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HIDDEN SYMMETRIES: NEW PERSPECTIVES FOR $d = 11$ SUPERGRAVITY^{*)}

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ABSTRACT

We review the recent construction of a new version of $d = 11$ supergravity with local $SU(8)$ invariance.

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1. INTRODUCTION

Eleven appears to be the maximum number of dimensions in which a consistent supergravity theory exists¹. For this reason, simple ($N = 1$) supergravity in eleven dimensions² is a possible candidate for the ultimate unification of fundamental interactions. Owing to the presence of a three-index gauge field in this theory, compactification to four dimensions occurs naturally³ and leads to effective $d = 4$ field theories whose low energy gauge group coincides with the isometry group of the seven-dimensional "internal" manifold M_7 by the standard Kaluza-Klein mechanism. However, extensive work over the past few years (see, for example, Ref. 4) has led to the realization that it is very difficult, if not impossible, to construct a model within this framework that correctly describes present-day physics, despite the existence of a large variety of possible ground state solutions with interesting properties. The two principal obstacles are the huge cosmological constant that is invariably obtained for the classical ground states and the no-go theorem of Ref. 5, according to which the fermions of the effective $d = 4$ low energy theory are always in vector-like representations of the isometry group (unless one invokes more exotic possibilities). Similar difficulties are encountered in the context of extended ($N > 2$) supergravities in four dimensions. There as well, fermions occur only in vector-like representations independently of whether the theory is in a symmetric or spontaneously broken phase.

A possible way out of this impasse is offered by the existence of "hidden symmetries" (and perhaps further hidden structures) of $d = 11$ supergravity, as well as possible extensions of this theory. One of the most unexpected and puzzling features of $N = 8$ supergravity in four dimensions is the appearance of a local chiral $SU(8)$ invariance of the $N = 8$ Lagrangian and a global non-compact $E_{7(+7)}$ invariance of the equations of motion⁶. The local $SU(8)$ invariance persists when local $SO(8)$ gauge interactions are introduced which,

however, break E_7 ⁷. These hidden symmetries were discovered "experimentally" when the reduction of $d = 11$ supergravity on the seven-torus was done in Ref. 6, and have so far not been understood in the framework of higher dimensions. In particular, it has not been clear whether their appearance reflects as yet undiscovered properties of the $d = 11$ theory or is just an artefact of the truncation which disappears when other compactifications are considered and/or the massive modes are retained. The answer to these questions is clearly of tantamount importance for the future destiny of the theory.

It is the main purpose of this article to review the progress that has recently been made towards a better understanding of these issues. We have recently shown that the $d = 11$ theory itself possesses hidden symmetries⁸; it is possible to rewrite all the transformation laws and the field equations in a form that is manifestly covariant under local $SU(8)$ in eleven dimensions. The fermions transform as representations of this $SU(8)$ which are chiral in the four-dimensional sense, notwithstanding the fact that the $SU(8)$ group is local in eleven dimensions. Moreover, the bosonic quantities that pertain to the spinless fields, which include the $SU(8)$ connections, constitute representations of the group $E_{7(+7)}$. Our construction is based on $d = 11$ supergravity rewritten in a certain way as a four-dimensional theory with fields that depend on seven extra co-ordinates. This theory is still equivalent to the full eleven-dimensional one, and there exists a natural reformulation of our results within the context of any non-trivial ground state solution.

To set the stage, we briefly recall that the $d = 11$ Lagrangian of Ref. 2 describes the interactions of an elfbein E_M^A (M and $A = 1, \dots, 11$ are curved and flat indices respectively), a 32-component Majorana vector spinor Ψ_M and a "three-index photon" A_{MNP} , which is subject to an Abelian gauge invariance $\delta A_{MNP} = \partial [M^A{}_{NP}]$ and enters the equations of motion only through its invariant field strength $F_{MNPQ} = 4! \partial [M^A{}_{NPQ}]$ (our conventions are those of Ref. 9 and are also summarized in Refs. 8 and 10). All these fields depend on eleven co-ordinates z^M . When the reduction on T^7 is performed⁶, one splits these into $d = 4$ and $d = 7$ co-ordinates x^μ and y^m respectively, and simply drops the dependence on the "internal" co-ordinates y^m ; fields of different spin in four dimensions are then obtained by picking suitable values of the indices. Thus, the spin-zero fields originate from three sources, namely 28 states from the metric g_{mn} , seven from $A_{\mu\nu m}$ (after a duality transformation) and 35 from A_{mnp} . It was one of the major achievements of Ref. 6 to show that the symmetry could not only be enlarged from the manifest $SO(7)$ to an expected $SO(8)$, but also to $SU(8)$, and that the 70 scalar and pseudoscalar states are described by the

coset space $E_{7(+7)}/SU(8)$. Naturally, the 70 scalars are transformed into each other by the action of $SU(8)$, but there is no symmetry in the original theory which mixes g_{mn} , $A_{\mu\nu m}$ and A_{mnp} into each other: hence the name "hidden symmetries". It is clear that these cannot be directly understood from the formulation given in Ref. 2.

Our strategy has therefore been to construct a new version of $d = 11$ supergravity in which these symmetries are no longer hidden, but manifest. The construction evolved from an attempt to understand the relation between gauged $N = 8$ supergravity and the $d = 11$ supergravity at the non-linear level, and several relevant steps have already been outlined in Ref. 10. The basic idea is to first restrict the tangent space group $SO(1,10)$ of $d = 11$ supergravity to $SO(1,3) \times SO(7)$ by a partial gauge choice and then to enlarge it to $SO(1,3) \times SU(8)$ by the introduction of new gauge degrees of freedom. In contrast to the construction in Ref. 6, which followed a similar pattern, all physical degrees of freedom of the $d = 11$ theory are retained here. It is furthermore necessary to redefine the fields of $d = 11$ supergravity according to the "standard" prescription^{6,10}. Although these redefinitions may look like technicalities somewhat beside the main point, they do play an essential role, and therefore we briefly summarize them here. One first uses the local $SO(1,10)$ invariance of the theory to fix a gauge where the elfbein assumes the form

$$E_M^A = \begin{bmatrix} \Delta^{-1/2} e_\mu^\alpha & B_\mu{}^n e_n{}^a \\ 0 & e_m{}^a \end{bmatrix} \quad (1.1)$$

The tangent space group is reduced to $SO(1,3) \times SO(7)$ in this way. Compensating rotations are needed in the supersymmetry variations and co-ordinate reparametrizations in order to maintain the gauge choice (1.1). Moreover, we have already included a Weyl-rescaling factor

$$\Delta(x, y) \equiv \det e_m{}^a(x, y) \quad (1.2)$$

in (1.1), which is just the factor needed for the canonical normalization of the $d = 4$ Einstein action. It is also possible to perform the Weyl rescaling with respect to a non-trivial background by replacing the full siebenbein in

(1.2) by the deviation S_a^b from the background $\hat{e}_m^a$, i.e.,

$$S_a^b(x, y) = \hat{e}_a^m(y) e_m^b(x, y) \quad (1.3)$$

The fermionic fields have to be redefined in an analogous manner. It is convenient to use fields with $d = 11$ flat indices, in terms of which the redefined fields are given by

$$\begin{aligned} \psi_\mu &= e_\mu^\alpha \Delta^{-1/4} e^{-\frac{i}{4}\pi\gamma^5} (\psi_\alpha - \gamma^5 \gamma_\alpha \Gamma^a \psi_a) \\ \psi_a &= \Delta^{-1/4} e^{-\frac{i}{4}\pi\gamma^5} \psi_a \end{aligned} \quad (1.4)$$

The redefinitions (1.1)-(1.4) are necessary before one can introduce the new $SU(8)$ gauge degrees of freedom. This procedure is described in Section 2. One of the main advantages of the new formulation of $d = 11$ supergravity is that the original problem of establishing the consistency of the truncation to the "massless" sector in principle can now be solved completely, whereas it looked prohibitively difficult in the old formulation. Some aspects of this issue are dealt with in Section 3.

It should be noted that there exist earlier attempts to understand the origin of hidden symmetries. The reductions of $d = 11$ supergravity to dimensions other than four were studied in Ref. 11, and this investigation has revealed further remarkable properties of this theory: for instance, the reduction to d space-time dimensions gives rise to the whole series of exceptional groups E_{11-d} . We will briefly comment on the possible relation between this observation and our present construction in Section 4. Finally, the possible usefulness of Cartan-Maurer equations in the present context was stressed in Ref. 12.

2. COMPLEXIFICATION

To realize $SU(8)$ in $d = 11$ supergravity, one must evidently complexify all fields of the theory. This involves eliminating quantities like the siebenbein e_m^a which cannot possibly transform under $SU(8)$ and replacing them by

others on which SU(8) can be realized. For the fermions, the complexification is accomplished by noting that chiral SU(8) can be realized on the eight-dimensional spinor representation of SO(7) through the matrices Γ_{mn} , $\Gamma_{m8} \equiv i\Gamma_m$ and $\gamma^5 \Gamma_{mnp}$ ⁶. The various expressions can be further simplified by the use of chiral notation. We employ the letters A,B,C,... to denote spin-7 indices which are then promoted to chiral SU(8) indices. For the gravitino field ψ_μ , these are introduced in such a manner that

$$\gamma^5 \psi_\mu^A = \psi_\mu^A, \quad \gamma^5 \psi_{\mu A} = -\psi_{\mu A} \quad (2.1)$$

For the redefined spin- $\frac{1}{2}$ fields, one first eliminates the $d = 7$ vector index by switching to the combination $\Gamma_{[AB}^a \psi_{aC]}$ ⁶ and then defines¹⁰

$$\chi^{ABC} \equiv \frac{3\sqrt{2}}{4} i (1 + \gamma^5) \Gamma_{[AB}^a \psi_{aC]} \quad (2.2)$$

$$\chi_{ABC} \equiv \frac{3\sqrt{2}}{4} i (1 - \gamma^5) \Gamma_{[AB}^a \psi_{aC]}$$

The fermion fields ψ_μ^A and χ^{ABC} thus transform as the 8- and 56-dimensional representations of chiral SU(8), respectively.

To identify the proper SU(8)-covariant bosonic quantities is a more difficult task. The analysis of Ref. 10 suggests that the siebenbein must be replaced by the antisymmetric tensor⁸

$$e^m_{AB} \equiv i \Delta^{-1/2} e_a^m \Gamma_{AB}^a \quad (2.3)$$

which is, however, not SU(8) covariant. We now redefine the fields ψ_μ^A and χ^{ABC} and the supersymmetry parameters ϵ^A by means of a local (x- and y-dependent) SU(8) transformation Φ^A_B ; the degrees of freedom contained in Φ can then be used to promote (2.3) to a proper SU(8) tensor, viz.

$$e^m_{AB} = i \Delta^{-1/2} e_a^m \Gamma_{CD}^a \bar{\Phi}^C_A \bar{\Phi}^D_B \quad (2.4)$$

In order to avoid the introduction of new degrees of freedom, we let Φ be subject to a local (x- and y-dependent) SU(8) group, according to

$$\bar{\Phi}^A_B \rightarrow \bar{\Phi}^A_C U^C_B \quad (2.5)$$

so that by going to a special gauge ($\Phi = 1$) we recover (2.3). After extracting Φ from the fermion fields and the supersymmetry parameter, these quantities and (2.4) will transform covariantly under the local SU(8) induced by (2.5) according to their index structure [note that the complex conjugate of (2.4) has upper indices, i.e., $e^{mAB} \equiv (e^m_{AB})^*$]. The remaining bosonic quantities are not so readily identified. It requires a careful analysis of the transformation laws and a bit of inspiration to find them. They are⁸

$$\begin{aligned} B_{\mu A}^B \equiv & \bar{\Phi}^C_A \left\{ \frac{1}{2} \Omega_{\mu ab} \Gamma_{CD}^{ab} + \frac{\sqrt{2}}{12} \Delta^{-1/2} e_{\mu}^{\alpha} F_{\alpha abc} \Gamma_{CD}^{abc} \right. \\ & \left. - \frac{\sqrt{2}}{12} \Delta^{-1/2} e_{\mu}^{\delta} \varepsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma a} \Gamma_{CD}^a - 2 \delta_{CD} \partial_{\mu} \right\} \bar{\Phi}_D^B \end{aligned} \quad (2.6)$$

$$\begin{aligned} \mathcal{H}_{\mu}^{ABCD} \equiv & \left\{ \Omega_{\mu ab} \Gamma_{EF}^a \Gamma_{GH}^b - \right. \\ & - \frac{\sqrt{2}}{36} \Delta^{-1/2} e_{\mu}^{\delta} \varepsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma a} \Gamma_{EF}^b \Gamma_{GH}^{ba} + \\ & \left. + \frac{\sqrt{2}}{6} \Delta^{-1/2} e_{\mu}^{\alpha} F_{\alpha abc} \Gamma_{EF}^a \Gamma_{GH}^{bc} \right\} \cdot \\ & \cdot \bar{\Phi}_E^A \bar{\Phi}_F^B \bar{\Phi}_G^C \bar{\Phi}_H^D \end{aligned} \quad (2.7)$$

$$\begin{aligned} B_{mA}{}^B &\equiv \Phi^C{}_A \left\{ \frac{\sqrt{2}}{14} i f e_{ma} \Gamma_{CD}^a + \frac{1}{2} e_a{}^n \partial_m e_{nb} \Gamma_{CD}^{ab} \right. \\ &\quad \left. + \frac{\sqrt{2}}{48} e_m{}^d F_{mabc} \Gamma_{CD}^{abc} - 2 \delta_{CD} \partial_m \right\} \Phi_D{}^B \end{aligned} \quad (2.8)$$

$$\begin{aligned} \mathcal{A}_m{}^{ABCD} &\equiv \left\{ e_a{}^n \partial_m e_{nb} \Gamma_{EF}^a \Gamma_{GH}^b + \right. \\ &\quad \left. + \frac{\sqrt{2}}{42} i f e_{ma} \Gamma_{EF}^b \Gamma_{GH}^{ba} + \right. \\ &\quad \left. + \frac{\sqrt{2}}{24} e_m{}^d F_{dabc} \Gamma_{EF}^a \Gamma_{GH}^{bc} \right\} \Phi_E^A \Phi_F^B \Phi_G^C \Phi_H^D \end{aligned} \quad (2.9)$$

$$\begin{aligned} \mathcal{G}_{\alpha\beta AB}^+ &\equiv \left\{ \left(-\frac{i}{16} \Delta^{1/2} \Omega_{\alpha\beta a} + \frac{i}{8} \Delta^{-1/2} e_a{}^m e_{[\alpha}{}^n \partial_m e_{\beta]n} \right) \Gamma_{CD}^a \right. \\ &\quad \left. + \frac{\sqrt{2}}{32} i \Delta^{-1/2} F_{\alpha\beta ab} \Gamma_{CD}^{ab} \right\}_+ \Phi^C{}_A \Phi^D{}_B \end{aligned} \quad (2.10)$$

where F is the four-index field strength with $d = 11$ tangent-space indices, and $\mathcal{G}_{\alpha\beta AB}^+$ is self-dual in indices $[\alpha\beta]$ [the antiself-dual tensor is $\mathcal{G}_{\alpha\beta}^{-AB} \equiv (\mathcal{G}_{\alpha\beta AB}^+)^*$]. Furthermore

$$f(x, y) \equiv -\frac{i}{24} \varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta}(x, y) \quad (2.11)$$

$$\mathcal{D}_\mu \equiv \partial_\mu - B_\mu{}^m \partial_m \quad (2.12)$$

and the relevant coefficients of anholonomy are given by

$$\Omega_{\mu ab} \equiv e_a^m \partial_\mu e_{mb} - e_a^m \partial_m B_\mu{}^n e_{nb} \quad (2.13)$$

$$\Omega_{\alpha\beta a} \equiv 2 e_{[\alpha}{}^\mu e_{\beta]}{}^\nu \partial_\mu B_\nu{}^m e_{ma} \quad (2.14)$$

The transformation (2.5) now induces corresponding SU(8) transformations on the quantities (2.6)-(2.10). $\mathcal{B}_{\mu A}{}^B$ and $\mathcal{B}_{mA}{}^B$ transform as gauge fields associated with x- and y-dependent SU(8) transformations. The complex tensors $\mathcal{F}_\mu$ and $\mathcal{F}_m$ are self-dual in the indices [ABCD], i.e.,

$$\mathcal{F}_M{}^{ABCD} = \frac{1}{24} \epsilon^{ABCDEFGH} \mathcal{F}_M{}_{EFGH} \quad (M = \mu, m) \quad (2.15)$$

The tensors $\mathcal{G}_{\alpha\beta}{}^{AB}$ and $\mathcal{G}_{\alpha\beta}{}^{-AB}$ are antisymmetric in [AB] and transform in the 28- and $\overline{28}$ -representation of SU(8).

Evidently, the introduction of complex quantities forces us to give up the framework of Riemannian geometry. It is remarkable that $\mathcal{B}_\mu$, $\mathcal{F}_\mu$ and $\mathcal{B}_m$, $\mathcal{F}_m$ take the form of a gauge connection of the exceptional group E_7 in eleven dimensions. Moreover, the quantities $(e_m{}^{AB}, e_{mAB})$ can be assigned to the fundamental 56-dimensional representation of E_7 . Thus, they look like a generalized vielbein on a seven-dimensional base-manifold with tangent space group E_7 . The relations which generalize the usual vielbein postulate of Riemannian geometry read

$$\begin{aligned} \partial_\mu e_m{}^{AB} + \partial_n B_\mu{}^m e_n{}^{AB} + \frac{1}{2} \partial_n B_\mu{}^n e_m{}^{AB} + \\ + \mathcal{B}_\mu{}^C [A e_m{}^B]_C - \frac{3}{4} \mathcal{F}_\mu{}_{ABCD} e_m{}^{CD} = 0 \end{aligned} \quad (2.16)$$

$$\partial_m e_n{}^{AB} + \mathcal{B}_m{}^C [A e_n{}^B]_C - \frac{3}{4} \mathcal{F}_m{}_{ABCD} e_n{}^{CD} = 0 \quad (2.17)$$

One can now show that the supersymmetry variations and the equations of motion can be cast into a manifestly $SU(8)$ covariant form; the relevant expressions have been given in Ref. 8 and will not be repeated here. However, it should be stressed that the rather tedious calculations necessary to establish these results provide a non-trivial consistency check on the ideas described here, since it is not immediately evident that the expressions obtained from $d = 11$ supergravity after all the necessary redefinitions (1.1)-(1.4) and (2.1), (2.2) can be reassembled into manifestly $SU(8)$ -invariant or $SU(8)$ -covariant ones by means of no other than the quantities introduced above.

There are several noteworthy and unusual features in our construction. First of all, the $SO(7)$ part in (2.8) is not the usual $SO(7)$ spin connection $\omega_{mab} \Gamma^{ab}$ as one might have naively expected. In ordinary Riemannian geometry, the vielbein is constant with respect to a covariant derivative that contains the spin connection and the Christoffel symbol. This relation is contained in (2.17), but the two gauge connections are here broken up and their "constituents" (which are functions of e_a^m and its first derivative) are then distributed between the new quantities $\mathcal{B}_m$ and $\mathcal{A}_m$. Thus, the relation (2.17) is still covariant under general co-ordinate transformations in the seven internal dimensions. However, this covariance is no longer manifest, and one may ask how general co-ordinate transformations are realized in terms of the new quantities. It is here that E_7 seems to play an important role. Obviously, the quantity $e_a^n \partial_m e_{nb}$ appearing in (2.8) and (2.9) is not a vector under general co-ordinate transformations, but when one works out these transformations on $\mathcal{B}_m$ and $\mathcal{A}_m$, it turns out that, curiously, they can be written in the form

$$\begin{aligned} \delta_{g.c.t.} \mathcal{B}_m &= - \partial_m \xi^n \mathcal{B}_n - \xi^n \partial_n \mathcal{B}_m + \\ &\quad + \delta_{E_7} \mathcal{B}_m \\ \delta_{g.c.t.} \mathcal{A}_m &= - \partial_m \xi^n \mathcal{A}_n - \xi^n \partial_n \mathcal{A}_m \\ &\quad + \delta_{E_7} \mathcal{A}_m \end{aligned} \quad (2.18)$$

where ξ^m is an infinitesimal co-ordinate transformation and δ_{E_7} represents a variation by which $\mathcal{B}_m$ and $\mathcal{A}_m$ transform like gauge connections of a local E_7 group. Another important aspect is that (2.17) cannot be solved for $\mathcal{B}_m$ and

$\mathcal{F}_m$ in terms of e^m_{AB} and its complex conjugate unlike in ordinary Riemannian geometry, because the equations now contain additional degrees of freedom coming from the field strengths F_{MNPQ} . It must be emphasized that, at this point, it is by no means guaranteed that the quantities $\mathcal{B}_\mu$ and $\mathcal{B}_m$ really constitute SU(8) connections because the field strengths F_{MNPQ} which appear in the definitions (2.6) and (2.8) are related to each other by the Bianchi identities

$$\partial_{[M} F_{NPQR]} = 0 \quad (2.19)$$

For instance, the $\Gamma^{(3)}$ component of $\mathcal{B}_m$ in (2.8) contains the field strength F_{mabc} , which is antisymmetric in all four indices and further constrained by (2.19), whereas a bona fide gauge connection should not satisfy any constraint of this type. Thus, an inconsistency will inevitably arise unless one can cast these constraints into a manifestly SU(8)-covariant (or invariant) form. Fortunately, this is possible in our case, as we will now illustrate by giving an example (for a complete discussion, the reader is referred to Ref. 8). For this purpose, we consider a particular component of the SU(8) field strength in eleven dimensions, namely

$$\begin{aligned} \mathcal{F}_{\mu m} A^B &\equiv \partial_\mu \mathcal{B}_m A^B - \partial_m \mathcal{B}_\mu A^B - \\ &- \partial_m \mathcal{B}_\mu{}^n \mathcal{B}_n A^B + \frac{1}{2} [\mathcal{B}_\mu, \mathcal{B}_m] A^B \end{aligned} \quad (2.20)$$

Here the third term is somewhat unusual, but necessary for the field strength to be SU(8)-covariant, since

$$[\partial_\mu, \partial_m] = \partial_m \mathcal{B}_\mu{}^n \partial_n \quad (2.21)$$

as follows from the definition (2.12). The relevant identity reads ^{*)}

$$\begin{aligned} e^m C(A \{ \mathcal{F}_{\mu m}{}^B{}_C + \frac{3}{8} \mathcal{A}_\mu{}^{B)DEF} \mathcal{A}_m CDEF - \\ - \frac{3}{8} \mathcal{A}_m{}^{B)DEF} \mathcal{A}_\mu CDEF \} = 0 \end{aligned} \quad (2.22)$$

*) It is again curious that the expression in brackets may be viewed as the $[\mu m]$ component of an E_7 field strength in eleven dimensions.

which is manifestly SU(8)-covariant. To prove (2.22), one goes to the special gauge $\Phi = 1$ in (2.6)-(2.9) and evaluates the $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$ components of the expression in brackets in (2.22) separately. In this gauge, the $\Gamma^{(2)}$ component turns out to vanish. The $\Gamma^{(3)}$ component becomes proportional to

$$\left\{ \partial_\mu F_{mnpq} - 4 \partial_m F_{\mu npq} - B_\mu{}^r \partial_r F_{mnpq} + \right. \\ \left. + 4 B_\mu{}^r \partial_m F_{rnpq} \right\} \Gamma_{BC}^{npq} \quad (2.23)$$

Multiplying by e^{mCA} in the gauge (2.3) and symmetrizing in the indices (AB), one checks that the resulting expression vanishes by the Bianchi identities $\partial_{[m} F_{npqr]} = \partial_{[\mu} F_{npqr]} = 0$. Finally, the remaining contribution to (2.22) is found to be proportional to (of course, $E \equiv \det E_M^A$)

$$E^{-1} \partial_\mu (E F^{\nu\rho\sigma\tau}) + E^{-1} \partial_m (E F^{\nu\rho\sigma m}) + \\ + \frac{\sqrt{2}}{144} i \varepsilon^{\nu\rho\sigma\tau} \varepsilon^{mnpqrst} F_{\tau mnp} F_{qrst} \quad (2.24)$$

which vanishes by the equations of motion. Remarkably, the SU(8) tensor (2.22) contains both Bianchi identities and field equations as its components, and these are rotated into each other by the action of SU(8). It is thus no longer possible to distinguish between constraints on the one hand and equations of motion on the other, and the equivalence of the new version of $d = 11$ supergravity with the original one only holds for the equations of motion, whereas the two versions have a different off-shell structure.

Needless to say, this may have important consequences for a possible off-shell formulation of the theory and the construction of invariant higher order counterterms.

3. CONSISTENCY OF THE S^7 TRUNCATION

As a by-product of our results, the consistency to all orders of the truncation of supergravity compactified on S^7 ¹³ to its massless sector^{13,14} is now almost manifest. The resulting theory is generally believed to coincide with gauged $N = 8$ supergravity, but so far this claim has only been partially verified^{13-19,10}. Obviously, the S^7 truncation is more complicated than the

^T one of Ref. 6 because the zero modes, which constitute the massless $N = 8$ multiplet, now acquire a non-trivial dependence on the internal co-ordinates y^m . It was already pointed out in Ref. 16 that a straightforward substitution of the massless ansätze into the transformation rules leads to an apparent inconsistency due to a mismatch in the y -dependence. However, a consistent truncation to the massless sector is only possible if the y -dependence precisely matches in the transformation laws, since otherwise the massive modes, which have been discarded in the truncation, reappear through the supersymmetry transformations. The arguments of Ref. 10 show that the only possible redefinition by which one may achieve consistency of the transformation rules to all orders must take the form of a field-dependent $SU(8)$ rotation, which acts uniformly on all fermionic fields in accordance with the results of the preceding section (the necessary redefinition of the supersymmetry transformation parameter required to achieve consistency to the lowest non-trivial order had already been worked out in Ref. 17). We note that consistency of the supersymmetry transformations encompasses other notions of consistency, such as requiring consistency of the truncated field equations, and is therefore the strongest requirement.

There is no room here to deal with all aspects of this problem, and we will therefore concentrate our discussion on the particular form taken by the generalized vielbein (2.3) in the S^7 truncation. The scalar sector of $N = 8$ supergravity is characterized by the "sechsfundfünfzigbein"⁶ (our notation and conventions are those of Ref. 7)

$$U(x) = \begin{bmatrix} u_{ij}{}^{IJ}(x) & v_{ij}{}^{IJ}(x) \\ v^{ij}{}_{IJ}(x) & u^{ij}{}_{IJ}(x) \end{bmatrix} \quad (3.1)$$

which is an element of the coset space $E_7/SU(8)$. u and v are 28×28 matrices and the indices $[ij]$ transform under local $SU(8)$. The crucial quantities for the S^7 compactification are the (y -dependent) Killing spinors $\eta^1_A(y)$, since all the massless ansätze can be expressed through them^{13,14}. Of special importance are the 28 Killing vectors on S^7 which can be represented in the form¹³

$$K^{\circ m IJ} = i \bar{e}_a^m(y) \bar{\gamma}^I(y) \Gamma^a \gamma^J(y) \quad (3.2)$$

where $\bar{e}_a^m$ is the background siebenbein on the round S^7 . In our case, the Killing spinors also serve to turn "flat" $SU(8)$ indices into "curved" ones; for instance, one defines

$$u_{AB}^{IJ}(x,y) \equiv \gamma_A^i(y) \gamma_B^j(y) u_{ij}^{IJ}(x), \text{ etc.} \quad (3.3)$$

The central result then is that the generalized vielbein e_{AB}^m of (2.4) is given by the simple formula^{10,18,19}

$$\begin{aligned} e_{AB}^m &= i \Delta^{-1/2} e_a^m (\bar{\Phi}^T \Gamma^a \Phi)_{AB} = \\ &= K^{\circ m IJ} (u_{AB}^{IJ} + v_{AB IJ}) \end{aligned} \quad (3.4)$$

where $\Phi = \Phi(x,y)$ is the $SU(8)$ rotation required to make the transformation laws consistent to all orders¹⁰, and $e_a^m = e_a^m(x,y)$ is the full siebenbein containing scalar and pseudoscalar fluctuations in the massless sector. Both quantities are implicitly determined through (3.4) as functions of the $d = 4$ quantities u and v . To prove that the second line of (3.4) can really be rewritten in the form (2.4), we only have to verify the $SU(8)$ covariant "Clifford property"⁸

$$e_{AC}^m e^{nCB} + e_{AC}^n e^{mCB} = 2 \Delta^{-1} g^{mn} \delta_A^B \quad (3.5)$$

The existence of Φ and e_a^m then follows from a complexified version of the well-known uniqueness theorems for the representations of Clifford algebras²⁰. Equation (3.5) is true provided that the left-hand side vanishes when traced with an arbitrary traceless matrix Λ_B^A ; it is sufficient to assume Λ to be anti-Hermitian, since any Hermitian matrix can be rendered thus by multiplication with i . Inserting the second line of (3.4) into the left-hand side of (3.5), and tracing with an anti-Hermitian matrix $\Lambda_{AB}^{CD} \equiv \delta_{[A}^C \Lambda_{B]}^D$, we get

$$\begin{aligned} & \left\{ u_{IJ}{}^{AB} \wedge_{AB}{}^{CD} u_{CD}{}^{KL} + v_{IJ}{}^{AB} \wedge_{AB}{}^{CD} v_{CD}{}^{KL} + \right. \\ & \left. + u_{IJ}{}^{AB} \wedge_{AB}{}^{CD} v_{CD}{}^{KL} + v_{IJ}{}^{AB} \wedge_{AB}{}^{CD} u_{CD}{}^{KL} \right\} \cdot \\ & \cdot (K^{\circ m IJ} K^{\circ n KL} + K^{\circ n IJ} K^{\circ m KL}) \end{aligned} \quad (3.6)$$

We now use the antihermiticity of Λ , i.e., $\Lambda_{AB}{}^{CD} = -\Lambda^{CD}{}_{AB}$ in conjunction with the E_7 Lie algebra relations⁸

$$\begin{aligned} & (u \wedge u)_{IJ}{}^{KL} - (v \wedge v)_{IJ}{}^{KL} = \\ & = \frac{2}{3} \delta_{[I}^{[K} \left\{ (u \wedge u)_{JJM}{}^{LJM]} - (v \wedge v)_{JJM}{}^{LJM]} \right\} \end{aligned} \quad (3.7)$$

and

$$(u \wedge v)_{IJKL} - (v \wedge u)_{IJKL} = \quad (3.8)$$

= antisymmetric and complex self-dual in $[IJKL]^*$)

where the contractions over $SU(8)$ indices are no longer explicitly indicated. Substituting these relations into (3.6), we see that (3.7) leads to an expression containing

$$K^{\circ m IJ} K^{\circ n IL} + K^{\circ n IJ} K^{\circ m IL} = 2 \delta^{JL} g^{\circ mn} \quad (3.9)$$

which yields zero because the trace over J and L of the expression in brackets of (3.7) vanishes again by the properties of the E_7 Lie algebra. Similarly, the remainder reduces to an expression proportional to

*) I.e., the real part is self-dual, whereas the imaginary part is antiself-dual.

$$K^{\circ m} [IJ K^{\circ n} KL] J_m [u \wedge v - v \wedge u] [IJKL] \quad (3.10)$$

which vanishes because the first factor is (real) self-dual while the second is (real) antiself-dual. Thus, the E_7 properties of the 56-bein (3.1) play a crucial role in establishing (3.5) and thereby the consistency of (3.4). This result is enough to prove the consistency of the truncated spin-1 transformation law to all orders (a different argument for the case of purely scalar fluctuations may be found in Ref. 19).

Another important consequence of (3.4) is the full non-linear ansatz for the $d = 7$ metric for arbitrary scalar and pseudoscalar fluctuations. It is given by the implicit formula¹⁸

$$8 \Delta^{-1} g^{mn} = K^{\circ m} IJ K^{\circ n} KL \cdot (u_{ij} IJ + v_{ij} IJ)(u^{ij} KL + v^{ij} KL) \quad (3.11)$$

which follows directly from (3.4). We have "tested" (3.11)¹⁸ by substituting the vacuum expectation values of u and v at the various critical points of the $N = 8$ potential²¹. One indeed recovers the correct metrics of various known solutions of $d = 11$ supergravity and obtains some new ones in this manner^{*}).

It is also possible in principle to obtain the other non-linear ansätze as well and to establish the consistency of the S^7 truncation to all orders. The crucial point is that the supersymmetry variations of $d = 11$ supergravity in the form given in Refs. 8 and 10 have the same form as the $d = 4$ transformations^{6,7}, and many of the non-linear results can be directly read off by comparing them. For instance, after performing the (x - and y -dependent) $SU(8)$ rotation Φ on the fermions, the massless ansätze for the fermions are given by the simple formulae

$$\begin{aligned} \psi_{\mu}^A(x, y) &= \psi_{\mu}^i(x) \eta_A^i(y) \\ \chi^{ABC}(x, y) &= \chi^{ijk}(x) \eta_A^i(y) \eta_B^j(y) \eta_C^k(y) \end{aligned} \quad (3.12)$$

^{*}) Results for the $SU(3) \times U(1)$ invariant critical point have also been obtained recently²²).

to all orders. It is useful to realize that the $d = 11$ theory in the formulation of Ref. 2 and the $d = 4$ theory of Refs. 6 and 7 correspond to two different gauge choices for e^m_{AB} , namely (2.4) with $\Phi = 1$ and Φ determined by (3.4), respectively. This mismatch of gauges accounts for the apparent inconsistency first noticed in Ref. 16. In the new formulation of $d = 11$ supergravity, we find many relations whose (non-linear) analogues in gauged $N = 8$ supergravity are easily recognized. For instance, substituting (3.4) into (2.16) and using the Killing vector ansatz

$$B_\mu{}^m(x, y) = K^{\circ m IJ}(y) A_\mu^{IJ}(x) \quad (3.13)$$

one finds the following $d = 4$ relation⁸

$$D_\mu(u_{ij}{}^{IJ} + v_{ij IJ}) = \frac{3}{4} \mathcal{R}_{\mu ijkl}(u^{kl}{}_{IJ} + v^{kl IJ}) \quad (3.14)$$

where D_μ is the $SO(8) \times SU(8)$ covariant derivative defined in Ref. 7. Equation (3.14) was also found in the context of $N = 8$ supergravity, but there it was arrived at in a completely different way⁷. Similarly, (2.22) becomes just one of the so-called "T-identities"⁷ in the S^7 truncation. Finally, one realizes that the additional terms in the supersymmetry variations induced by the gauging of $SO(8)$ in $N = 8$ supergravity are essentially nothing but $SU(8)$ -covariant derivatives in the internal dimensions. The E_7 character of these terms, which was noticed in Ref. 16, can likewise be understood.

4. OUTLOOK

The systematic investigation of the supersymmetry transformation laws of $d = 11$ supergravity and the comparison with those of $d = 4$ supergravity has led us to a new version of this theory with many intriguing features. In particular, the still mysterious role played by E_7 in eleven dimensions remains to be elucidated. Although there is probably more structure to be discovered, the examples discussed in the foregoing section should be enough to convince the reader of the usefulness of the results obtained so far. Gauged $N = 8$ supergravity may be viewed as an effective "low energy approximation" of the full $d = 11$ theory, and in view of its complicated structure, it is remarkable that all aspects of its emergence from the $d = 11$ theory can now

be studied. Conventional Kaluza-Klein theories have rarely been examined beyond the linearized level, and therefore the results described here may be useful in other contexts, even if neither $N = 8$ supergravity nor simple supergravity in eleven dimensions are relevant to physics.

The existence of an $SU(8)$ -invariant version of $d = 11$ supergravity also opens new perspectives for the chirality problem mentioned in the introduction. Following the conjecture of Ref. 6, one may speculate that at least part of the local $SU(8)$ becomes dynamical at the quantum level. An important topic for future study then would be to examine whether the $SU(8)$ invariant version of $d = 11$ supergravity may be afflicted with anomalies. Previous results on the possible occurrence of anomalies for the $d = 4$ theory²³ cannot be taken over here, because the $SU(8)$ current J_M has now eleven components and its divergence $\partial^\mu J_\mu + \partial^m J_m$ must be evaluated in eleven dimensions. If this also turns out to vanish for the quantized theory, the compactifications to four dimensions with $\langle \partial^m J_m \rangle = 0$ are perhaps singled out.

Finally, we return to the remarks made at the end of the Introduction. One may contemplate the possibility of reducing $d = 11$ supergravity to other dimensions than $d = 4$. Since this gives rise to different hidden symmetries¹¹, it is conceivable that repeating the procedure of Ref. 8 for these reductions might lead to yet more versions of $d = 11$ supergravity; for some recent speculations on these matters, see also Ref. 24. It is worthwhile to point out that the rank of the hidden group is always equal to the number of compactified dimensions. This feature is reminiscent of compactification in string theories²⁵. Perhaps one should not exclude the existence of a theory beyond $d = 11$ supergravity in which these coincidences are naturally explained.

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