

- 1) G, P_0, P, S and K_0 are semisimple;
- 2) We have assumed (3.8) for reasons to be explained later.

So far, we have simplified the original problem of finding G, S, R, H to the extent that it is now formulated entirely as a single sequence of embedding. An important step is to find all P_0, K_0 pairs centralizing each other. This centralizing property is a very restrictive one. Clearly, when $P_0 \times K_0$ is a maximal subgroup, P_0 and K_0 do centralize each other. Given such a P_0 , we have to find all semisimple subgroups P of P_0 which satisfy conditions (3.4). As is proved in Appendix A, an efficient algorithm to determine the possible groups P is the following:

- (i) Remove a single dot from the Dynkin diagram of P_0 . The remaining part of the diagram gives a semisimple group P with the property $C_{P_0}(U_1) = P$.
- (ii) It is easy to determine $C_{P_0}(P)$ from the extended Dynkin diagram of P_0 . One

finds that it is always U_1 except if the removed dot detaches the extra (most negative) root from the diagram, in which case $C_{P_0}(P) = SU_2$.

In our search for interesting models, we have to know the centralizers of K and K_0 in H . In particular, we shall focus on the case $K_0 = SU_3^C$, $C_H(K)$ being Abelian and $C_H(K_0)$ being non-Abelian, because this ensures the unification of the U_1 electromagnetic within some non-Abelian electroweak group (not necessarily $SU_2 \times U_1$). If there is a non-Abelian factor of $C_H(K_0)$ which commutes with $U_1^{e.m.}$, then that factor should be contained in $C_H(K)$ which, however, is assumed to be Abelian. In principle, there is a possibility where this simple argument does not ensure unification, namely if $C_H(K)$ should turn out to be non-Abelian. If this happens, one has to check further for unification. However, in practice, we did not encounter one single case where this happened, at least with $K_0 = SU_3$. In view of Eq. (3.6), we are led to consider $C_{P_0}(R)$. Furthermore, we also require $\text{rank}(S) = \text{rank}(R)$, which seems to be a necessary condition to have parity violation^{11),8)}.

To check whether a particular choice of $S(\subset P)$ allows for unification, it is easiest to take first $R = U_1^{\text{rank}(S)}$, i.e., R is the maximal torus in S . If $C_{P_0}(R)$ is Abelian, then there is no possibility for unification with this choice of S and with any other choice of R . Indeed, since $r(R) = r(S)$, clearly $R \supseteq U_1^{r(S)}$ and therefore $C_{P_0}(R) \subseteq C_{P_0}(U_1^{r(S)})$. Finally, all these results can be assembled into a simple recipe to construct all possible models, starting with a simple group G satisfying all the requirements. This recipe consists of the following five steps:

- 1) Find all regular $SU_3 \leftrightarrow P_0$ subgroups of G where P_0 is semisimple.
(As we shall see in the next section, this is a very severe constraint and it eliminates virtually all SU_3 embeddings in the classical groups.)
- 2) Given a P_0 in step 1, find all semisimple subgroups P of P_0 with $C_{P_0}(P) = U_1$ and $C_G(P) = K$ (by removing dots from the Dynkin diagram of P_0 , with the exception of the dot whose removal detaches the extra root of the extended Dynkin diagram of P_0).
- 3) Determine all semisimple $S \subset P$ such that $C_P(S) = 0$ and $C_G(S) = K$.
- 4) Take the maximal torus $R = U_1^{r(S)}$ in S and check if $C_{P_0}(R)$ is non-Abelian,

and also if $C_P(R)$ is Abelian (otherwise investigate further). If this is the case, find all other R subgroups of S with these properties. Compute H .

After these four steps, the construction of the bosonic sector is complete.

- 5) Compute the spectrum of the surviving fermions as outlined in the previous section.

4. - EMBEDDINGS IN CLASSICAL ALGEBRAS

In this section a demonstration will be given that there exist no embeddings of $SU_3 \times U_1$ in any classical algebra such that the centralizer of $SU_3 \times U_1$ is $U_1 \times P$ (P semisimple) and has $SU_3 \times U_1$ only as its centralizer. The treatment given is more general than strictly required, but it is hoped that the techniques presented will prove useful in later applications. [One might, for example, wish to extend the present work by incorporating the weak SU_2 algebra in the original embedding.]

The primary step is to identify particular classes of embedded SU_3 algebras, and in the case of embedding in SU_N , it is sufficient to specify the SU_3 decomposition of the fundamental $\underline{N}$ multiplet. If a given SU_3 multiplet $\underline{M}^i$ appears N_i times in this decomposition, then the centralizer C_1 of that SU_3 will contain an SU_{N_i} . Also, subject only to respecting the tracelessness of the original generators, the centralizer will contain a U_1 factor distinguishing these $\underline{M}_i$ multiplets ($\underline{N}_i$) from the remaining $N - \sum N_i$ singlets of SU_{N_i} . That is to say that the centralizer of the SU_3 is

$$C_1 = \left[\prod_{N_i > 1} SU_{N_i} \right] \times \left[\prod_{i=1}^{\ell-1} U_i \right] \quad (4.1)$$

where ℓ is the number of distinct SU_3 multiplets in the N multiplet, and

$$\underline{N} = \sum_{i=1}^{\ell} (\underline{N}_i, \underline{M}_i) a_i \quad (4.2)$$

where a_i symbolizes the $(\ell-1)$ labels associated with the U_1 factors. Now, therefore, the product of the conjugate multiplet $\bar{\underline{N}}$ with $\underline{N}$ itself takes the form

$$\underline{N} \times N = \sum_{i,j} (\underline{N}_i \times N_j, \underline{M}_i \times M_j) a_j - a_i \quad (4.3)$$

and, noting that SU_3 singlets arise only in the product of each multiplet $\underline{N}_1$ with its conjugate (and then once only), the structure of the centralizer C_1 of SU_3 given in (4.1) above is confirmed explicitly since the product of each fundamental $\underline{N}_1$ with its conjugate decomposes into the sum of the adjoint of that SU_{N_1} and a singlet. The next step is to select the subset of cases in which the centralizer C_2 of the centralizer C_1 is the SU_3 itself, and not some larger algebra. Equation (4.3) shows that the C_1 singlet given by an $\underline{N}_1 \underline{N}_1$ product gives rise to elements of C_2 from all SU_3 multiplets in the decomposition of $\underline{M}_1 \underline{M}_1$, so that if any $\underline{M}_1$ is larger than a triplet then C_2 is larger than SU_3 , and the embedding may be rejected. By the observation that each $\underline{N}_1 \underline{N}_1$ product yields a C_1 singlet, attention may be further restricted to partitions of $\underline{N}$ involving only a single type of SU_3 multiplet. (This conclusion may also be reached by requiring C_1 to be semisimple, so that ℓ must be unity to avoid the U_1 factors.) Without loss of generality, the fundamental now decomposes under $SU_3 \times SU_M$, say, as

$$\underline{N} = (\underline{3}, \underline{M}) \quad (4.4)$$

and therefore the adjoint as

$$\underline{N}^2 - 1 = (\underline{8}, \underline{1}) + (\underline{1}, \underline{M}^2 - 1) + (\underline{8}, \underline{M}^2 - 1) \quad (4.5)$$

which reveals the important feature that only one type of SU_M non-singlet appears in this case. It follows that if a further decomposition of SU_M under a subalgebra $P \times U_1$ is made so as to include an overall singlet arising from the second term in (4.5), then a corresponding SU_3 octet appears as a contribution to C_2 arising from the third term of that equation. Hence this final case is excluded, and the possibility of taking G to be SU_N is eliminated.

When G is taken to be SO_N , it is sufficient to consider the (real) N dimensional vector representation and to study the adjoint by examining the antisymmetric part of the product of two such vectors. As Dynkin notes¹⁴⁾, the embeddings in this case may not be fixed completely until the content of two independent representations (e.g., vector and spinor) is specified. These ambiguities, which are related to outer automorphisms, are irrelevant in the present context

since the argument presented eliminates the possibilities, irrespective of the further specification of the embeddings. Proceeding as in the previous case, one may write

$$\underline{N} = \sum_r (\underline{N}_r, \underline{R})_0 + \sum_c [(\underline{N}_c, \underline{C})_1 + (\bar{\underline{N}}_c, \bar{\underline{C}})_{-1}] \quad (4.6)$$

where again the multiplet structure and U_1 labels of C_1 have been anticipated, and where $\underline{R}$ and $\underline{C}$ denote respectively real and complex representations of SU_3 ; the ranges of summation are not important this time and have been omitted. Examination of the antisymmetric part of the product of $\underline{N}$ by itself for SU_3 singlets is again simplified by the observation that such singlets appear once (and only once) in each product of an SU_3 multiplet with its conjugate (and in the symmetric part of that product if the representation is real). Thus, since the product has the form

$$\begin{aligned} [\underline{N} \times \underline{N}]_A &= \sum_{r, N_r > 1} [(\underline{N}_r \times \underline{N}_r)_A, (\underline{R} \times \underline{R})_S]_0 \\ &+ \sum_{r, R > 1} [(\underline{N}_r \times \underline{N}_r)_S, (\underline{R} \times \underline{R})_A] + \sum_c [(\bar{\underline{N}}_c \times \underline{N}_c), (\bar{\underline{C}} \times \underline{C})]_0 + \dots \end{aligned} \quad (4.7)$$

where the remaining terms have non-zero U_1 labels and contain no SU_3 singlets, the centralizer is identified as

$$C_1 = \left[\prod_{r, N_r > 1} SO_{N_r} \right] \times \left[\prod_{c, N_c > 1} SU_{N_c} \right] \times \left[\prod_c U_1 \right] \quad (4.8)$$

and the notation adopted is now seen to be appropriate. Inspection of (4.7) for C_1 singlets reveals that all the multiplets of $[\underline{R} \times \underline{R}]_A$ and of $(\bar{\underline{C}} \times \underline{C})$ will give generators of C_2 for each $\underline{R}$ and $\underline{C}$, so that there are no non-trivial embeddings in which C_2 is no larger than SU_3 itself.

The analysis of symplectic cases follows in a similar way to the above, and if G is Sp_{2N} , it suffices to specify the SU_3 content

$$2\underline{N} = \sum_r (2\underline{N}_r, \underline{R})_0 + \sum_c [(\underline{N}_c, \underline{C})_1 + (\bar{\underline{N}}_c, \bar{\underline{C}})_{-1}] \quad (4.9)$$

of the real even-dimensional vector $\underline{2N}$, where again the C_1 structure is anticipated in the notation. Notice that the number $(2N_r)$ of any given real SU_3 representation appearing must be even. This is easily seen from a generalization of the familiar result that, although SO_8 contains an embedded SU_3 with the eight-vector identified with an SU_3 octet, there is no similar embedding in Sp_8 because the only SU_3 singlet in $\underline{8} \times \underline{8}$ is in the symmetric part. Thus specifically, if the non-singular antisymmetric invariant tensor which defines the algebra of Sp_{2N} is denoted by J_{AB} , where the indices take values in the range 1 to $2N$, then (in component notation) the product $(\underline{2N})_A \times (\underline{2N})_B$ must contain a piece proportional to J_{AB} in its reduction. However, restricting attention to the sub-range of indices corresponding to a single real SU_3 representation $\underline{R}$, the invariant tensor must have the structure

$$J_{AB} = j_{\alpha\beta} \delta_{ab} \quad (4.10)$$

with the obvious notation in which $\underline{2N}_r$ and $\underline{R}$ have components $(\underline{2N}_r)_\alpha$ and $(\underline{R})_a$ respectively, since δ_{ab} is the only SU_3 singlet in $\underline{R} \times \underline{R}$. Now, since δ_{ab} is symmetric, $j_{\alpha\beta}$ is antisymmetric and thus is singular (and hence so is J_{AB}) unless $(2N_r)$ is even as indicated. To continue the discussion, it is convenient to work in a representation which emphasizes a particular $SU_N \times U_1$ substructure of Sp_{2N} in which

$$\underline{2N} = \underline{N}_1 + \overline{\underline{N}}_{-1} \quad (4.11)$$

where the U_1 labels are explicitly exhibited. Now, if J_{AB} is taken to have the matrix form

$$J = (-i\tau_2) \times 1^{(N)} \quad (4.12)$$

where the $1^{(N)}$ is N by N and τ_2 is the familiar Pauli matrix, then the set

$$\tau \times \underline{S}^{(N)} \quad \text{and} \quad 1 \times \underline{A}^{(N)}$$

gives a matrix representation of the Sp_{2N} generators where $\underline{S}^{(N)}$ and $\underline{A}^{(N)}$ are all the symmetric and antisymmetric Hermitian N by N matrices respectively. In this

basis the set of matrices

$$\tau_3 \times \underline{S}^{(N)} \quad \text{and} \quad 1 \times \underline{A}^{(N)}$$

generates the $SU_N \times U_1$ subalgebra, and the decomposition given in (4.11) is manifest. Now, since the SU_3 decomposition of the $2N$ vector specifies the embedding (up to conjugation), it is sufficient to specify the SU_3 content of the $\underline{N}$ vector of SU_N ; that is to say, one can take the SU_3 to lie within this SU_N without loss of generality. [In this basis, the anticipated $\underline{2N}_r$ multiplet of Sp_{2N_r} in (4.9) decomposes into

$$\underline{2N}_r = \underline{N}_{r-1} + \underline{N}_{r-1} \quad (4.13)$$

and it is the sum of this U_1 label with the one previously displayed in (4.9) which is to be identified with the one introduced in (4.11).] With the exact form of J , and its form in each subspace, now specified, it is simple to discover C_1 and C_2 by examining the structure of the adjoint as revealed by the symmetric part of the product

$$\begin{aligned} [\underline{2N} \times \underline{2N}]_s = & \sum [(\underline{2N}_r \times \underline{2N}_r)_s, (\underline{R} \times \underline{R})_s]_0 \\ & + \sum_{r \geq 1, R \geq 1} [(\underline{2N}_r \times \underline{2N}_r)_A, (\underline{R} \times \underline{R})]_0 + \sum_c [(\underline{N}_c \times \underline{N}_c), (\underline{C} \times \underline{C})]_0 + \dots \end{aligned} \quad (4.14)$$

since this can be spanned by the generators multiplied by $(-i\tau_2) \times 1^{(N)}$ in this basis. As previously, the identification of SU_3 singlets gives C_1 , this time as

$$C_1 = \left[\prod_r Sp_{2N_r} \right] \times \left[\prod_{c, N_c \geq 1} SU_{N_c} \right] \times \prod_c [U_1] \quad (4.15)$$

and similarly all non-trivial embeddings are at once excluded if C_2 is not to be larger than SU_3 itself, for reasons corresponding to those in the previous case.

It is now clear that any $SU_3 \times U_1$ embeddings with the properties described in Section 2 can only be found in exceptional algebras; the study of these algebras is the topic of the next section. Already it is clear, however, that of the infinitely many examples one might have expected, only a small number (at most) can exist.

5. - MODELS IN THE EXCEPTIONAL ALGEBRAS

We now proceed to analyze the embeddings of SU_3 and $SU_3 \times U_1$ in the exceptional algebras. As we shall see, only a few embeddings of SU_3 exist in E_7 and in E_8 satisfying $C_G(SU_3) = P_0$ (P_0 is semisimple) and $C_G(P_0) = SU_3$. It is surprising that our framework turns out to be so restrictive, but given the extremely large degree of arbitrariness one meets in most model-building schemes, this is a theoretically desirable feature. Also the fact that E_7 and E_8 emerged naturally is rather interesting, since these groups were already suggested as possible unifying gauge groups for quite different reasons⁷⁾.

The first step in our scheme is to find all SU_3 subalgebras in F_4 , E_6 , E_7 , E_8 with $SU_3 \leftrightarrow P_0$. (We discarded G_2 as it is too small to contain $SU_3 \times U_1$.) Unfortunately, the powerful techniques for identifying the possible subalgebras and their centralizers used in the previous section are not applicable in the exceptional algebras. Therefore, we had to find all different (non-conjugate) SU_3 's going through the maximal subalgebras¹⁴⁾⁻¹⁶⁾. This is a straightforward but tedious task, and we refer to Appendix B, where we summarize our results in Tables 1a, 1b, 1c and 1d. We mention here that it was Dynkin who listed simple subalgebras of the exceptional algebras in his important paper¹⁴⁾. Part of our work can be regarded as a check of his SU_3 list. Apart from one case in E_8 , we fully agree with his results. We also call attention to a somewhat subtle point in Appendix B concerning linear equivalence and conjugacy.

We now list all $SU_3 \times P_0$ subalgebras with $SU_3 \leftrightarrow P_0$ and P_0 semisimple in the exceptional algebras F_4 , E_6 , E_7 , E_8 :

$$F_4 : \quad SU_3 \times SU_3 \quad (R.M.) \quad (5.1)$$

$$E_6 : \quad \begin{array}{ll} a) & SU_3 \times SU_3 \times SU_3 \\ b) & SU_3 \times G_2 \end{array} \quad \begin{array}{l} (R.M.) \\ (S.M.) \end{array} \quad (5.2)$$

$$E_7 : \quad \begin{array}{ll} a) & SU_3 \times SU_6 \\ b) & SU_3 \times SU_3 \times SU_2 \end{array} \quad \begin{array}{l} (R.M.) \\ (S.N.M.) \end{array} \quad (5.3)$$

$$E_8 : \quad \begin{array}{ll} a) & SU_3 \times E_6 \\ b) & SU_3 \times SU_2 \\ c) & SU_3 \times SU_3 \times G_2 \end{array} \quad \begin{array}{l} (R.M.) \\ (S.M.) \\ (S.N.M.) \end{array} \quad (5.4)$$

R.M. stands for regular maximal, S.M. for special maximal and N.M. for non-maximal. Equations (5.1) - (5.4) are immediately established from Tables 1a - 1d in Appendix B. From the list above, it is clear that there are indeed embeddings of a product group $SU_3 \times P_0$ which are not maximal; nevertheless, the factors do centralize each other.

Following the recipe given in Section 3, we now have to find all semisimple subalgebras P of P_0 with $C_{P_0}(P) = U_1$ and $C_G(P) = SU_3 \times U_1$. The branching rule of the adjoint of F_4 in (5.1) is

$$52 = (8, 1) + (1, 8) + (6, \bar{3}) + (\bar{6}, 3) \quad (5.5)$$

We must be careful, since the two SU_3 's are not symmetric, so it matters which SU_3 we identify with P_0 . Using the result of Appendix A, it is clear that P should be SU_2 , but since $C_{F_4}(SU_2^{(1)}) = SU_4$ and $C_{F_4}(SU_2^{(2)}) = S_{P_6}$, there is no possible model in F_4 .

In case (5.2a), the decomposition of the adjoint of E_6 is the following:

$$78 = (8, 1, 1) + (1, 8, 1) + (1, 1, 8) + (3, 3, \bar{3}) + (\bar{3}, \bar{3}, 3) \quad (5.6a)$$

Here, $P = SU_3^{(2)} \times SU_2^{(3)}$ and $C_{E_6}(SU_3 \times SU_2) = SU_3 \times U_1$, so we can go on. We should now take all $S \subseteq P$ with $C_{E_6}(S) = SU_3 \times U_1$. We have the following three possibilities: $S_0 = P$, $S_1 = SU_2^{(m)} \times SU_2^{(3)}$, $S_3 = SU_2^{(diag)}$, where $SU_2^{(m)}$ is the maximal SU_2 subalgebra of $SU_3^{(2)}$ ($3 = 3$) and $SU_2^{(diag)}$ is the diagonal SU_2 subalgebra of $SU_2^{(m)} \times SU_2^{(3)}$. So we found acceptable sequences of $P_0 > P > S_i$ for the first time. However, in the last step things go wrong, since in all three cases $C_{P_0}(R_i) = U_1^4$, which means that there is no unification. (R_i 's are the maximal tori in S_i , $i = 0, 1, 2$.) For (5.2b), the relevant branching rule is:

$$78 = (8, 1) + (1, 14) + (8, 7) \quad (5.6b)$$

Now P could be one of the two (different) maximal SU_2 's in G_2 . Since $C_{G_2}(SU_2^{(1)}) = SU_2^{(2)}$, there is no electromagnetic U_1 , so this case is also unacceptable.

Let us consider (5.3b) in E_7 first. The $SU_3^{(1)} \times SU_3^{(2)} \times SU_2$ content of the adjoint of E_7 is:

$$133 = (1, 1, 3) + (8, 1, 1) + (1, 8, 1) + (6, \bar{3}, 1) + (\bar{6}, 3, 1) + (8, 1, 3) \\ + (3, 3, 3) + (\bar{3}, \bar{3}, 3) \quad (5.7a)$$

Here $P_0 = SU_3^{(2)} \times SU_2$ and the only semisimple P with $C_{P_0}(P) = U_1$ is $SU_2^{(2)} \times SU_2$; however, as $C_{E_7}(SU_2^{(2)} \times SU_2) = Sp_6$, (5.3b) should be disregarded. For case (5.3a), we have the following branching rule:

$$133 = (8, 1) + (1, 35) + (3, 15) + (\bar{3}, 15) \quad (5.7b)$$

Since $P_0 = SU_6$, we can obviously take for P the following three algebras:

$$P_1 = SU_5 \quad P_1 \rightarrow SU_3 \times U_1 \quad (5.8a)$$

$$P_2 = SU_4 \times SU_2 \quad P_2 \rightarrow SU_4 \quad (5.8b)$$

$$P_3 = SU_3 \times SU_3 \quad P_3 \rightarrow SU_3 \times U_1 \quad (5.8c)$$

(5.8b) should be immediately discarded as $C_{E_7}(P_2) = SU_4$. Applying the usual steps in our analysis, we get the following results for (5.8a). There are two possible algebras for S ($\subset SU_5$):

$$S_1 = SO_5 \quad (5.9a)$$

$$S_2 = SO_3 \quad (5.9b)$$

The embedding of S_1 (resp. S_2) in SU_5 is characterized by $5 = 5(5)$. The interesting R subalgebras of S_1 [that is, for which $C_{SU_6}(r)$ is non-Abelian so that unification does occur] are:

$$R_1 = SO_4 \quad C_{SU_6}(R_1) = SU_2 \times U_1 \quad (5.10a)$$

$$R_2 = SU_2 \times U_1 \quad C_{SU_6}(R_2) = SU_2 \times U_1 \times U_1 \quad (5.10b)$$

$$R_3 = U_1 \times U_1 \quad C_{SU_6}(R_3) = SU_2 \times U_1^4 \quad (5.10c)$$

It is easy to verify that $C_P(R_i)$ is Abelian in all three cases. This means that we found models where $C_H(SU_3)$ is non-Abelian while $C_H(SU_3 \times U_1)$ is Abelian, so the electromagnetic U_1 is part of a non-Abelian group (we have unification). The H groups are found to be

$$C_{E_7}(R_i) = H_i$$

$$H_1 = SU_4 \times SU_2 \quad (5.11a)$$

$$H_2 = SO_7 \times SU_2 \times U_1 \quad (5.11b)$$

$$H_3 = SO_8 \times SU_2 \times U_1 \times U_1 \quad (5.11c)$$

Since $R_1 \supset R_2 \supset R_3$, clearly $H_3 \supset H_2 \supset H_1$. We have constructed models where the bosonic sector satisfies all the constraints we imposed. Next we consider the spectrum of symmetric (surviving) fermions.

The coset corresponding to case (5.10a) is SO_5/SO_4 , which is four-dimensional so $R = SO_4$ ($d = \dim S/R$). The spinor of SO_4 is

$$4_S = (2, 1) + (1, 2) \quad (5.12)$$

Decomposing the fundamental representation of E_7 , the 56, under SO_4 , we find that no spinor of SO_4 appears in the branching, hence there would be no surviving fermions at all! This is true for any representation of E_7 . The next coset space is six-dimensional; it is given as $SO_5/SO_3 \times U_1$, corresponding to (5.10b).

The embedding of R into SO_6 is specified as $SO_6 \supset SU_3 \times U_1^{(1)} \supset SU_2 \times U_1^{(2)} \times U_1^{(1)}$, and R should be identified with $SU_2 \times U_1^{(2)}$. Then the spinor of SO_6 has the branching rule

$$4 = 2_1 + 1_{-2} + 1_0 \quad (5.13)$$

where the lower labels denote $U_1^{(2)}$ values. The 56 is decomposed under $(SU_3^C \times SO_3) \times (U_1 \times U_1^{e.m.})$ as

$$\begin{aligned}
 56 = & (3, 2)(1, 1) + (3, 2)(-1, 1) + (3, 1)(0, 1) + (3, 1)(0, -5) \\
 & + (\bar{3}, 2)(1, -1) + (\bar{3}, 2)(-1, -1) + (\bar{3}, 1)(0, -1) + (\bar{3}, 1)(0, 5) \\
 & + (1, 2)(1, -3) + (1, 2)(-1, -3) + (1, 1)(0, -3) + (1, 1)(2, -3) \\
 & + (1, 1)(-2, -3) + (1, 2)(1, 3) + (1, 2)(-1, 3) \\
 & + (1, 1)(0, 3) + (1, 1)(2, 3) + (1, 1)(-2, 3) + (1, 3)(0, -3) \\
 & + (1, 3)(0, 3)
 \end{aligned}$$

(5.14)

From (5.14) we conclude that almost everything survives from the 56 (except the SO_3 triplet pieces). We have singlets of SU_3^C with electric charges ± 3 and triplets with charges 1 and -5. By imposing reality, state doubling can be avoided. Parity is violated. Although other multiplets can give more realistic charges, no example without exotic particles exists. We mention here that since our coset space is not symmetric, most of the surviving fermions will get masses after dimensional reduction due to the geometrical mass terms.

The coset of (5.10c) is eight-dimensional and the calculations carried out analogously to the above show that there are no surviving fermions at all. To complete the analysis of (5.8a) we consider the other possibility for S, (5.9b). The H group is found to be

$$H = SO_8 \times SU_2 \times U_1 \times U_1$$

and there is unification. However, as it can be easily seen, there will be no surviving fermions again.

Let us now consider the next choice for P corresponding to (5.8c). Then the only interesting S is $SO_3 \times SO_3$, and when $R_1 = U_1 \times U_1$ the H group is again $SO_8 \times SU_2 \times U_1 \times U_1$, and there are no surviving fermions. A last possibility where $R_2 = SO_3 \times U_1$ is uninteresting in view of the fact that S and R have a factor in common. The argument for this is spelt out in Appendix C; it turns out that there is no unification in this case and $H = SU_4 \times U_1 \times U_1$. This completes the analysis for E_7 .

We now discuss the largest exceptional algebra, E_8 . Obviously, (5.4b) should be immediately discarded. The branching rule of the adjoint of E_8 for (5.4c) is:

$$248 = (8, 1, 1) + (1, 8, 1) + (6, \bar{3}, 1) + (\bar{6}, 3, 1) + (1, 1, 14) + (8, 1, 7) + (3, 3, 7) + (\bar{3}, \bar{3}, 7) \quad (5.15)$$

Again, the order of the SU_3 's is important; clearly SU_3^C should be identified with the first SU_3 in (5.15), so $P_0 = SU_3^{(2)} \times G_2$. It is not hard to verify that there is no semisimple $P \subset P_0$, such that $C_{P_0}(P) = U_1$ and $C_{E_8}(P) = SU_3 \times U_1$. This leaves us with (5.4a), which is analogous to case (5.3a) in E_7 .

The decomposition of the 248 reads

$$248 = (8, 1) + (1, 78) + (3, 27) + (\bar{3}, \bar{27}) \quad (5.16)$$

Applying the algorithm of Appendix A, we find the following candidates for P : SO_{10} , $SU_2 \times SU_5$ and $SU_2 \times SU_3 \times SU_3$. The first case can be excluded because its centralizer in $E_8 = SU(4) \times U_1 \supset SU_3 \times U_1$. For the two remaining choices of P , we now list the relevant possibilities for S and R :

$$P^{(1)} = SU_2 \times SU_5 \quad (5.17)$$

$$S^{(1)} = SU_2 \times SO_3$$

$$R_1 = U_1 \times SO_4$$

$$R_2 = U_1 \times SU_2 \times U_1 \quad (5.17a)$$

$$R_3 = U_1 \times U_1 \times U_1$$

$$S^{(2)} = SU_2 \times SO_3$$

$$R = U_1 \times U_1 \quad (5.17b)$$

$$S^{(3)} = SU_2^{(diag)}$$

$$R = U_1 \quad (5.17c)$$

$$P^{(2)} = SU_2 \times SU_3 \times SU_3$$

$$(5.18)$$

$$\begin{aligned} S^{(1)} &= SU_2 \times SO_3 \times SO_3 \\ R &= U_1 \times U_1 \times U_1 \end{aligned} \quad (5.18a)$$

$$\begin{aligned} S^{(2)} &= SO_3^{(1+2)} \times SO_3^{(3)} \\ R &= U_1 \times U_1 \end{aligned} \quad (5.18b)$$

Here some comments are in order. First of all, we omitted all those choices for R where no unification occurs [i.e., where $C_{E_6}(R)$ is Abelian]. Secondly, we omitted all cases where S and R have a factor in common. As is shown in Appendix C, we can discard these cases because either the whole problem (including fermions) can be embedded in a proper subgroup of G , or there is no unification. In other words, (5.17) and (5.18) give all irreducible E_8 models for which the bosonic sector satisfies our criteria.

With respect to the cases (5.17a), (5.17b) and (5.18a), one more observation should be made. If we write $P^{(1)} = SU_2 \times P_1$ and $P^{(2)} = SU_2 \times P_3$, then P_1 and P_3 are the same as those appearing in Eqs. (5.8a) and (5.8c) respectively of the E_7 analysis. It is not hard to show that the H groups in E_8 are simply of the form $U_1 \times H^{(E_7)}$ because $C_{E_8}(U_1 \times R^{(E_7)}) = U_1 \times C_{E_7}(R^{(E_7)}) = U_1 \times H^{(E_7)}$. Furthermore, since $C_{E_6}(U_1 \times R^{(E_7)}) = U_1 \times C_{SU_6}(R^{(E_7)})$, we have that $C_H(E_8)(SU_3) = U_1 \times C_H(E_7)(SU_3)$. In other words, the bosonic sector is a trivial extension of the corresponding E_7 models. However, because the internal spaces do have a different dimensionality for the corresponding E_7 and E_8 cases, one does not expect such a trivial extension to exist in the fermion sector. We will comment on this question when we give the results for the fermions in the E_8 models to which we now turn.

For the cases R_1 and R_3 of (5.17a), one finds that no fermions survive. For the case R_2 , we have the eight-dimensional coset space $SU_2 \times SO_5/U_1 \times SU_2 \times U_1$. The embedding of $R_2 \subset SO_8$ is then characterized by the inclusions

$$SO_8 \supset SU_4 \times U_1 \supset SU_2 \times SU_2 \times U_1 \times U_1 \supset SU_2 \times (U_1 \times U_1) \times U_1 \supset SU_2 \times U_1 \times U_1 \quad (5.19)$$

The resulting branching rule for the SO_8 spinor under R_2 is then:

$$2(1, 1) + 2(-1, -1) + 1(0, 1) + 1(0, -1) + 1(2, -1) + 1(-2, 1) \quad (5.20)$$

The fermions which are in the 248 of E_8 have the decomposition (5.16) under $SU_3 \times E_6$, and the relevant E_6 representations decompose under $R_2 \times U_1^{e.m.}$ as ^{*}

$$\begin{aligned} 27 = & 1(0, \pm 1, -5) + 1(0, \pm 1, 1) + 2(\pm 1, \pm 1, 1) \\ & + 1(0, 0, -2) + 1(0, 0, +4) + 1(\pm 2, 0, -2) \\ & + 2(\pm 1, 0, -2) + 2(\pm 1, 0, +4) + 3(0, 0, -2) \end{aligned} \quad (5.21a)$$

and

$$\begin{aligned} 78 = & 1(0, \pm 1, -3) + 1(0, \pm 1, 3) + 1(\pm 2, \pm 1, -3) + 1(\pm 2, \pm 1, 3) + 2(\pm 1, \pm 1, -3) \\ & + 2(\pm 1, \pm 1, 3) + 4[1(0, 0, 0)] + 1(0, \pm 2, 0) \\ & + 1(\pm 2, 0, 0) + 1(0, 0, 6) + 1(0, 0, -6) + 2[2(\pm 1, 0, 0)] \quad (5.21b) \\ & + 2(\pm 1, 0, 6) + 2(\pm 1, 0, -6) + 2[3(0, 0, 0)] + 3(0, \pm 1, 3) + 3(0, \pm 1, -3) \\ & + 3(\pm 2, 0, 0) \end{aligned}$$

The surviving fermions are triplets of colour with charges 1 and -5 and colour singlets with charges ± 3 . As expected, there is no parity violation in this case where we started with a real gauge group and we reduce over an eight-dimensional manifold. Comparing this model with the corresponding E_7 model as specified in (5.13) and (5.14), we see that although more fermions survive, their quantum numbers are the same; there are just more copies distinguished by the new U_1 quantum number.

The next model of interest corresponds to (5.17b), with the four-dimensional internal space $SU_2 \times SO_3/U_1 \times U_1$. The spinor of SO_4 transforms under $R = U_1 \times U_1$ as

$$(2, 1) \longrightarrow (1, 1) + (-1, -1) \quad (5.22)$$

The relevant E_6 representations in (5.16) have the following branching rules under $S^{(2)} \times U_1^{e.m.}$:

$$27 = (2, 1)(-5) + (2, 5)(1) + (1, 5)(4) + (1, 3+7)(-2) \quad (5.23a)$$

^{*}) Reference 16) appears to give the decomposition of the $\overline{27}$ in the $SU_2 \times SU_6$ subgroup of E_6 . This is only important in making comparisons with other models.

and

$$\begin{aligned} 78 = & (2, 3+7)(3) + (2, 3+7)(-3) + (3, 1)(0) \\ & + (1, 1+3+5+7+9)(0) + (1, 5)(6) + (1, 5)(-6) \end{aligned} \quad (5.23b)$$

Because $S^{(2)} \approx SO_4$, it is easy to see that there are no surviving fermions. It is amusing to note that in the corresponding E_7 model, there were no surviving fermions either.

The model corresponding to (5.17c) is new in the sense that even its bosonic sector has no counterpart in E_7 . The coset space is just a two-sphere SU_2/U_1 , and each even representation under $S^{(3)} = SU_2 = SO_3$ gives a surviving fermion. The fermion content decomposes under $S^{(3)} \times U_1^{e.m.}$ as

$$27 = 2(-5) + (6+4)(1) + 5(4) + (3+7)(-2) \quad (5.24a)$$

and

$$\begin{aligned} 78 = & (2+4+6+8)(3) + (2+4+6+8)(-3) + (1+3+3+5+7+9)(0) \\ & + 5(6) + 5(-6) \end{aligned} \quad (5.24b)$$

The surviving theory has triplets with charges -5 and 1, and singlets with charge ± 3 [coming from the 2, 4, 6 and 8 in (5.24b)]. Once more, geometric mass terms will be absent. Parity is violated. State doubling can be avoided by imposing reality.

In the penultimate case (5.18a), it turns out that no fermions survive at all. This is somewhat unfortunate, because the coset space is six-dimensional and hence parity violation would have been a possibility.

Finally we discuss the case (5.18b), which has no bosonic counterpart in E_7 . The coset space is $SO_3 \times SO_3/U_1 \times U_1$. The branching rule for the fermions under $S^{(2)} \times U_1^{e.m.} = SO_3^{(1+2)} \times SO_3^{(3)} \times U_1^{e.m.}$ is:

$$\begin{aligned} 27 = & (2+4, 1)(-1) + (2, 3)(1) + (3, 1)(2) \\ & + (1, 3)(-2) + (3, 3)(0) \end{aligned} \quad (5.25a)$$

and

$$\begin{aligned}
 7\theta = & (2, 1)(3) + (2, 1)(-3) + (1 + 3 + 3 + 5, 1)(0) \\
 & + (1, 3+5)(0) + (3, 3)(-2) + (3, 3)(2) \\
 & + (4+2, 3)(1) + (4+2, 3)(-1)
 \end{aligned}
 \tag{5.25b}$$

Again, because $S^{(2)} \simeq SO_4$ and the R embedding is unique, one immediately infers that there are no surviving fermions.

We have completed our analysis of all candidates for a realistic model. Unfortunately, none of them was realistic. Only rarely was parity violated, the $U_1^{e.m.}$ charges did not usually conform to the usual quark-lepton assignments, and exotic particles were found whenever these basic requirements were met. We hope to present the least fanciful of these models in detail at a later date¹⁷⁾.

6. - CONCLUSIONS

We have analyzed a wide class of models in an attempt to find the standard $SU_3 \times U_1$ model by applying the coset space dimensional reduction technique to a higher dimensional gauge theory. The fact that our conclusion is a negative one forces us to think about other possibilities which go beyond our attempt. One may think of removing the restriction that S can be faithfully embedded in G. Although there is no argument against this possibility in principle, the computation will be severely complicated because one has to compute the surviving Higgs fields explicitly and then minimize the potential to find out the spontaneous symmetry breaking pattern. It is not excluded that tools can be developed which would (partially) solve this problem for the general case, however. Another direction in which one might go is to take H to be a grand unified group and K to be the standard $SU_3 \times SU_2 \times U_1$ model. Here an additional symmetry breaking mechanism has to be invented to do the breaking of the weak group. Finally G can be taken in the form of a direct product of two or more simple groups. Our methods should be at least partially applicable in this context.

APPENDIX A: THE U_1 ALGORITHM

In this Appendix, we develop an algorithm to obtain all semisimple algebras $P \subset P_0$ which satisfy

$$C_{P_0}(U_1) = U_1 \times P \quad (A.1)$$

and

$$C_{P_0}(P) = U_1 \quad (A.2)$$

We first construct the solution to (A.1). Let us consider a simple root system of P_0

$$\{ \vec{\gamma}_i ; i = 1, \dots, \text{rank}(P_0) \}$$

and the generators of the Cartan subalgebra

$$\{ \vec{H} \} = \{ H^k ; k = 1, \dots, \text{rank}(P_0) \}$$

Then we have the standard commutation relations

$$[H^k, H^l] = 0 \quad (A.3a)$$

$$[\vec{H}, E_{\pm \gamma_i}] = \pm \vec{\gamma}_i E_{\pm \gamma_i} \quad (A.3b)$$

Without loss of generality, we may now choose the U_1 generator λ in the Cartan subalgebra of P_0 as

$$\lambda = \vec{j} \cdot \vec{H} \quad (A.4)$$

We choose $\vec{\lambda}$ such that it is orthogonal to all but a single simple root, that is

$$\vec{j} \cdot \vec{\gamma}_i = \delta_{in} \quad (A.5)$$

Then we have

$$\begin{aligned} [\lambda, E_{\pm \gamma_i}] &= \pm \vec{j} \cdot \vec{\gamma}_i E_{\pm \gamma_i} = \pm \delta_{in} E_{\pm \gamma_n} \\ [\vec{j}, \vec{H}] &= 0 \end{aligned} \quad (A.6)$$

Therefore, the centralizer of λ , $C_{P_0}(\lambda)$, is generated by

$$\left\{ \vec{H}, E_{\pm \alpha} ; \alpha = \sum_i K_i \gamma_i \quad \text{with} \quad K_n = 0 \right\} \quad (A.7)$$

The subalgebra spanned by (A.7) corresponds to $U_1 \times P$, where P is the subalgebra with a Dynkin diagram obtained from that of P_0 by removing the dot corresponding to the simple root γ_n .

We now show that (A.5) is a necessary condition for (A.1) to hold. Suppose $\vec{\lambda}$ is not orthogonal to all but a single simple root $\vec{\gamma}_n$, then there exists at least one other root $\vec{\gamma}_p$, such that

$$\vec{\lambda} \cdot \vec{\gamma}_p \neq 0 \quad \vec{\lambda} \cdot \vec{\gamma}_n = 1$$

By the same arguments as before, one obtains in this case that the centralizer $C_{P_0}(\lambda)$ contains at least two U_1 factors, so that P cannot be semisimple and hence (A.1) is not satisfied. So we learn from (A.5) that there are no more than rank (P_0) solutions to (A.1), often less because of symmetries of the Dynkin diagram itself.

Next we have to determine for which of the subalgebras P just obtained, the second condition (A.2) holds. The answer to this question is easily obtained from the extended Dynkin diagram of P_0 . One reads off that $C_{P_0}(P)$ is indeed U_1 , except if the removal of γ_n from the extended Dynkin diagram would detach the extra (largest negative) root from the diagram. In that case, we have $C_{P_0}(P) = SU_2$ and $P \times SU_2$ is a maximal regular subalgebra of P_0 . This then completes our algorithm for extracting subalgebras $P \subset P_0$ which do satisfy (A.1) and (A.2).

APPENDIX B: ON THE CLASSIFICATION OF ALL SU_3 SUBALGEBRAS OF THE
EXCEPTIONAL LIE ALGEBRAS

A systematic procedure for obtaining all SU_3 subalgebras of a simple algebra G is to enumerate all possible sequences of maximal subalgebras

$$G > G_1 > G_2 > \dots > SU_3$$

The maximal subalgebras of semisimple algebras are completely classified. The basic paper which is the culmination of the work of many outstanding mathematicians was written by Dynkin¹⁴⁾. For practical calculations, one needs the branching rules of representations of G_1 under G_{i+1} . These can be found in Refs. 15) and 16).

Our results for all SU_3 subalgebras of the algebras F_4 , E_6 , E_7 and E_8 are contained in Tables 1a, 1b, 1c and 1d respectively. These tables contain the following information:

- 1) The first two columns give the branching rules for the adjoint and fundamental representations (for E_8 these two coincide) under the different SU_3 embeddings.
- 2) In the third column, we list the smallest regular^{*)} subalgebras (up to conjugacy) in which the SU_3 in question is embedded.
- 3) The columns with headings C_1 , C_2 respectively contain the centralizers

$$SU_3 \rightarrow C_1 \longleftrightarrow C_2$$

in G . (For us, the only interesting cases are those when $C_2 = SU_3$.)

- 4) In the last column, we list all possible ways in which the SU_3 algebra can be reached via the maximal subalgebras of G .

To specify the inclusions, one has to make use of the other tables as well. Let us illustrate the necessary procedure by the following example in E_7 . From

*) A subalgebra $A \subset G$ is called regular if the root system of A forms a subset of the root system of G . A subalgebra $A \subset G$ is maximal if there is no proper subalgebra $B \subset G$ such that $A \subset B$.

Table 1c, we learn, for example, that the SU_3 given by number 7 has two inclusions: $B(1)$, $D(1,1)$. The letters B and D refer to the entries (rows) of Table 2 and specify the maximal subalgebras, M , in which the SU_3 is contained. So in the present example in Table 2 (in the column E_7), in rows B and D we find $M = SU_8$ and $M = SU_3 \times SU_6$ respectively. The way our SU_3 is embedded in the maximal subalgebras M is specified by the numbers between parentheses. For example, $B(1)$ means the first SU_3 embedding in SU_8 which is found in Table 3 to be the $8 \rightarrow 8$ embedding.

If M contains several factors, the situation can be more complicated. In the present case $D(1,1)$ means that we look for the first embedding in the first factor (this embedding is trivial here since the first factor is an SU_3 as well), and the first embedding in the second factor (here the $6 \rightarrow 6$ embedding in SU_6) and our SU_3 is now the diagonal subalgebra $SU_3^{(1+2)} = SU_3^{(1)} \times SU_3^{(2)}$. However, a diagonal subalgebra can usually be chosen in several different ways, which are generally not equivalent. For example, one finds the inclusion $D(1,1)$ also in row 9 of Table 1c. To be specific, a diagonal embedding of SU_3 in $SU_3 \times SU_3$ is given by

$$(3, 1) + (1, 3) \rightarrow 3 + 3$$

while the other one is given by

$$(3, 1) + (1, 3) \rightarrow 3 + \bar{3}$$

For completeness, we remark that, for example, $D(0,4)$ means that the SU_3 is entirely within the SU_6 factor.

As was mentioned earlier, Dynkin listed all SU_3 subalgebras in the exceptional subalgebras, so we have to argue why we reproduced his results here. Our main motivation was to compute the centralizers C_1 , C_2 which would have been rather hard to obtain from Dynkin's results. Since we also list the inclusions, we hope that our tables will be easier for practical calculations. In fact, we even discovered that in the E_8 case there is an SU_3 subalgebra missing from Dynkin's list (Table 1d, entry No. 9). Apart from this omission, we fully agree with his results.

Finally, a comment is in order as to why we have entries 10 and 12 separately. Clearly, their SU_3 content is identical and they also figure together in Dynkin's list. However, as is also clear from Table 1d, the SU_3 's are different,

that is, they are not conjugate since their C_2 's are different. Conjugacy here means that two embeddings are equivalent under inner or outer automorphisms. [Inner automorphisms of the algebra correspond to Weyl reflections, outer automorphisms to the discrete symmetry group of the Dynkin diagram (the factor group). Notice that the interchange of inequivalent spinors of orthogonal groups can lead to distinct embeddings. Thus $C(0,3)$ appears in both cases 1 and 6 of Table 1c because SO_{12} has distinct 32 and 32' spinors.]

One may think that two embeddings in G are equivalent if the branching rules for any representation of G are the same for both embeddings. (By Dynkin this property is called L-equivalence of two subalgebras.) As is shown in Ref. 14), this is indeed the case for all classical algebras, i.e., L-equivalence implies conjugacy. However, in the exceptional algebras this is not necessarily the case for which the SU_3 's mentioned above provide an example. To check that our two SU_3 's are indeed L-equivalent, we verified that they also have the same branching rule for the next smallest dimensional representation of E_8 , the 3875. This is sufficient to guarantee L-equivalence.

In fact, the branching rules are different under $SU_3 \times SU_2$. For the embedding No. 12, we have

$$248 = (8,1) + (1,3) + (8,7) + (10,5) + (1\bar{0},5) + (27,3)$$

whereas for the other embedding (No. 10), we find

$$248 = 4(8,1) + 2(8,2) + (1,3) + 2(10,2) + 2(1\bar{0},2) + (10,1) + (1\bar{0},1) + 3(27,1)$$

Indeed, in spite of the fact that their SU_3 content is identical, the embeddings are not equivalent because of a subtle conspiracy of the SU_3 and SU_2 quantum numbers. We feel that this interesting phenomenon should be better understood and it also shows how careful one must be when working with exceptional algebras.

APPENDIX C: LEMMA ON REDUCIBILITY

Lemma: If S is semisimple but not simple, and R and S have at least one simple factor $\tilde{N}$ in common, then either (i) the whole problem is reducible to a subgroup $\tilde{G} = C_G(\tilde{N})$ of G , or (ii) $C_{P_0}(R)$ contains U_1 as a separate factor and no unification occurs.

Proof: Let us define the following centralizers: $\tilde{G} = C_G(\tilde{N})$, $\tilde{P}_0 = C_{P_0}(\tilde{N})$, $\tilde{P} = C_P(\tilde{N})$, where $S = \tilde{N} \times \tilde{S}$ and $R = \tilde{N} \times \tilde{R}$ by assumption. It is clear that

$$C_G(S) = C_G(\tilde{N} \times \tilde{S}) = C_{\tilde{G}}(\tilde{S}) = K \quad (C.1a)$$

A similar argument yields that

$$C_G(R) = C_{\tilde{G}}(\tilde{R}) = H \quad (C.1b)$$

$$C_{P_0}(S) = C_{\tilde{P}_0}(\tilde{S}) = U_1 \quad (C.1c)$$

$$C_{P_0}(R) = C_{\tilde{P}_0}(\tilde{R}) \quad (C.1d)$$

This last identity implies that

$$C_H(K_0) = C_{\tilde{P}_0}(\tilde{R}) \quad (C.2)$$

which should be non-Abelian for unification to occur.

Furthermore, since $P \supseteq \tilde{N} \times \tilde{P}$, we have that

$$K = C_G(P) \subseteq C_G(\tilde{N} \times \tilde{P}) = C_{\tilde{G}}(\tilde{P}) \quad (C.3a)$$

On the other hand, because $\tilde{P} \supseteq \tilde{S}$,

$$C_{\tilde{G}}(\tilde{P}) \subseteq C_{\tilde{G}}(\tilde{S}) = K \quad (C.3b)$$

where we used (C.1a) in the last step. From (C.3), it follows that

$$C_{\tilde{G}}(\tilde{P}) = K \quad (C.4)$$

Applying the same argument to $P \subset P_0$, one obtains

$$C_{\tilde{P}_0}(\tilde{P}) = U_1 \quad (C.5)$$

Consequently, there are two possibilities for $\tilde{P}_0$:

$$i) \quad \tilde{P}_0 \supset \tilde{P} \times U_1.$$

Using (C.4), we get

$$C_{\tilde{G}}(\tilde{P}_0) \subset C_{\tilde{G}}(\tilde{P} \times U_1) = K \equiv K_0 \times U_1$$

On the other hand,

$$C_{\tilde{G}}(\tilde{P}_0) = C_{\tilde{G}}(\tilde{N} \times \tilde{P}_0) \supseteq C_{\tilde{G}}(\tilde{P}_0) \equiv K_0$$

so that indeed $C_{\tilde{G}}(\tilde{P}_0) = K_0$, and the whole bosonic sector can be entirely embedded in $\tilde{G}$.

$$ii) \quad \tilde{P}_0 = \tilde{P} \times U_1.$$

Now we find

$$C_{\tilde{P}_0}(\tilde{P}) = C_{\tilde{P}}(\tilde{P}) \times U_1$$

In view of (C.2), this result implies that no unification occurs.

ACKNOWLEDGEMENTS

We would like to thank L. Michel, J.-P. Derendinger and M. Surridge for interesting discussions, and S. Ferrara for reading the manuscript.

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Nr	52	26	Min. incl. reg. subalg.	C_1	C_2	Inclusions
1	$8 + 3(6) + 3(\bar{6}) + 8(1)$	$8 + 3(3) + 3(\bar{3})$	SU_3	SU_3	SU_3	$B(1,0), C(0,1)$
2	$8 + 6(3) + 6(\bar{3}) + 8(1)$	$3(3) + 3(\bar{3}) + 8(1)$	SU_3	SU_3	SU_3	$B(0,1), A(2), D(0,1)$
3	$15 + \bar{15} + 3 + \bar{3} + 8 + 8$	$8 + 6 + \bar{6} + 3 + \bar{3}$	$SU_3 \times SU_3$	0	F_4	$B(1,1)$
4	$10 + \bar{10} + 4(8)$	$3(8) + 2(1)$	$SO_8, SU_3 \times SU_3$	0	F_4	$A(1), B(1,1)$

Table 1a: SU_3 embeddings in F_4

Nr	78	27	Min. incl. reg. subalgebra	C_1	C_2	Inclusions
1	$2(8) + 3(6+\bar{6}) + 3(3+\bar{3}) + 8(1)$	$8 + 3(3+\bar{3}) + 1$	$SU_3 \times SU_3$	SU_3	$SU_3 \times SU_3$	$C(1,1,0); F(1); G(1); B(2); C(0,1,1); C(1,0,1)$
2	$8 + 9(3+\bar{3}) + 16(1)$	$3(3+\bar{3}) + 9(1)$	SU_3	$SU_3 \times SU_3$	SU_3	$C(1,0,0); C(0,1,0); C(0,0,1); B(4); A(2); G(2); H(0,1)$
3	$15 + \bar{15} + 3(8) + 6 + \bar{6} + 2(3+\bar{3})$	$8 + 6 + \bar{6} + 3 + \bar{3} + 1$	$SU_3 \times SU_3 \times SU_3$	0	E_6	$G(3); H(1,1); E(1); C(1,1,1)$
4	$10 + \bar{10} + 7(8) + 2(1)$	$3(8) + 3(1)$	$SU_3 \times SU_3 \times SU_3, SO_8$	$U_1 \times U_1$	$SO_8 \times U_1 \times U_1$	$G(4); C(1,1,1); A(1)$
5	$27 + 2(10+\bar{10}) + 8 + 3(1)$	$15 + 2(\bar{6})$	SU_6	SU_2	SU_6	$B(1)$
6	$8(8) + 14(1)$	$\bar{6} + 7(3)$	$SU_3 \times SU_3$	C_2	SU_3	$B(3); H(1,0); C(0,1,1); C(1,1,0); C(1,0,1)$
7	$35 + \bar{35} + 8$	27	E_6	0	E_6	$D(1)$

Table 1b: SU_3 embeddings in E_6

Nr	133	56	Min. incl. reg. subalg.	C_1	C_2	Inclusions
1	$4(8)+3(6)+3(6)+9(3)+9(3)+1(1)$	$2(8)+6(3)+6(3)+4(1)$	$SU_3 \times SU_3$	$SU_3 \times SU_2$	SU_3	$A(1,0), B(3), C(0,3), D(0,3), E(0,1), F(0,1), D(1,4)$
2	$8+15(3)+15(3)+35(1)$	$6(3)+6(3)+20(1)$	SU_3	SU_6	SU_3	$A(2,0), B(5), C(0,4), D(1,0), D(0,4), F(0,2), G(1,0)$
3	$15+15+5(8)+3(6)+3(6)+4(3)+4(3)+3(1)$	$2(8)+2(6)+2(6)+2(3)+2(3)+4(1)$	$SU_3 \times SU_3 \times SU_3$	SU_2	E_6	$A(3,0), D(1,2), D(1,3), F(0,3), G(1,1)$
4	$10+10+13(8)+9(1)$	$6(8)+8(1)$	$SO_8, SU_3 \times SU_3 \times SU_3$	$SU_2 \times SU_2 \times SU_2$	SO_8	$A(4,0), C(0,1), D(1,3), F(0,4)$
5	$27+15+15+2(10)+2(10)+2(6)+2(6)+8+4(1)$	$15+15+2(6)+2(6)+2(1)$	SU_6	$SU_2 \times U_1$	$SU_6 \times U_1$	$A(5,0), B(2)$
6	$8(8)+6+6+7(3)+7(3)+15(1)$	$6+6+7(3)+7(3)+2(1)$	$SU_3 \times SU_3$	$C_2 \times U_1$	$SU_3 \times U_1$	$A(6,0), B(4), D(0,2), D(1,4), G(0,1), C(0,3)$
7	$3(27)+10+10+4(8)$	$2(10)+2(10)+2(8)$	$SU_8, SU_6 \times SU_3$	0	E_7	$B(1), D(1,1)$
8	$27+3(15)+3(15)+8+8(1)$	$10+10+3(6+6)$	SU_6	SU_3	SU_6	$C(0,2), D(0,1)$
9	$27+2+2+4+15+15+2(8)+6+6$	$15+15+10+10+3+3$	$SU_6 \times SU_3$	0	E_7	$D(1,1)$
10	$35+35+2(27)+8+1$	$2(27)+2(1)$	E_6	U_1	E_6	$A(7,0)$
11	125+8	$28+28$	E_7	0	E_7	$H(1)$

Table 1c: SU_3 embeddings in E_7

Nr	248	Min. incl. reg. subalgebra	C_1	C_2	Inclusions
1	$8(8)+3(6+6)+21(3+3)+22(1)$	$SU_3 \times SU_3$	$C_2 \times SU_3$	SU_3	$A(5), B(1,1), C(0,1), C(0,6), C(1,2), D(0,1), D(0,6), E(7), E(8), F(0,1), F(1,2)$
2	$8+27(3+3)+78(1)$	SU_3	E_6	SU_3	$A(6), B(0,1), B(1,0), C(0,2), C(1,0), D(0,2), E(9), F(0,2), F(1,0)$
3	$9(8)+15+15+7(6+6)+8(3+3)+14(1)$	$SU_3 \times SU_3 \times SU_3$	C_2	F_4	$C(0,3), C(1,1), C(1,6), D(0,3), F(0,3), F(1,1), E(6)$
4	$25(8)+10+10+28(1)$	$SU_3 \times SU_3 \times SU_3, SO_8$	SO_8	SO_8	$A(3), C(0,4), C(1,6), D(0,4), E(5), F(0,4)$
5	$8(8)+3(15+15)+10+10+3(6+6)+6(3+3)+2(1)$	$SU_3 \times SU_3 \times SU_3 \times SU_3, SO_8 \times SU_3$	$U_1 \times U_1$	$SO_8 \times SU_3 \times U_1 \times U_1$	$A(2), C(1,3), C(1,4), F(1,3), F(1,4)$
6	$8+27+3(15+15)+2(10+10)+6(6+6)+11(1)$	SU_6	$SU_3 \times SU_2$	SU_6	$A(4), C(0,5), D(0,5), D(0,8), E(4)$
7	$8+35+35+6(27)+8(1)$	E_6	SU_3	E_6	$C(0,7), D(0,10)$
8	$2(8)+27+2+2+3(15+15)+2(10+10)+6+6+2(3+3)+3(1)$	$SU_6 \times SU_3$	SU_2	E_7	$C(1,5), D(0,9), E(2)$
9	$2(8)+4+2+4+35+35+2+2+15+15$	$SU_3 \times E_6$	0	E_8	$C(1,7)$
10	$8(8)+3(27)+5(10+10)+3(1)$	$SU_3 \times SU_6, SO_8 \times SO_8, SU_8$	SU_2	E_7	$A(1), C(1,5), D(0,7), E(3)$
11	$8+125+2(28+28)+3(1)$	E_7	SU_2	E_7	$D(0,11)$
12	$8(8)+3(27)+5(10+10)+3(1)$	$SU_8, SO_8 \times SO_8$	SU_2	SU_3	$E(1), G(0,1), A(1)$

Table 1d: SU_3 embeddings in E_8

	G_2	F_4	E_6	E_7	E_8
A	SU_3	SO_9	$SO_{10} \times U_1$	$E_6 \times U_1$	SO_{16}
B	$SU_2 \times SU_2$	$SU_3 \times SU_3$	$SU_2 \times SU_6$	SU_8	$SU_5 \times SU_5$
C	SU_2	$SU_2 \times Sp_6$	$SU_3 \times SU_3 \times SU_3$	$SU_2 \times SO_{12}$	$SU_3 \times E_6$
D		$SU_2 \times G_2$	SU_3	$SU_3 \times SU_6$	$SU_2 \times E_7$
E		SU_2	G_2	$SU_2 \times G_2$	SU_9
F			Sp_8	$SU_2 \times F_4$	$G_2 \times F_4$
G			F_4	$G_2 \times Sp_6$	$SU_2 \times SU_3$
H			$SU_3 \times G_2$	SU_3	Sp_4
I				SU_2	SU_2
J				SU_2	SU_2
K					SU_2

Table 2. Maximal subgroups of the exceptional groups

Nr	SO_8	SO_9	SO_{10}	SO_{12}	SO_{16}	SU_6	SU_8	SU_9
1	8	8 + 1	8 + 2(1)	8 + 4(1)	2(8)	6	8	8 + 1
2	$3 + \bar{3} + 2(1)$	$3 + \bar{3} + 3(1)$	$3 + \bar{3} + 4(1)$	$6 + \bar{6}$	$8 + 3 + \bar{3} + 2(1)$	$3 + \bar{3}$	$6 + 2(1)$	$6 + 3$
3				$2(3) + 2(\bar{3})$	$8 + 8(1)$	$3 + 3$	$3 + \bar{3} + 2(1)$	$\bar{6} + 3$
4				$3 + \bar{3} + 6(1)$	$6 + \bar{6} + 4(1)$	$3 + 3(1)$	$3 + 3 + 2(1)$	$6 + 3(1)$
5					$2(3) + 2(\bar{3}) + 4(1)$		$3 + 5(1)$	$3(3)$
6					$3 + \bar{3} + 10(1)$			$2(3) + \bar{3}$
7								$2(3) + 3(1)$
8								$3 + \bar{3} + 3(1)$
9								$3 + 6(1)$

Table 3: SU_3 branching rule of the vector representations of some classical groups.