Top-Quark Mass in Grand-Unified Theories with Intermediate Scales

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The relation $m_t/m_c \approx m_b/m_s$, which predicts a top-quark mass close to 40 GeV, is derived in the context of SO(10) (and E_6) grand-unified theories with suitable intermediate scales.

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There are indications from the work of Arnison *et al.* (UA1 experiment)¹ that the top (t) quark is found at a mass somewhere between 30 and 50 GeV. Given this information can one construct realistic models which predict a t -quark mass within the above range?²

The main purpose of this Letter is to address this question in the context of grand-unified theories (GUTs). We will show that it is possible to construct GUTs based on SO(10) and E_6 gauge groups in which the following relation holds to a good approximation:

$$m_t/m_c \approx m_b/m_s. \quad (1)$$

Here m_c , m_s , m_t , and m_b denote the masses of the quarks of the second and third generation. Equation (1) predicts a top-quark mass close to 40 GeV. For instance, using $m_s(1 \text{ GeV}) = 0.15 \text{ GeV}$, $m_c(2m_c) = 1.2 \text{ GeV}$, and $m_b(2m_b) = 4.5 \text{ GeV}$, one obtains $m_t(m_t) = 38 \text{ GeV}$. Uncertainties arise from the relatively large errors in the quoted values of the masses of the known quarks. The desirability of Eq. (1) in the context of the Kobayashi-Maskawa matrix has been discussed by Stech³ and Wolfenstein.³ Our derivation of (1) heavily relies on the existence of suitable intermediate mass scales. In addition to predicting (1), the approach we follow allows us to reproduce the asymptotic relation $m_b \approx m_\tau$, while avoiding wrong relations like $m_\mu = m_s$ and $m_d = m_e$. A specific SO(10) axion model exhibiting these and other interesting features is discussed in some detail.

In the SO(10) model⁴ the fermions (quarks and leptons) of each generation belong to the 16-dimensional spinor representation. The SO(10) symmetry implies that the scalar fields having Yukawa couplings belong to the **10** (vector), **126** (fifth-rank antisymmetric tensor), and **120** (third-rank antisymmetric tensor) representations. The Yukawa couplings of the **10** and the **126** are symmetric in the generation indices, while the **120** must have antisymmetric couplings. If only a single real **10** were present the masses of the charge

$+\frac{2}{3}$ and charge $-\frac{1}{3}$ quarks would be equal which, of course, is unacceptable. Thus, at least a complex **10** of Higgs bosons is needed. However, acceptable fermion mass matrices for the first two generations require, in addition, contributions from the **126** and **120** of Higgs bosons. In general, there are two independent Yukawa couplings for each complex Higgs boson in a real SO(10) representation, in which case a prediction of the top-quark mass is not possible. In order to remedy this situation and obtain a unique Yukawa coupling for each Higgs representation we impose an additional U(1) symmetry on the model which acts on the fermions [U(1) charge 1, say] and the scalar fields.

The Yukawa couplings involving the **10**, **126**, and the **120** can be schematically written as follows:

$$L_{\text{Yuk}} = f_{ij} \mathbf{16}_i \mathbf{16}_j \mathbf{10} + h_{ij} \mathbf{16}_i \mathbf{16}_j \mathbf{126}^* + k_{ij} \mathbf{16}_i \mathbf{16}_j \mathbf{120}. \quad (2)$$

Here i, j are generation indices, f and h are symmetric matrices, and k is an antisymmetric matrix. There are, in all, eight color-singlet SU(2)_L doublets with electrically neutral components in the **10**, **126**, and **120** representations. With respect to the Pati-Salam subgroup SU(4)_c $\otimes$ SU(2)_L $\otimes$ SU(2)_R⁵ of SO(10), they belong to (1, 2, 2) in **10** and **120**, and (15, 2, 2) in **126** and **120**. In an SU(2)_L $\otimes$ SU(2)_R matrix representation,

$$\begin{aligned} \phi_1 = \phi_{10}(1, 2, 2) &= \begin{pmatrix} \mathbf{10}^u & 0 \\ 0 & \mathbf{10}^d \end{pmatrix}, \\ \phi_2 = \phi_{126^*}(15, 2, 2) &= \begin{pmatrix} \mathbf{126}^u & 0 \\ 0 & \mathbf{126}^d \end{pmatrix}, \\ \phi_3 = \phi_{120}(1, 2, 2) &= \begin{pmatrix} \mathbf{120}_{1,2,2}^u & 0 \\ 0 & \mathbf{120}_{1,2,2}^d \end{pmatrix}, \\ \phi_4 = \phi_{120}(15, 2, 2) &= \begin{pmatrix} \mathbf{120}_{15,2,2}^u & 0 \\ 0 & \mathbf{120}_{15,2,2}^d \end{pmatrix}. \end{aligned} \quad (3)$$

Here the superscripts u (for up) and d (for down) refer to the scalar components that couple to the up-type and the down-type quarks (and leptons), respectively. The mass matrices M_U, M_D , and M_L for charge $\frac{2}{3}$ quarks, charge $-\frac{1}{3}$ quarks, and the charged leptons, respectively, are given at the GUT scale M_X by

$$\begin{aligned} M_U &= \frac{1}{2}\sqrt{2}f\langle\mathbf{10}\rangle^u + \frac{1}{6}\sqrt{6}h\langle\mathbf{126}\rangle^u + \frac{1}{2}\sqrt{2}k\langle\mathbf{120}\rangle_{1,2,2}^u + \frac{1}{6}k\langle\mathbf{120}\rangle_{15,2,2}^u, \\ M_D &= \frac{1}{2}\sqrt{2}f\langle\mathbf{10}\rangle^d + \frac{1}{6}\sqrt{6}h\langle\mathbf{126}\rangle^d + \frac{1}{2}\sqrt{2}k + \langle\mathbf{120}\rangle_{1,2,2}^d + \frac{1}{6}\sqrt{6}k\langle\mathbf{120}\rangle_{15,2,2}^d, \\ M_L &= \frac{1}{2}\sqrt{2}f\langle\mathbf{10}\rangle^d - \frac{1}{2}\sqrt{6}h\langle\mathbf{126}\rangle^d + \frac{1}{2}\sqrt{2}k\langle\mathbf{120}\rangle_{1,2,2}^d - \frac{1}{2}\sqrt{6}k\langle\mathbf{120}\rangle_{15,2,2}^d, \end{aligned} \quad (4)$$

where f , h , and k are the three matrices introduced in (2).

At first sight (4) seems to (and does!) contain so many parameters that any prediction about fermion masses appears unlikely. However, we will see that in some models the order of magnitude of certain vacuum expectation values (vev's) in (3) is controlled by ratios of symmetry-breaking scales.⁶

In the limit in which the contribution only of the Higgs $\mathbf{10}$ to the fermion mass matrices is considered one obtains at the GUT scale⁷ M_X

$$M_U^0 \propto M_D^0 \quad (5)$$

and

$$M_D^0 = M_L^0. \quad (6)$$

Relation (5) implies (among other things)

$$m_t^0/m_c^0 = m_b^0/m_s^0 \quad (7)$$

and relation (6) gives

$$m_b^0 = m_\tau^0, \quad (8)$$

$$m_s^0 = m_\mu^0, \quad (9)$$

$$m_d^0 = m_e^0. \quad (10)$$

Here the superscript zero is to remind us of the above approximation. Although relation (8) can be regarded as fairly successful, (9) and (10) clearly cannot. Hence, relation (7) is not yet very meaningful. We need to consider additional Higgs-boson ($\mathbf{126}$ and $\mathbf{120}$) contributions which should help cure the bad predictions involving fermions of the first and the second generation, and also generate nonzero mixing angles which are excluded by (5).

In order to see how the relation $m_\mu = m_s$ is modified while (8) is kept, we consider the $\mathbf{126}$ of Higgs fields. The primary vev of the $\mathbf{126}$ is in the $(\mathbf{10}^*, \mathbf{1}, \mathbf{3})$ direction under $SU(4)_c \otimes SU(2)_L \otimes SU(2)_R$. It breaks both the $B-L$ and the left-right symmetries in $SO(10)$, and gives the right-handed neutrino some suitable large mass. However, when the $SU(4)_c$ symmetry is broken, the weak doublets in the $(\mathbf{15}, \mathbf{2}, \mathbf{2})$ of $\mathbf{126}$ can mix with the $(\mathbf{1}, \mathbf{2}, \mathbf{2})$ of $\mathbf{10}$ and acquire an in-

duced vev.⁶ The induced vev would be of the same order of magnitude as the primary vev of $\mathbf{10}$, unless the $SU(4)_c$ symmetry is broken at some scale $M_c < M_X$, in which case it is suppressed by one or more powers of M_c/M_X relative to $\langle\mathbf{10}\rangle$. Clearly, if M_c is sufficiently below M_X , the induced vev can be employed to break the relation $m_\mu = m_s$. However, the relation $m_\tau = m_b$ is left essentially intact.

If we can arrange, preferably through symmetry considerations, that to a good approximation $\langle\mathbf{126}\rangle^u \propto \langle\mathbf{10}\rangle^u$ and $\langle\mathbf{126}\rangle^d \propto \langle\mathbf{10}\rangle^d$, so that $\langle\mathbf{126}\rangle^u/\langle\mathbf{126}\rangle^d = \langle\mathbf{10}\rangle^u/\langle\mathbf{10}\rangle^d$, then it would follow from (4) that the up- and down-quark mass matrices are proportional to each other as in (5), but (6) is violated. Moreover, we now obtain relation (1) which predicts a top-quark mass of about 40 GeV.

The mixing angles are still zero. However, the proportionality of $\langle\mathbf{126}\rangle^u$ to $\langle\mathbf{10}\rangle^u$ and $\langle\mathbf{126}\rangle^d$ to $\langle\mathbf{10}\rangle^d$ is only approximate. If there are subleading terms that break it, the up and the down mass matrices will no longer be proportional and mixing angles will be induced.

Mixing can also be introduced through a $\mathbf{120}$ Higgs boson. This field usually is not needed for any symmetry breaking in $SO(10)$. Nevertheless, we can assume that it is present. As for the $\mathbf{126}$, the four $SU(2)_L$ doublets in the $\mathbf{120}$ can acquire induced vev's. The $(\mathbf{1}, \mathbf{2}, \mathbf{2})$'s in $\mathbf{120}$ have in general induced vev's that are unsuppressed. If we can arrange that $\langle\mathbf{120}\rangle^u/\langle\mathbf{120}\rangle^d = \langle\mathbf{10}\rangle^u/\langle\mathbf{10}\rangle^d$, then all we have achieved so far will be maintained. In particular, relation (1) would still hold and the mixing angles vanish. As in the case of the $\mathbf{126}$, the $(\mathbf{15}, \mathbf{2}, \mathbf{2})$ doublets in $\mathbf{120}$ can acquire induced vev's suppressed by power(s) of M_c/M_X . Once again, if the leading components of the induced vev's $\langle\mathbf{120}\rangle^u$ and $\langle\mathbf{120}\rangle^d$ are proportional to $\langle\mathbf{10}\rangle^u$ and $\langle\mathbf{10}\rangle^d$, respectively, the relation (1) will hold. The subleading terms can break the proportionality between up- and down-quark mass matrices and lead to nonzero mixing angles.

To see how all this can be implemented, let us discuss a specific $SO(10)$ model to which we append a global Peccei-Quinn [$U(1)_{PQ}$] symmetry.⁸ There are two intermediate scales, M_c and $M_{PQ} \simeq M_R$, at which $SU(4)_c$ and $SU(2)_R \otimes U(1)_{B-L} \otimes U(1)_{PQ}$, respectively, are broken. The complex symmetry-breaking

pattern and the Higgs representations are as follows:

$$\begin{aligned}
 \text{SO}(10) \otimes \text{U}(1)_{\text{PQ}} &\xrightarrow[M_X]{210(1, 1, 1)_0} \text{SU}(4)_c \otimes \text{SU}(2)_L \otimes \text{SU}(2)_R \otimes \text{U}(1)_{\text{PQ}} \\
 &\xrightarrow[M_c]{45_c(15, 1, 1)_0} \text{SU}(3)_c \otimes \text{SU}(2)_L \otimes \text{SU}(2)_R \otimes \text{U}(1)_{B-L} \otimes \text{U}(1)_{\text{PQ}} \\
 &\xrightarrow[M_{\text{PQ}} \approx M_R]{45_{\text{PQ}}(15, 1, 1)_4, 126(10^*, 1, 3)_2} \text{SU}(3)_c \otimes \text{SU}(2)_L \otimes \text{U}(1) \xrightarrow[M_W]{10(1, 2, 2)_{-2}} \text{SU}(3)_c \otimes \text{U}(1)_{\text{em}}. \quad (11)
 \end{aligned}$$

Only the primary vev's of the Higgs fields are given in (11). The subscripts attached to the $\text{SU}(4)_c \otimes \text{SU}(2)_L \otimes \text{SU}(2)_R$ representations denote the $\text{U}(1)_{\text{PQ}}$ charges.

Standard renormalization-group arguments⁹ can be employed to constrain the mass scales M_X , M_c , and M_{PQ} . Let M_W denote the weak boson mass. Then, with $\alpha_s(M_W) \approx 0.1$ and $\sin^2 \theta_W(M_W) \approx -0.23$, one could have $M_X/M_c \approx 50$, $M_c/M_{\text{PQ}} \approx 40$, and $M_{\text{PQ}}/M_W \approx 10^{10}$. The proton lifetime is larger by about two orders of magnitude compared to the $\text{SU}(5)$ prediction. Also the energy density in axions in this example is close to the critical energy density of the Universe.¹⁰ We shall now see that the presence of the intermediate scales enhances the predictive power of the $\text{SO}(10)$ model as far as fermion masses are concerned.

We fine tune the parameters of the Higgs potential such that a suitable linear combination of $\langle 10 \rangle^u$ and $\langle 10 \rangle^d$ acquires vacuum expectation value of the order of the weak scale. Let us now examine how the doublets in 126 acquire a vev. Consider, for example, the Higgs coupling

$$210 \times 45_c \times 126 \times 10. \quad (12)$$

The $(15, 2, 2)$ component in the 126 has mass which is typically of order M_X . From this and (12) it follows that the induced vev's are

$$\langle 126 \rangle^u \sim (M_c/M_X) \langle 10 \rangle^u, \quad (13)$$

$$\langle 126 \rangle^d \sim (M_c/M_X) \langle 10 \rangle^d. \quad (14)$$

Thus, not only are the induced vev's suppressed relative to the $\langle 10 \rangle$, as expected, but also

$$\langle 126 \rangle^u / \langle 126 \rangle^d = \langle 10 \rangle^u / \langle 10 \rangle^d. \quad (15)$$

Note that for relation (15) to hold it is essential that both $\langle 210 \rangle$ and $\langle 45_c \rangle$ do not break $\text{SU}(2)_R \otimes \text{U}(1)_{\text{PQ}}$.

To see how corrections to (15) arise, consider the coupling

$$210 \times 45_{\text{PQ}} \times 126 \times 10. \quad (16)$$

In this case one finds

$$\langle 126 \rangle^u \sim (M_{\text{PQ}}/M_X) \langle 10 \rangle^d, \quad (17)$$

$$\langle 126 \rangle^d \sim (M_{\text{PQ}}/M_X) \langle 10 \rangle^u. \quad (18)$$

Note the necessity of introducing a new scale M_{PQ} below M_c in order that (15) hold to a good approximation. The subleading terms in (17) and (18) are suppressed by M_{PQ}/M_c relative to the leading terms in (13) and (14). One can verify that this also holds for the remaining Higgs couplings involving the 126 .

The above arguments can similarly be repeated for couplings involving the 120 of Higgs fields. For instance, from the existence of the Higgs coupling

$$210 \times 120 \times 10 \quad (19)$$

(here and below the dimensionful coupling is $\sim M_X$) one can deduce that the induced vev's of $(1, 2, 2)$ in 120 are unsuppressed relative to $\langle 10 \rangle$. Moreover, one finds that

$$\langle 120 \rangle_{1,2,2}^u / \langle 120 \rangle_{1,2,2}^d = \langle 10 \rangle^u / \langle 10 \rangle^d. \quad (20)$$

Similarly from the coupling

$$210 \times 45_c \times 120 \times 10^* \text{ (or } 45_c \times 120 \times 10^*), \quad (21)$$

one obtains for the induced vev's of $(15, 2, 2)$ in 120 ,

$$\langle 120 \rangle_{15,2,2}^u \sim (M_c/M_X) \langle 10 \rangle^u, \quad (22)$$

$$\langle 120 \rangle_{15,2,2}^d \sim (M_c/M_X) \langle 10 \rangle^d, \quad (23)$$

with

$$\langle 120 \rangle_{15,2,2}^u / \langle 120 \rangle_{15,2,2}^d = \langle 10 \rangle^u / \langle 10 \rangle^d. \quad (24)$$

The subleading terms in this case come from couplings like

$$45_{\text{PQ}} \times 120 \times 10, \quad (25)$$

which give

$$\langle 120 \rangle_{15,2,2}^u \sim (M_{\text{PQ}}/M_X) \langle 10 \rangle^d, \quad (26)$$

$$\langle 120 \rangle_{15,2,2}^d \sim (M_{\text{PQ}}/M_X) \langle 10 \rangle^u. \quad (27)$$

Combining (15), (20), (24), and (4), we conclude that to a good approximation, the up-quark mass matrix M_U is proportional to the down-quark mass matrix M_D . Thus, relation (1) holds in this approximation.

We have seen above that the proportionality between M_U and M_D is broken by subdominant effects such as those in (17), (18) and (26), (27). Let us de-

fine α, β, γ through

$$\alpha = \langle 10 \rangle^u / \langle 10 \rangle^d, \quad (28)$$

$$\beta = \langle 126 \rangle^u / \langle 126 \rangle^d, \quad (29)$$

$$\gamma = \langle 120 \rangle_{15,2,2}^u / \langle 120 \rangle_{15,2,2}^d. \quad (30)$$

Using the subdominant terms one can obtain acceptable mixing between generations if, for instance, $|\alpha - \beta| = 1$, $|\beta - \gamma| = 3$. With this choice of parameters the top-quark mass prediction remains essentially unchanged since $|\alpha - \beta| \ll \alpha, \beta$. Note that the most difficult mixing to arrange is the one between the second and third generations. With the hierarchy of scales chosen, a little calculation shows that one cannot get mixing angles between the second and third generations larger than about 0.05, which may be regarded as a prediction of the model. However, mixing between the first and second generations could easily be large. The correct Cabibbo angle can, therefore, be obtained, although not predicted.

In the limit $M_{PQ} \rightarrow M_R \rightarrow 0$, in which case $\alpha = \beta = \gamma$, the mixing between the generations tends to zero and $m_u/m_d = m_t/m_b$, which is clearly wrong. This is not true, however, when mixing is generated. As a result of the smallness of the masses of the first generation and the large mixing with the second, the relation $m_u > m_d$ can easily be broken and the correct m_u/m_d ratio be obtained. That this is possible may be seen by consideration of the limiting case in which the first-generation Yukawa couplings vanish. One then finds $m_u = 0$ and $m_d = \sin^2 \theta_C m_s$ (θ_C being the Cabibbo angle). Clearly, appropriately small first-generation Yukawa couplings yield the correct m_u/m_d ratio.

Finally, we wish to make some remarks about the extension of our discussion so far to the E_6 model.¹¹ The fermions of each generation belong to the 27-dimensional complex representation of E_6 . Under the $SO(10) \otimes U(1)$ subgroup of E_6 ,

$$27 = 16_1 \oplus 10_{-2} \oplus 1_4. \quad (31)$$

The $U(1)$ charges are the subscripts attached to the various $SO(10)$ representations. For our purposes the important point that follows from (31) is that this $U(1)$ subgroup of E_6 now plays the role that $U(1)_{PQ}$ played in our previous discussion. Thus, we could imagine breaking the E_6 symmetry down to $SO(10)$

$\otimes U(1)$ at some superheavy scale M_s . We are then left with an effective "low"-energy ($\ll M_s$) $SO(10) \otimes U(1)$ theory. Proceeding as before, we end up once again with relation (1).

To conclude, we have argued that the presence of certain symmetry-breaking scales in $SO(10)$ (and E_6) grand-unified theories enables one to derive a prediction for the top-quark mass [relation (1)] which can be confronted with the ongoing UA1 experiment.

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