CERN-TH.4154/85

FERMION MASS PREDICTIONS FROM HIGHER DIMENSIONS

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ABSTRACT

Higher dimensional theories predict ratios between Yukawa-couplings and gauge couplings in the resulting four-dimensional gauge theory. After spontaneous symmetry breaking of weak interactions, these relations are converted into relations between particle masses. We calculate explicitly the Yukawa couplings resulting from a six-dimensional $SO(12)$ gauge theory. We argue that the fermion masses follow the pattern $m_t \gg m_b, m_c, m_t \gg m_s, m_\mu \gg m_d, m_u, m_e$.

CERN-TH.4154/85
April 1985

1. INTRODUCTION

An explanation of the observed masses of quarks and leptons as well as the matrix describing mixing angles and CP violation is one of the old theoretical problems in particle physics. In four-dimensional unified gauge theories these quantities are related to Yukawa couplings between fermions and scalars. Unfortunately, even in grand unified theories the matrices of Yukawa couplings contain many free parameters. A few relations between fermion masses may be predicted^{1),2)}, but an understanding of the full spectrum seems difficult in this context.

Higher dimensional theories have typically only a few free parameters. The action for superstring theories³⁾, for example, has no free dimensionless parameter at all. If such theories lead to a realistic four-dimensional model after dimensional reduction, the spectrum of fermion masses should be highly predictable. Four-dimensional gauge fields and scalars often correspond to different components of the same higher dimensional field. The existence of relations between Yukawa couplings and gauge couplings should therefore not surprise. Even if higher dimensional models have a few free parameters, we expect relations among fermion masses and a prediction on the relative scale of fermion masses compared to the mass of the W-boson.

In the next section we outline the steps leading to such predictions in a general framework. The higher dimensional theory may either be pure gravity coupled to spinors or include other fields like gauge fields in higher dimensions or scalars. The discussion covers the field theory limit of a higher dimensional string theory as well as possible supersymmetries. Fermion mass predictions can be obtained if the model admits an approximate ground state with $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry leading to a certain number of massless four-dimensional quarks and leptons with the correct chiral quantum numbers. The requirement that all quarks and charged leptons acquire a non-vanishing mass from spontaneous breaking of $SU(2)_L \times U(1)_Y$ imposes certain topological constraints on candidate ground states. We argue that the problem of fermion masses and the gauge hierarchy problem are partly decoupled. Similar to grand unified theories, fermion mass ratios may be predicted even though the explanation of a small symmetry breaking scale may remain unsatisfactory.

In Sections 3, 4 and 5 we discuss fermion mass ratios in the context of a six-dimensional $SO(12)$ gauge theory^{4),5)}. We calculate all Yukawa couplings for

the chiral quarks and leptons on a class of classical solutions where internal space is a sphere with gauge fields in a monopole configuration. Although the model is not complete - six dimensional scalars have to be included - the spectrum of fermion masses reproduces some features of the observed mass spectrum for certain solutions. In the limit of an unbroken $U(1)_R \times U(1)_q$ symmetry, three sorts of scalars contribute to fermion masses: H_1 couples only to the top quark, H_2 contributes to m_b , m_c and m_t and h induces masses for s and μ . These scalars have different quantum numbers so that their mixing may be small. This suggests the pattern $m_t \gg m_b, m_c, m_\tau \gg m_s, m_\mu \gg m_d, m_u, m_e$. We give our conclusions in Section 6 and two Appendices display the conventions used in this paper for $SO(12)$ generators in the spinor representations and for generalized spherical harmonics.

2. QUARK AND LEPTON MASSES FROM HIGHER DIMENSIONS

Our starting point for a calculation of fermion masses should be a model which leads to chiral fermions in four dimensions. There must be solutions of higher dimensional field equations with maximal four-dimensional symmetry [Poincaré symmetry or (anti-)de Sitter symmetry in case of a small cosmological constant] and gauge symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$. The gauge symmetry corresponds to isometries of internal space or to special higher dimensional gauge transformations. The effective four-dimensional theory is obtained by integrating over internal space. The harmonic expansion of spinors on this solution should lead to massless fermions with the chiral quantum numbers of quarks and leptons. Chirality implies that quarks and leptons can only get a mass of the order of the scale of spontaneous symmetry breaking of $SU(2)_L \times U(1)_Y$ or smaller.

The number of chiral four-dimensional fermions is given by an index⁶⁾, the chirality index N_C . This index measures the number of generations. [For an example with $SO(10)$ symmetry, N_{16} counts the number of unpaired left-handed four-dimensional fermions in the 16 representation.] A non-vanishing chirality index with respect to $SU(3)_C \times SU(2)_L \times U(1)_Y$ guarantees that the number of massless chiral fermion generations is stable against small changes of higher dimensional parameters or small deformations in the solution as long as $SU(3)_C \times SU(2)_L \times U(1)_Y$ remains unbroken. In higher dimensional theories, massless fermions have now a similar status as massless gauge bosons: the zero mass is guaranteed by symmetry plus topology and any mass term involves the breaking of symmetry. In general, the chirality index is not a property of internal space

alone but involves the geometry of full higher dimensional space-time⁷⁾. In the case of a direct product topology $M^4 \times$ compact internal space, the chirality index is given by the character valued index on internal space alone. Two general results are useful to remember: in higher dimensional Riemannian geometry without gauge fields the properties of the Lorentz algebra imply⁸⁾ that chiral fermions are only possible for $d = 2 \bmod 4$. Only $d = 2 \bmod 8$ can lead to a realistic spectrum without doubling of fermions with the same mass. In Riemannian geometry with higher dimensional gauge fields, the same criteria imply $d = 2 \bmod 8$, $d = 6 \bmod 8$ or $d = 2 \bmod 2$, depending whether the fermions are in a real, pseudoreal or complex representation of the gauge group. The second result implies^{9),10)} that in Riemannian geometry without gauge fields the chirality index vanishes for all possible symmetry groups and all representations if the ground state has topology $M^4 \times$ compact internal space.

Three different ways are known to obtain a non-vanishing chirality index. The most popular starts with gauge fields in higher dimensions. Several models leading to chiral fermions are known^{11)-13),9),4)}. Among the anomaly-free models are the field theoretical limit of $SO(32)$ or $E_8 \times E_8$ superstrings³⁾ in 10 dimensions or the six-dimensional $SO(12)$ model^{4),5)} which we will discuss later. The second possibility starts with higher dimensional Riemannian gravity alone and abandons the requirement that the ground state should be $M^4 \times$ compact internal space. Chiral fermions are found if internal space is non-compact^{14),7)} or if geometry does not have the standard direct product form four-dimensional space $\times$ internal space⁷⁾. The simplest model starts with 18-dimensional Riemannian gravity coupled to a Majorana-Weyl spinor. This model has gravitational anomalies¹⁵⁾. In a first step of dimensional reduction, it leads to the anomaly-free chiral six-dimensional $SO(12)$ model⁴⁾. The third alternative also remains in the context of gravity alone. Chiral fermions are obtained from generalization^{16),14)} of Riemannian geometry.

There are several models leading to chiral fermions in standard generations of unification groups like $SU(5)$, $SO(10)$ or E_6 or subgroups of them like $SU(3)_C \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$, etc. This gives the required chiral quantum numbers for quarks and leptons. At this stage, quarks and leptons are all massless and we now have to investigate the mechanism by which they get mass. In a realistic model, colour and electric charge remain unbroken and all masses must be due to spontaneous breaking of $SU(2)_L \times U(1)_Y$. Let us denote the scale of $SU(2)_L \times U(1)_Y$ breaking by $\langle \phi \rangle$. In a realistic model one needs $\langle \phi \rangle \approx 174$ GeV. This

scale is very small compared to the Planck mass which is roughly the characteristic mass scale of all viable higher dimensional models. It is not clear how to find a natural explanation for such a small scale in higher dimensional theories and we will not tackle this problem - the gauge hierarchy problem - in this paper. We only note at this point that in higher dimensional models with a few free parameters one expects that the parameters can be adjusted to give a small value of $\langle\phi\rangle$, similar to grand unified models. Assuming a small scale $\langle\phi\rangle$, it can be shown⁶⁾ that the masses of all quarks and leptons are of order $\langle\phi\rangle$ or smaller.

In higher dimensional models, spontaneous symmetry breaking corresponds to a continuous deformation of geometry and bosonic matter field configuration which reduces the symmetry of a given state. Since all quarks and charged leptons are observed to be massive, we require that all chiral fermions (with the possible exception of neutrinos) must get a mass from such a continuous deformation breaking $SU(2)_L \times U(1)_Y$. This requirement is not trivial and places topological restrictions on candidate $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric states. The first restriction^{6),4)} is that the Atiyah-Singer index¹⁷⁾ on this state must vanish. This index is a purely topological index counting the difference in the number of zero modes for the two inequivalent internal Weyl spinors. (We always assume an even number of dimensions.) If this index is different from zero one always remains with the corresponding number of massless fermions which cannot get mass by any continuous deformation. An equivalent formulation of this problem requires that the harmonic expansion of bosonic degrees of freedom on the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric state must contain four-dimensional scalars with the appropriate quantum numbers to admit Yukawa couplings to the chiral quarks and leptons. A topological obstruction like a non-vanishing Atiyah-Singer index implies that certain fermions cannot have Yukawa couplings to any of the scalars in the harmonic expansion. [For a simple example in six-dimensional Einstein-Maxwell theory, see Ref. 4).] There can be additional topological obstructions if the higher dimensional model has more than one irreducible fermion representation $\phi_1 \dots \phi_i \dots \phi_n$ which are independent in the sense that the action has discrete symmetries of the sort $\Gamma_i: \phi_j \rightarrow \phi_j$ for $j \neq i$, $\phi_i \rightarrow -\phi_i$. (This is the case if there are more than one fermion representation with minimal gravitational and gauge interactions.) We then have to require that the Atiyah-Singer index vanishes for each representation separately. The total number of quarks and leptons are in a vectorlike representation of $SU(3)_C \times U(1)_{em}$. Nevertheless, this group admits complex representations. In the above case we need a vanishing chirality index with respect to $SU(3)_C \times U(1)_{em}$ for each fermion representation separately. Otherwise $SU(3)_C \times U(1)_{em}$ symmetry

forbids mass terms for some of the chiral fermions⁵⁾. We note, however, that symmetries of the type Γ_i may be broken by couplings to higher dimensional scalar fields.

Fortunately, a vanishing Atiyah-Singer index does not exclude the chirality index being different from zero. A non-zero chirality index is possible even on a trivial topology in Riemannian gravity⁷⁾ or for a topologically trivial configuration of higher dimensional gauge fields⁵⁾. In fact, the chirality index is not a property of topology alone. It is only defined with respect to a group of symmetries G and is a property of G -deformation-classes⁶⁾ of geometries. (Things are even more subtle if space-time does not have geometry $M^4 \times$ compact internal space. The chirality index may change⁷⁾ within a given G -deformation class.) A G -deformation class consists of all states (geometries plus gauge field configurations, for example) which can be obtained from each other by continuous deformations, leaving G unbroken. The states with a given topology may contain different G -deformation classes and the chirality index may change from one to the other. A good example⁴⁾ are $SO(12)$ monopoles on S^2 . They are all topologically trivial, but they can be "unwound" only by using $SU(3)_C \times SU(2)_L \times U(1)_Y$ non-singlet degrees of freedom. The different $SU(3)_C \times SU(2)_L \times U(1)_Y$ -deformation classes often lead to a different chirality index.

This points to a problem in models which combine ideas of grand unification and dimensional reduction. This combination was proposed early¹⁸⁾ because of two advantages: the quarks and leptons form simple representations of unification groups like $SU(5)$, $SO(10)$ or E_6 and the ratio between the gauge couplings of $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ fits well the experimentally observed values. However, the grand unified group must be broken to $SU(3)_C \times SU(2)_L \times U(1)_Y$ at a scale comparable to the typical scale of compactification. Spontaneous symmetry breaking in the context of higher dimensional theories can lead to several unusual features⁵⁾. For example, the number of chiral fermion generations may change in the course of the breaking $SU(5) \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y$ if the broken and unbroken solutions correspond to different $SU(3)_C \times SU(2)_L \times U(1)_Y$ -deformation classes! Also, the Yukawa couplings often do not obey the relations of grand unified symmetry^{5), 19)}. In general, one first has to find the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric approximate ground state which differs from the true ground state only by corrections proportional to the scale $\langle \phi \rangle$ of weak symmetry breaking. The content of chiral fermions and their Yukawa couplings have to be calculated in this approximate ground state.

Let us now assume that the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric approximate ground state is known. The fermion mass problem can then be split into several pieces. First one has to identify the chiral quarks and leptons. The discussion of their masses can neglect all heavy states in the harmonic expansion of spinors. All effects of mixing with heavy fermions are suppressed⁶⁾ by a factor $\langle \phi \rangle L_0$ compared to typical fermion masses of order $\langle \phi \rangle$. Here L_0 is the characteristic length scale of internal space. Next one has to identify the weak Higgs doublet ϕ whose vacuum expectation value drives the breaking of $SU(2)_L \times U(1)_Y$. In general this doublet is a linear combination of many four-dimensional scalar fields with the appropriate quantum numbers. The third step involves the calculation of the Yukawa couplings of ϕ to the chiral fermions. For a given approximate ground state they are uniquely determined in dependence of the parameters of the higher dimensional action. Typically, Yukawa couplings are a function of the four-dimensional gauge coupling g . Knowledge of Yukawa couplings allows predictions on fermion mass ratios like m_c/m_b , etc., at the scale L_0^{-1} . The ratio between fermion masses and the mass M_W of the W-boson is also predicted. Finally, one has to find the scale $\langle \phi \rangle$. This last step - the solution of the gauge hierarchy problem - may be the most difficult. Fortunately, it can be decoupled to a large extent from the problem of fermion mass predictions.

In a realistic model, it may not be easy to find the correct $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric approximate ground state. What can we learn from other known solutions which typically have a symmetry larger than $SU(3)_C \times SU(2)_L \times U(1)_Y$? If such a solution is in a different $SU(3)_C \times SU(2)_L \times U(1)_Y$ -deformation class from the approximate ground state, a discussion of fermion mass relations for such a solution has nothing to do with the final fermion mass relations. The chiral fermions of such a solution are not related to the quarks and leptons of the approximate ground state - even if the number were the same. If a solution is in the same deformation class as the approximate ground state, there is a continuous one-to-one connection between $SU(3)_C \times SU(2)_L \times U(1)_Y$ representations in the harmonic expansion on this solution and on the approximate ground state. Especially quarks and leptons on both states are continuously related. The same is true for possible candidates for the weak Higgs doublet. This permits at least a qualitative understanding of certain features of the fermion mass matrix by studying the representation content of fields for the known solution and evaluating their Yukawa couplings. Two problems exist for quantitative fermion mass predictions from such a solution: first, the harmonics for the chiral fermions may be different for this solution and the approximate ground state. This corresponds to a mixing with "heavy" harmonics by going from one state to the other. The same is true for the Higgs scalars. Even if all

harmonics were approximately the same, the Yukawa couplings might change between the two states due to terms of the type $\bar{\psi}\psi\phi\phi_0$. Here ϕ_0 is a $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet scalar field whose vacuum expectation value differs for the two states. There is, however, an important special case where all these difficulties are minimized: this is the case if the spontaneous symmetry breaking relating the known solution to the approximate ground state happens at a scale M_I sufficiently small compared to L_0^{-1} . All corrections to predictions on fermion masses are then suppressed by factors $M_I L_0$. In the remainder of this paper we will illustrate all these ideas in the context of a simple model.

3. YUKAWA COUPLINGS FROM A SIX-DIMENSIONAL $SO(12)$ GAUGE THEORY

As our specific example we will discuss the six-dimensional $SO(12)$ model^{4),5)}. This incorporates many characteristic features of higher dimensional gauge theories which are expected to be shared by more complicated models like those based on $E_8 \times E_8$ in ten dimensions obtained from superstrings³⁾. It is possible that it can be obtained⁵⁾ from the $E_8 \times E_8$ model in a first step of dimensional reduction. It may also be obtained from pure gravity in 18 dimensions⁴⁾. Our model is simple enough to allow an explicit evaluation of the relevant Yukawa couplings.

The Majorana-Weyl spinors ψ_1 and ψ_2 of this model have positive and negative six-dimensional helicity respectively and they belong to the inequivalent spinor representations 32_1 and 32_2 of $SO(12)$. The model is free of all anomalies¹⁵⁾. All Yukawa couplings are contained in the six-dimensional covariant kinetic term for the fermions:

$$\begin{aligned} \mathcal{L}_\psi &= i \hat{g}_6^{1/2} \bar{\psi}_1 \hat{\gamma}^{\hat{\mu}} (\nabla_{\hat{\mu}} - i \hat{g} \hat{A}_{\hat{\mu}}^{(1)}) \psi_1 \\ &\quad + i \hat{g}_6^{1/2} \bar{\psi}_2 \hat{\gamma}^{\hat{\mu}} (\nabla_{\hat{\mu}} - i \hat{g} \hat{A}_{\hat{\mu}}^{(2)}) \psi_2 \\ &= \mathcal{L}_{\psi_1} + \mathcal{L}_{\psi_2} \end{aligned} \tag{1}$$

Here $\nabla_{\hat{\mu}}$ is the six-dimensional covariant derivative, $\hat{g}_6 = -\det \hat{g}_{\hat{\mu}\hat{\nu}}$ [we use conventions of Ref. 4)] and

$$\hat{A}_{\hat{\mu}}^{(1,2)} = \frac{1}{2} \hat{A}_{\hat{\mu}}^{AB} T_{AB}^{(1,2)} ; A, B = 1 \dots 12 \quad (2)$$

with $T_{AB}^{(1,2)}$ the $SO(12)$ generators in the spinor representations 32_1 and 32_2 (compare Appendix A). We use six-dimensional signature $\eta_{\hat{m}\hat{n}} = \text{diag}(1, -1, -1, -1, -1, -1)$ and

$$\begin{aligned} \{\hat{\gamma}^{\hat{m}}, \hat{\gamma}^{\hat{n}}\} &= 2 \eta^{\hat{m}\hat{n}} \\ \hat{\gamma}^{\hat{m}} &= \gamma^m ; m = 0, 1, 2, 3 \\ \hat{\gamma}^{\hat{4}} &= \gamma^5 \Gamma^1 = i \gamma^5 \otimes \tau_1 \\ \hat{\gamma}^{\hat{5}} &= \gamma^5 \Gamma^2 = i \gamma^5 \otimes \tau_2 \\ \gamma^5 &= i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \end{aligned} \quad (3)$$

Here γ^m are the usual four-dimensional Dirac matrices. The Pauli-matrices τ_1 and τ_2 act on internal spinor indices. (Two-dimensional Dirac spinors have two components.)

Let us now assume the existence of a classical solution with Poincaré invariance and $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry. This solution is considered to be the approximate ground state of the theory up to small effects of spontaneous $SU(2)_L \times U(1)_Y$ breaking. Poincaré invariance implies, for the vielbein and the gauge field in the ground state^{20),21)}

$$\hat{e}_{\hat{\mu}}^{\hat{m}} = \begin{pmatrix} \sigma^{\mu\nu}(y) \delta_{\mu}^m & 0 \\ 0 & \hat{e}_a^{\hat{m}}(y) \end{pmatrix} ; \hat{A}_{\hat{\mu}}^{\hat{a}} = \begin{pmatrix} 0 \\ \hat{A}_a^{\hat{a}}(y) \end{pmatrix} \quad (4)$$

Known classical solutions of this type for the six-dimensional $SO(12)$ Einstein-Yang-Mills theory include monopole solutions^{11),13),4),5)} on an internal sphere S^2 or generalization to non-compact internal space²¹⁾. All these solutions lead to symmetries larger than $SU(3)_C \times SU(2)_L \times U(1)_Y$, but additional classical solutions are expected once six-dimensional scalar fields are included into the action^{4),5)}. The spinor representations of $SO(12)$ contain fields with the usual quantum numbers of quarks and leptons as well as mirror particles. [With

respect to the $SO(10)$ subgroup, both 32_1 and 32_2 transform as $16 + \overline{16}$.] For all known classical solutions one finds an asymmetry between standard fermions and mirror fermions in the harmonic expansion, provided a certain monopole number $n^{4),5)}$ is different from zero. The model accounts for n generations of chiral quarks and leptons in the resulting effective four-dimensional theory.

Except for the massless chiral modes, all other infinitely many fermionic modes in the harmonic expansion have masses $\gtrsim 10^{17}$ GeV. We concentrate on Yukawa couplings of the chiral fermions and truncate the harmonic expansion

$$\psi(y, x) = \psi^0 = \psi_{0j}(y) \psi^{0j}(x) \quad (5)$$

Here j is a multi-index denoting the different quantum numbers of the chiral spinors. The chiral spinors form a complex representation of the $SO(12)$ subgroup left invariant by the ground state. The only possible Yukawa couplings involve scalars in non-trivial representations of this subgroup. They are contained in the internal components $\hat{A}_\alpha$ of the $SO(12)$ gauge fields and belong to the 66-dimensional adjoint representation of $SO(12)$. We denote

$$\bar{A}_\alpha = \hat{e}_\alpha^\alpha (\hat{A}_\alpha - \hat{A}_\alpha) \quad (6)$$

$$\bar{A}_\pm = \frac{1}{\sqrt{2}} (\bar{A}_1 \mp i \bar{A}_2)$$

The harmonic expansion for the internal components of $SO(12)$ gauge fields reads^{13),5)}

$$\bar{A}_\pm(y, x) = \bar{b}_{\pm i}(y) \phi^{\pm i}(x) \quad (7)$$

The index i denotes the various quantum numbers of the four-dimensional scalar fields $\phi^{\pm i}(x)$.

Retaining only those scalars, the fermionic action (1) for the chiral fermions reduces to

$$\mathcal{L}_{\psi_{1,2}} = \hat{g}_6^{1/2} \bar{\psi}_{1,2}^0 \left\{ i \sigma^{-1/2} \gamma^\mu \partial_\mu + i \sqrt{2} \hat{g} \gamma^5 (\bar{A}_+ \tau_+ + \bar{A}_- \tau_-) \right\} \psi_{1,2}^0 \quad (8)$$

Here we have defined

$$\tau_{\pm} = \frac{1}{2} (\tau_1 \mp i \tau_2) \quad (9)$$

The first term in Eq. (8) leads to the standard kinetic term whereas the second contains all Yukawa couplings. There is no mixing between ϕ_1 and ϕ_2 and we can discuss the Yukawa couplings separately for both spinors.

We define internal and four-dimensional Weyl spinors

$$\psi_{oj}^{\pm}(y) = \frac{1}{2} (1 \pm \tau_3) \psi_{oj}(y) \quad (10)$$

$$\psi_{L,R}^{oj}(x) = \frac{1}{2} (1 \pm \gamma^5) \psi^{oj}(x)$$

Using the six-dimensional Weyl constraints

$$\begin{aligned} (\gamma^5 \otimes \tau_3) \psi_1 &= \psi_1 \\ (\gamma^5 \otimes \tau_3) \psi_2 &= -\psi_2 \end{aligned} \quad (11)$$

and integrating over the coordinates y of internal space, one finds for the four-dimensional Yukawa couplings of ϕ_1

$$\begin{aligned} \mathcal{L}_{Yuk}^{(1)} &= \bar{\psi}_R^{o\bar{k}}(x) i \gamma^5 h_{\bar{k}ji}^- \varphi^{-i}(x) \psi_L^{oj}(x) \\ &+ \bar{\psi}_L^{oj}(x) i \gamma^5 h_{j\bar{k}i}^+ \varphi^{+i}(x) \psi_R^{o\bar{k}}(x) \end{aligned} \quad (12)$$

with Yukawa couplings

$$\begin{aligned} h_{\bar{k}ji}^- &= \sqrt{2} \bar{g} \int d^2 y g_2^{1/2} \sigma^2(\psi_{o\bar{k}}^-(y))^{\dagger} b_{-i}(y) \tau_- \psi_{oj}^+(y) \\ h_{j\bar{k}i}^+ &= \sqrt{2} \bar{g} \int d^2 y g_2^{1/2} \sigma^2(\psi_{oj}^+(y))^{\dagger} b_{+i}(y) \tau_+ \psi_{o\bar{k}}^-(y) \end{aligned} \quad (13)$$

For the spinor ϕ_2 the subscripts L and R designing four-dimensional helicity are interchanged.

The spinors ψ_1 and ψ_2 are subject to a six-dimensional Majorana constraint

$$B_4^{-1} B_2^{-1} B_{12}^{-1} \psi_{1,2}^* = \psi_{1,2} \quad (14)$$

The SO(12) charge conjugation matrix B_{12} acts on SO(12) indices and fulfils (see Appendix A)

$$B_{12}^* B_{12} = -1 \quad (15)$$

The four- and two-dimensional charge conjugation matrices are defined²²⁾

$$(\gamma^m)^* = -B_4 \gamma^m B_4^{-1}, \quad B_4^* B_4 = 1 \quad (16)$$

$$(\Gamma^a)^* = B_2 \Gamma^a B_2^{-1}, \quad B_2^* B_2 = -1$$

Internal charge conjugation maps every complex representation in the harmonics of $\psi^+(y)$ into a complex conjugate representation in $\psi^-(y)$ and vice versa⁸⁾:

$$\mathcal{C}_D \psi_{oj}^+(y) \equiv B_2^{-1} B_{12}^{-1} (\psi_{oj}^+(y))^* = \psi_{o\bar{j}}^-(y) \quad (17)$$

The Majorana constraint (14) therefore identifies

$$\mathcal{C}_4 \psi^{oj}(x) \equiv B_4^{-1} (\psi^{oj}(x))^* = \psi^{o\bar{j}}(x) \quad (18)$$

and one has

$$\bar{\psi}_R^{o\bar{k}}(x) = \tilde{\psi}_L^{ok}(x) \equiv (\psi_L^{ok}(x))^T B_4 \gamma^0 \quad (19)$$

This allows us to bring the Yukawa couplings of ψ_1 and ψ_2 into a manifestly symmetric form

$$\begin{aligned} \mathcal{L}_{Yuk}^{(1)} &= \tilde{\psi}_L^{ok}(x) i \gamma^5 h_{kji}^{(1)} \varphi^{-i}(x) \psi_L^{oj}(x) + h.c. \\ \mathcal{L}_{Yuk}^{(2)} &= \tilde{\psi}_L^{ok}(x) i \gamma^5 h_{kji}^{(2)} \varphi^{+i}(x) \psi_L^{oj}(x) + h.c. \end{aligned} \quad (20)$$

$$h_{kji}^{(1)} = \sqrt{2} \bar{g} \int d^2 y g_2^{1/2} \sigma^2 (\psi_{0k}^+)^T B_{12} b_{-i} \psi_{0j}^+ = h_{jki}^{(1)} = h_{\bar{k}ji}^{(1)}$$

$$h_{kji}^{(2)} = -\sqrt{2} \bar{g} \int d^2 y g_2^{1/2} \sigma^2 (\psi_{0k}^-)^T B_{12} b_{+i} \psi_{0j}^- = h_{jki}^{(2)} = h_{\bar{k}ji}^{(2)} \quad (21)$$

Here we have used the fact that the two terms in (12) are Hermitian conjugate to each other. We also have taken an explicit representation $B_2 = \Gamma^2 = i\tau_2$. In the above form it becomes apparent that the factor $i\gamma^5$ in (20) can be absorbed by a suitable chiral phase rotation of $\phi_L^{oj}(x)$ and we may drop it in the future.

We also note that the harmonics b_{+i} and b_{-i} are related by Hermitian conjugation

$$(b_{+i})^\dagger = b_{-i} \quad (22)$$

This identifies the four-dimensional fields

$$(\varphi^{+i}(x))^* = \varphi^{-i}(x) \quad (23)$$

For an evaluation of the Yukawa couplings h_{kji} we have to specify the normalization of $\psi_{oj}^\pm(y)$ and $b_{\pm i}(y)$. It is chosen so that the four-dimensional fields $\phi_{L,R}^{oj}(x)$ and $\phi^{\pm i}(x)$ have the standard normalization of their kinetic terms. For the fermionic harmonics this implies [compare (8)]

$$\int d^2 y g_2^{1/2} \sigma^{3/2} (\psi_{0k}^+)^T \psi_{0j}^+ \quad (24)$$

$$= \int d^2 y g_2^{1/2} \sigma^{3/2} (\psi_{0k}^-)^T \psi_{0j}^- = \frac{1}{2} \delta_{kj}$$

The factor $\frac{1}{2}$ is due to the identification of $\mathcal{C}_4 \psi^{oj}$ and $\psi^{o\bar{j}}$ due to the Majorana constraint⁸⁾. With this convention, the four-dimensional kinetic term only contains independent left-handed fields

$$\mathcal{L}_{kin}^F = i \bar{\psi}_L^{oj}(x) \gamma^\mu \partial_\mu \psi_L^{oj}(x) \quad (25)$$

Inserting the ansätze (6) and (7) into the six-dimensional kinetic term for the $SO(12)$ gauge bosons integrated over internal coordinates

$$\mathcal{L}_G = -\frac{1}{8} \int d^2y g_2^{1/2} \sigma^2 \text{Tr} \hat{G}_{\hat{\mu}\hat{\nu}} \hat{G}^{\hat{\mu}\hat{\nu}} \quad (26)$$

should give the standard kinetic term for the independent four-dimensional scalars $\phi^{+i}(x)$:

$$\mathcal{L}_{\text{kin}}^S = \partial^\mu (\phi^{+i}(x))^* \partial_\mu \phi^{+i}(x) \quad (27)$$

This gives the normalization

$$\int d^2y g_2^{1/2} \sigma \text{Tr} (\phi_{+i})^\dagger \phi_{+j} = 2 \delta_{ij} \quad (28)$$

Here and in Eq. (26) the trace is taken in the vector representation of $SO(12)$ [for conventions see 5)].

To proceed further, we will choose an explicit basis for the $SO(12)$ spinor representations. This is described in Appendix A. In Table 1 we give in this basis the Abelian quantum numbers for the fermions contained in 32_1 . For the fermions in 32_2 the sign of the eigenvalue of $T_{11,12}$ has to be reversed. Instead of the general notation $\psi_{oj}(y)$ for the chiral spinor harmonics we often will use the more intuitive notation $\tau(y)$ for the harmonics corresponding to the τ -lepton, etc. More generally, we use $e_j(y)$ for the harmonics of charged leptons in the 16 of $SO(10)$, etc.

We are interested in the Yukawa couplings of quarks and leptons to colour singlet, $SU(2)_L$ doublet scalar fields. The adjoint representation of $SO(12)$ contains a complex 10 plet of $SO(10)$. It has charge $q = \pm 1$ with respect to the Abelian symmetry in $SO(12)$ which commutes with $SO(10)$. This 10 plet contains two candidate Higgs doublets. It is sufficient to calculate the Yukawa couplings of the electrically neutral components of these Higgs fields which we denote by H_1 and H_2 . Abelian quantum numbers of H_1 and H_2 are given in Table 2. The scalars H_{1+} , H_{2+} with $q = 1$ and positive two-dimensional helicity (and their complex conjugates) couple to ordinary quarks and leptons. The scalars H_{1-} , H_{2-} with $q = 1$ and negative helicity couple to the mirror fermions⁵⁾. In our basis, the harmonics for H_{1+} and H_{2+} are (compare Appendix A)

$$b_{+H1i} = \frac{i}{2} (T_{3,11} + i T_{3,12} - i T_{4,11} + T_{4,12}) D_{1i}(y) = \tilde{H}_1 D_{1i}(y) \quad (29)$$

$$b_{+H2i} = -\frac{i}{2} (T_{3,11} + i T_{3,12} + i T_{4,11} - T_{4,12}) D_{2i}(y) = \tilde{H}_2 D_{2i}(y)$$

Here $D_i(y)$ are complex scalar functions of y and i now labels the harmonic expansion for these particular fields. The normalization (28) implies

$$\int d^2y g_2^{1/2} \sigma D_{1(2)i}^* D_{1(2)j} = \delta_{ij} \quad (30)$$

Using the explicit form of B_{12} one finds in our basis

$$B_{12} \tilde{H}_{1(2)} = \frac{1}{4} \tau_0 \otimes \tau_0 \otimes \{ \tau_0 \otimes \tau_1 \otimes \tau_3 \otimes \tau_3 - \tau_0 \otimes \tau_1 \otimes \tau_0 \otimes \tau_0 \\ \pm \tau_3 \otimes \tau_1 \otimes \tau_0 \otimes \tau_3 \mp \tau_3 \otimes \tau_1 \otimes \tau_3 \otimes \tau_0 \} \quad (31)$$

Inserting this expression into (21) reduces the Yukawa couplings to integrals over scalar functions.

Consider first the Yukawa couplings of the chiral left-handed quarks and leptons from the spinor ψ_2 . As required by the Abelian quantum numbers⁵⁾, H_1 couples only to u and ν whereas H_2 couples only to d and e . One finds the following non-vanishing Yukawa couplings

$$\begin{aligned} h_{u_k u_j^c H1i}^{(2)} &= \sqrt{2} \bar{g} \int d^2y g_2^{1/2} \sigma^2 u_k^- u_j^{c-} D_{1i} \\ h_{\nu_k \nu_j^c H1i}^{(2)} &= \sqrt{2} \bar{g} \int d^2y g_2^{1/2} \sigma^2 \nu_k^- \nu_j^{c-} D_{1i} \\ h_{d_k d_j^c H2i}^{(2)} &= \sqrt{2} \bar{g} \int d^2y g_2^{1/2} \sigma^2 d_k^- d_j^{c-} D_{2i} \\ h_{e_k e_j^c H2i}^{(2)} &= \sqrt{2} \bar{g} \int d^2y g_2^{1/2} \sigma^2 e_k^- e_j^{c-} D_{2i} \end{aligned} \quad (32)$$

The quarks and leptons from ψ_1 couple to the Hermitian conjugate of H_1 and H_2 ⁵⁾. Observing

$$b_{-\bar{H}_1 \bar{i}} = (b_{+H_1 i})^\dagger = \tilde{H}_1^\dagger D_{1i}^* \quad (33)$$

$$b_{-\bar{H}_2 \bar{i}} = (b_{+H_2 i})^\dagger = \tilde{H}_2^\dagger D_{2i}^*$$

$$B_{12} \tilde{H}_{1(2)}^\dagger = \frac{1}{4} \tau_0 \otimes \tau_0 \otimes \left\{ \tau_0 \otimes \tau_1 \otimes \tau_3 \otimes \tau_3 + \tau_0 \otimes \tau_1 \otimes \tau_0 \otimes \tau_0 \right. \\ \left. + \tau_3 \otimes \tau_1 \otimes \tau_0 \otimes \tau_3 + \tau_3 \otimes \tau_1 \otimes \tau_3 \otimes \tau_0 \right\} \quad (34)$$

one finds the Yukawa couplings

$$h_{u_k u_j^c \bar{H}_2 \bar{i}}^{(1)} = \sqrt{2} \bar{g} \int d^2 y g_2^{1/2} \sigma^2 u_k^+ u_j^{c+} D_{2i}^*$$

$$h_{\nu_k \nu_j^c \bar{H}_2 \bar{i}}^{(1)} = \sqrt{2} \bar{g} \int d^2 y g_2^{1/2} \sigma^2 \nu_k^+ \nu_j^{c+} D_{2i}^* \quad (35)$$

$$h_{d_k d_j^c \bar{H}_1 \bar{i}}^{(1)} = \sqrt{2} \bar{g} \int d^2 y g_2^{1/2} \sigma^2 d_k^+ d_j^{c+} D_{1i}^*$$

$$h_{e_k e_j^c \bar{H}_1 \bar{i}}^{(1)} = \sqrt{2} \bar{g} \int d^2 y g_2^{1/2} \sigma^2 e_k^+ e_j^{c+} D_{1i}^*$$

Among the four-dimensional fields $\phi^{+1}(x)$ we have discussed the electrically neutral components of weak doublet colour singlet fields $H_1^1(x)$ and $H_2^1(x)$. Let us now assume that these fields acquire vacuum expectation values $\langle H_1^1 \rangle$, $\langle H_2^1 \rangle$ breaking the weak $SU(2)_L \times U(1)_Y$ group. This induces a mass term for the quarks and leptons from (20):

$$\mathcal{L}_M^{(U)} = \tilde{u}_L^c (M_U)_{jk} u_L^k + h.c. = \bar{u}_R^j (M_U)_{jk} u_L^k + h.c. \quad (36)$$

and similar for d, ν and e. The mass matrices are given by

$$\begin{aligned}
 (M_U)_{jk} &= 2 \left\{ \sum_i h_{u_j^c u_k}^{(1)} \bar{H}_{2i} \langle H_2^i \rangle^* + \sum_i h_{u_j^c u_k}^{(2)} H_{1i} \langle H_1^i \rangle \right\} \\
 (M_V^{(0)})_{jk} &= 2 \left\{ \sum_i h_{\nu_j^c \nu_k}^{(1)} \bar{H}_{2i} \langle H_2^i \rangle^* + \sum_i h_{\nu_j^c \nu_k}^{(2)} H_{1i} \langle H_1^i \rangle \right\} \\
 (M_D)_{jk} &= 2 \left\{ \sum_i h_{d_j^c d_k}^{(1)} \bar{H}_{1i} \langle H_1^i \rangle^* + \sum_i h_{d_j^c d_k}^{(2)} H_{2i} \langle H_2^i \rangle \right\} \\
 (M_E)_{jk} &= 2 \left\{ \sum_i h_{e_j^c e_k}^{(1)} \bar{H}_{1i} \langle H_1^i \rangle^* + \sum_i h_{e_j^c e_k}^{(2)} H_{2i} \langle H_2^i \rangle \right\} \quad (37)
 \end{aligned}$$

Here the sum over j and k runs over all independent chiral spinors contained in $\psi^+(y)$ and $\psi^-(y)$ for the spinors ψ_1 and ψ_2 . The factor 2 arises since there are equal contributions from $h_{u_j^c u_k}^{(1)} H_{1i}$ and $h_{u_k^c u_j}^{(1)} \bar{H}_{1i}$, etc.

Vacuum expectation values (vev) of Higgs doublets also give mass to the W and Z bosons. In general, the neutral component of the low energy scalar doublet ϕ will be a mixture of different H_1^1 , H_2^1 and possibly even other doublets contained in a more complete model. We may denote

$$\langle H_1^i \rangle = \gamma_1^i \langle \phi \rangle \quad (38)$$

$$\langle H_2^i \rangle = \gamma_2^i \langle \phi \rangle$$

with

$$\sum_i |\gamma_1^i|^2 + \sum_i |\gamma_2^i|^2 \leq 1 \quad (39)$$

Equality holds if no other doublets are present. In a realistic model one needs

$$\langle \phi \rangle \approx 174 \text{ GeV} \quad (40)$$

and the mass of the W boson is

$$M_W^2 = \frac{1}{2} g^2 |\langle \phi \rangle|^2 \quad (41)$$

The weak gauge coupling g can be calculated at the compactification scale M_c from the six-dimensional kinetic term for the gauge fields (26)

$$g^{-2}(M_c) = \bar{g}^{-2} \int d^2 y g_2^{\prime\prime 2} = \bar{g}^{-2} V_2 \quad (42)$$

Once the Yukawa couplings (32) and (35) are calculated for a specific ground state, we can establish relations among fermion masses. If in addition some coefficients $\gamma_{1(2)}^i$ are known, relations between the fermion masses and the W boson mass can be established. If the low energy doublet ϕ is uniquely determined (all γ coefficients known), the total contribution of six-dimensional SO(12) gauge fields to the mass matrices of quarks and leptons is calculable. In the next section we will calculate the Yukawa couplings for some simple "trial ground states" and compare the results with the observed spectrum of fermion masses.

4. FERMION MASS RELATIONS

So far our discussion did not specify the properties of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric approximate ground state except the feature that it should lead to chiral quarks and leptons. We only have used algebraic properties related to six-dimensional Lorentz symmetry and SO(12) gauge invariance. Actual calculation of the Yukawa couplings h involves knowledge of the ground state geometry - the functions $\sigma(y)$ and $g_2^{\frac{1}{2}}(y)$ - and of the harmonics for chiral fermions and two-dimensional vectors $\hat{A}_\alpha$. As an example we will evaluate the Yukawa couplings on a geometry $M^4 \times S^2$ with gauge fields in a monopole configuration on S^2 . The most general $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric solution of this type is characterized⁵⁾ by three integers n , m and p with $n+p$ even. The number of chiral fermion generations is given by n . These solutions have at least a symmetry $SU(3)_C \times SU(2)_L \times U(1)_R \times U(1)_{B-L} \times U(1)_q \times SU(2)_G$. The Abelian symmetries $U(1)_R$ and $U(1)_{B-L}$ are subgroups of SO(10) with generators I_{3R} and $B-L$. The group $U(1)_q$ is the subgroup of SO(12) commuting with SO(10) and $SU(2)_G$ is related to isometries acting on S^2 . Both $U(1)_q$ and $SU(2)_G$ differentiate between fermions of different generations and we may call them generation symmetries in this sense. This does not imply that all fermions within a standard generation have the same $SU(2)_G \times U(1)_q$ transformation properties. Indeed, this is not the case whenever m or p differ from zero. These monopole solutions have not all required properties of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric approximate ground state since their symmetry is larger than $SU(3)_C \times SU(2)_L \times$

$U(1)_Y$. In a more realistic model, six-dimensional scalar fields have to be introduced⁴⁾ in order to break $U(1)_R \times U(1)_{B-L} \times U(1)_q \times SU(2)_G$ to $U(1)_Y$. Nevertheless, some of the qualitative features of the structure of the fermion mass matrices can be understood by evaluating Yukawa couplings on these solutions. We also will discuss the modifications expected for a more realistic approximate ground state.

For the monopole solution on S^2 we use polar coordinates θ and ϕ with

$$\sigma(\vartheta, \varphi) = 1 \quad (43)$$

$$g_z^{1/2}(\vartheta, \varphi) = L_0^2 \sin \vartheta \quad (44)$$

$$V_2 = 4\pi L_0^2, \quad g^2 = \bar{g}^2 / 4\pi L_0^2 \quad (45)$$

where the radius L_0 of S^2 can be calculated from the parameters of the six-dimensional action⁵⁾. The generalized spherical harmonics in presence of the monopole field are characterized by a "helicity" λ . One has

$$u_j^+(\vartheta, \varphi) = \frac{1}{\sqrt{2}} D_{\ell \lambda}^{(\lambda_u)}(\vartheta, \varphi) \quad (46)$$

$$\lambda_u = \frac{1}{2} - \frac{1}{2} N_u \quad (47)$$

and similar for other quarks and leptons. Properties of the harmonics $D_{\ell k}^{(\lambda)}$ are displayed in Appendix B. In Table 1 we have listed the values of N for the different fields contained in ψ_1 . The values of N for ψ_2 are obtained by reversing the sign of n . For the representation 32_1 , chiral harmonics $u^+(\theta, \phi)$ etc., exist⁵⁾ whenever $N > 0$ for

$$\ell = |\lambda| \quad (48)$$

$$k = -\ell, -\ell+1, \dots, \ell$$

Similarly, the "helicity" λ for harmonics $u^-(\theta, \phi)$ is

$$\lambda_u = -\frac{1}{2} - \frac{1}{2} N_u \quad (49)$$

The chiral harmonics $u^-(\theta, \phi)$ in 32_2 are obtained for $N < 0$, $\lambda = |\lambda|$. The generalized spherical harmonics for the Higgs scalars are

$$\begin{aligned} D_{1i}(\vartheta, \varphi) &= D_{\ell_2}^{(\lambda_1)}(\vartheta, \varphi) \\ D_{2i}(\vartheta, \varphi) &= D_{\ell_2}^{(\lambda_2)}(\vartheta, \varphi) \end{aligned} \quad (50)$$

$$\begin{aligned} \lambda_1 &= 1 - \frac{n}{2} + \frac{m}{2} \\ \lambda_2 &= 1 - \frac{n}{2} - \frac{m}{2} \end{aligned} \quad (51)$$

Using the integral over three-spherical harmonics in Appendix B, we can express the Yukawa couplings in terms of the four-dimensional gauge coupling g and the Wigner $3j$ symbols:

$$h_{u_2 u_1^c H_1 i}^{(2)} = g \left(\frac{(2\lambda_u + 1)(2\lambda_{u^c} + 1)(-2\lambda_1 + 1)}{2} \right)^{1/2} \begin{pmatrix} \lambda_u & \lambda_{u^c} & -\lambda_1 \\ \lambda_u & \lambda_{u^c} & \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_u & \lambda_{u^c} & -\lambda_1 \\ I_2 & I_j & I_i \end{pmatrix} \quad (52)$$

$$h_{u_2 u_1^c H_2 i}^{(1)} = (-1)^{I_i - \lambda_2} g \left(\frac{(-2\lambda_u + 1)(-2\lambda_{u^c} + 1)(-2\lambda_2 + 1)}{2} \right)^{1/2} \begin{pmatrix} -\lambda_u & -\lambda_{u^c} & -\lambda_2 \\ \lambda_u & \lambda_{u^c} & -\lambda_2 \end{pmatrix} \begin{pmatrix} -\lambda_u & -\lambda_{u^c} & -\lambda_2 \\ I_2 & I_j & -I_i \end{pmatrix}$$

Here I is the third component of $SU(2)_G$ angular momentum and the requirement $I_k + I_j + I_i = 0$ ($I_k + I_j - I_i = 0$) for non-vanishing $h^{(2)}$ ($h^{(1)}$) reflects conservation of I . The scalars $H_1(H_2)$ have non-vanishing Yukawa couplings to quarks and leptons whenever $\lambda_1 < 0$ ($\lambda_2 < 0$). In this case conservation of helicity ($\lambda_u + \lambda_{u^c} + \lambda_1 = 0$ for $h^{(2)}$, $\lambda_u + \lambda_{u^c} - \lambda_2 = 0$ for $h^{(1)}$) is automatically fulfilled for the couplings (32) and (35). Conservation of total angular momentum requires that only the lowest modes in the harmonic expansion of $H_1(H_2)$ with $\lambda = -\lambda_1$ ($\lambda = -\lambda_2$) have non-vanishing Yukawa couplings to the chiral quarks and leptons⁵⁾.

Let us investigate the $SU(5)$ symmetric solution characterized by $n = 3$, $p = 1$ and $m = 1$. The $SU(2)_G \times U(1)_q$ quantum numbers of the chiral fermions are⁵⁾

$$\begin{aligned}
 \begin{pmatrix} u_L \\ d_L \end{pmatrix} &: \quad \underline{2}_{1/2} + \underline{1}_{-1/2} \\
 u_L^c &: \quad \underline{2}_{1/2} + \underline{1}_{-1/2} \\
 d_L^c &: \quad - \quad \underline{3}_{-1/2} \\
 \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} &: \quad - \quad \underline{3}_{-1/2} \\
 \nu_L^c &: \quad \underline{4}_{1/2} - \\
 \bar{\nu}_L^c &: \quad - \quad \underline{1}_{1/2} \\
 e_L^c &: \quad \underline{2}_{1/2} + \underline{1}_{-1/2}
 \end{aligned} \tag{53}$$

[We have indicated the dimension of the $SU(2)_G$ representation with q as a subscript. The first column comes from the harmonic expansion of ψ_1 and the second from ψ_2 .] In Table 3 we give the quantum number I for the fields t' , c' ... as well as various fermion bilinears. (The prime indicates that these states are not necessarily eigenstates of the mass matrix.) The same quantum numbers for Higgs scalars can be found in Table 4. In this basis, the fermion mass matrices normalized to the mass of the W boson is found

$$\frac{M_U}{M_W} = \begin{pmatrix} t' & c' & u' \\ 2\gamma_1 & 0 & 0 \\ 0 & \frac{4}{13}(\gamma_2^-)^* & \frac{4}{16}(\gamma_2^0)^* \\ 0 & \frac{4}{16}(\gamma_2^0)^* & \frac{4}{13}(\gamma_2^+)^* \end{pmatrix} \begin{pmatrix} t^c \\ c^c \\ u^c \end{pmatrix} \tag{54}$$

$$\frac{M_D}{M_W} = \begin{pmatrix} b' & s' & d' \\ 2\gamma_2^- & 0 & 0 \\ -2\gamma_2^0 & 0 & 0 \\ 2\gamma_2^+ & 0 & 0 \end{pmatrix} \begin{pmatrix} b^c \\ s^c \\ d^c \end{pmatrix} \tag{55}$$

$$\frac{M_E}{M_W} = \begin{pmatrix} & \tau' & \mu' & e' \\ 2\gamma_2^- & -2\gamma_2^0 & 2\gamma_2^+ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tau^c \\ \mu^c \\ e^c \end{pmatrix} \quad (56)$$

The three fermion mass matrices are easily diagonalized. For generic γ_1 , γ_2^- , γ_2^0 and γ_2^+ the unitary transformations $(t, c, u) = (t', c', u')U_U^T$ and $(b, s, d) = (b', s', d')U_D^T$ are different and a non-zero Cabibbo angle between the first two generations is induced. We note that two charged leptons and two charge-1/3 quarks always remain massless. This is due⁵⁾ to the discrete symmetry $\Gamma: \phi_1 \rightarrow \phi_1, \phi_2 \rightarrow -\phi_2$. Let us suppose that the low energy Higgs doublet is essentially given by H_1 with a small admixture of H_2^- . In this limit

$$\gamma_1 \approx 1$$

$$\gamma_2^- = \frac{\alpha}{2} \approx \frac{\alpha}{2} \gamma_1 \quad (57)$$

$$\gamma_2^0 = \gamma_2^+ = 0$$

we have several relations for the fermion masses (at the scale L_0^{-1})

$$m_t = 2M_W \quad (58)$$

$$m_b = m_\tau = \alpha M_W \quad (59)$$

$$m_c = \sqrt{4/3} m_b \quad (60)$$

$$m_s = m_\mu = m_u = m_d = m_e = 0 \quad (61)$$

Also all mixing angles vanish and there is no CP violation. For $\alpha \approx 1/20$ the quarks and leptons divide into three groups: only the top quark has a mass of order M_W . The next group consists of b, c and τ with masses of a few GeV. The third group consists of s, μ , u, d and e which are massless at this level. These qualitative features seem to reflect the structure of the observed fermion mass matrices. The quantitative prediction $m_b = m_\tau$ is in satisfactory agreement

with experiment. However, the charm quark mass $m_c = \sqrt{4/3} m_b$ is too high by a factor of three.

We should remember, however, that the monopole solutions on S^2 are not a satisfactory approximate ground state. In a more realistic approximate ground state, the rotation symmetry on the sphere will be broken. Unless protected by some symmetry, the quantitative mass relations are expected to change. This is certainly the case for relations (58) and (60). In contrast, corrections to $m_b = m_\tau$ will be proportional to the scale where both $SU(5)$ and $SU(4)_C$ are spontaneously broken and may be suppressed²⁾. What are the conditions for the qualitative features like $m_t \gg m_b, m_c, m_\tau \gg m_s, m_\mu, m_d, m_u, m_e$ to persist? This brings us to the question why certain fermion masses are much smaller than others, the topic of the next section.

5. THE HIERARCHY OF FERMION MASSES

The ratio between the electron mass and the mass of the tau-lepton is about 3×10^{-4} . Similar small ratios are observed for m_u/m_t or m_d/m_b . Ratios like m_s/m_b or m_μ/m_τ are of order $1/40$ – $1/20$. We call this the hierarchy of fermion masses. How to understand such small numbers as 10^{-4} in a higher dimensional framework? In four-dimensional grand unified theories, Yukawa couplings are free parameters and we may take them as small as we want. In higher dimensional theories, however, the four-dimensional Yukawa couplings become calculable. As we have seen in the precedent section, they all tend to be of the same order as the gauge coupling g unless they vanish due to some symmetries. If both electron and tau-lepton have non-vanishing Yukawa couplings to the same Higgs scalars, there seems to be little chance to obtain $m_e \ll m_\tau$. [Here we work in a basis where the Higgs scalars are classified according to some symmetries which will be spontaneously broken like $SU(5) \times U(1) \times SU(2) \times U(1)$ in the preceding section. The low energy weak Higgs doublet will be a linear combination of such scalars.] Things are different if the Higgs scalar, whose vev gives the leading contribution to m_τ , does not couple to muons or electrons. Muon and electron must then get their masses from other Higgs scalars. The admixture of these scalars to the low energy doublet may be small ($\gamma_i \ll 1$) and a fermion mass hierarchy could be explained.

At first sight this seems only to shift the problem from explaining small Yukawa couplings to an explanation of small mixings γ_i . The second problem, however, is easier to treat. If two scalars have different quantum numbers with respect to some spontaneously broken symmetry G , their mixing must be proportional to some power of M_G , the scale where G is spontaneously broken. Typical mixing coefficients γ_i are then of order $(M_G L_O)^{\tilde{N}}$ with $\tilde{N}$ some model-dependent positive integer and L_O^{-1} the scale of spontaneous compactification. Even a moderate "fine structure" in the scales relevant for compactification ($M_G \approx 1/20 L_O^{-1}$, say) may lead to rather small γ_i and therefore rather small fermion mass ratios.

In four-dimensional grand unified theories the fermion mass hierarchy is usually related to the generation problem: the first generation has the lightest masses, the masses of the second generation are in an intermediate range and the third generation consists of the heavy quarks and leptons. This pattern is underlined by the smallness of the mixing angles between different generations. Looking a bit closer, however, the hierarchy of masses does not follow exactly the generation pattern. The fermion masses rather divide into four groups:

$$\begin{aligned}
 &m_t \\
 &m_b, m_c, m_\tau \\
 &m_s, m_\mu \\
 &m_d, m_u, m_e
 \end{aligned}
 \tag{62}$$

There is roughly a factor 1/20 between the overall mass scales of the different groups. This suggests that a small quantity $M_G L_O$ appears with different powers $\tilde{N}$ in the mixing coefficients γ_i . As we have seen in the previous section, there is no reason why the fermion mass hierarchy should follow exactly the generation pattern. The structure of the fermion mass matrices is rather dictated by the $SU(2)_G \times U(1)_q$ quantum numbers. In general, the various quarks and leptons within a generation have different transformation properties with respect to the generation group $SU(2)_G \times U(1)_q$.

Let us now come back to our model. There are three types of symmetries which can prevent mixings between different Higgs scalars in the limit where they remain unbroken. The first is the discrete symmetry T transforming $\phi_1 \rightarrow \phi_1$, $\phi_2 \rightarrow -\phi_2$. This tells us that the scalars contained in the harmonic expansion of the six-dimensional gauge bosons can only induce fermion masses

consistent with unbroken $SU(3)_C \times U(1)_{em}$ by pairing a quark u within ϕ_1 with an antiquark u^C also within ϕ_1 . Although for both spinors ϕ_1 and ϕ_2 together the number of u equals the number of u^C , there may be an asymmetry between quarks and antiquarks within ϕ_1 or ϕ_2 alone. Thus Γ symmetry can forbid mass terms, and in the above example with $n = 3$, $m = p = 1$, this is the case for s , μ , d and e . The second symmetry is $SU(2)_G \times U(1)_q$ generation symmetry or a suitable subgroup of it. For example, in the limit of unbroken $SU(2)_G$ the scalars H_2^- and H_1 cannot mix. Finally, there is the symmetry $U(1)_R$ which prevents mixing between H_1 and H_2 while allowing mixings between H_1^* and H_2 .

Before going on with a discussion how these different symmetries may be broken by the introduction of a six-dimensional scalar field⁴⁾, let us briefly investigate the structure of the mass matrices for other values of n , p and m . This will illustrate that a semi-realistic mass pattern as in the previous section ($m_t \gg m_b, m_c, m_\tau \gg m_s, m_\mu, m_d, m_u, m_e$) is far from trivial. We start with three generation models ($n = 3$): using the general results of Ref. 5) for the $SU(2)_G \times U(1)_q$ transformation properties of chiral fermions for various n , p and m , one finds that for $p \geq 3$ the symmetry Γ forbids mass terms for all charge-1/3 quarks or for all charge 2/3 quarks depending on $m > 0$ or $m < 0$. If we want that quarks other than top and charm acquire a mass consistent with Γ symmetry, we remain with $n = 3$, $p = 1$. Now we find that for $m \geq 2$ the field H_1 has no Yukawa couplings to chiral quarks. Both m_t and m_b are induced by the same field and therefore in the same order of magnitude. For $m < 0$ the symmetry Γ forbids masses for all charged leptons and relations like $m_b = m_\tau$ are excluded. The only remaining case with three generations is the example of the previous section with $n = 3$, $p = m = 1$. For four-generation models ($n = 4$), the case $p \geq 4$ leads again to a situation where either all up-type quarks or all down-type quarks remain massless due to Γ symmetry. For $p = 0$, Γ symmetry implies m complete massless generations. However, there is an additional discrete symmetry⁵⁾ for $p = 0$ which implies that the masses of charge 2/3 and charge-1/3 quarks are equal. Even if this symmetry is spontaneously broken by six-dimensional scalar fields, a hierarchy $m_t \gg m_b$ is difficult to understand. There may be an exception for $n = 4$, $p = 0$, $m = 3$ where all quarks and leptons of the first three generations must acquire their masses from effects violating Γ symmetry. What remains is the case $n = 4$, $p = 2$: for $m < 1$, no mass term is allowed for charged leptons. For $m \geq 3$, only H_2 has Yukawa couplings to chiral fermions. Only three up-type quarks, one down-type quark and between two and four charged leptons can acquire a mass consistent with Γ symmetry. There is no reason to expect relations like $m_b = m_\tau$. We are left with the $SU(5)$ symmetric

solution $n = 4$, $m = p = 2$. This leads to a similar structure of the mass matrix as the three-generation example in Section 4. However, mass relations now mainly involve fermions of the unobserved fourth generation. We conclude that the qualitative structure of the mass matrix is a very restrictive criterion for model building, leaving only very few possibilities in the present model.

In the remainder of this section we discuss some qualitative features of the breaking of Γ symmetry and $SU(2)_G \times U(1)_q$ symmetry for the solution $n = 3$, $p = m = 1$. We find that in addition to the mass splitting (58)-(61), there is a mechanism for a mass split between m_s, m_μ and m_d, m_u, m_e . This gives us some hope to obtain the mass pattern (62). The best candidates for a breaking of Γ symmetry are six-dimensional scalar fields with Yukawa couplings to the spinors ϕ_1 and ϕ_2 . Possible representations are the vector, third and fifth rank anti-symmetric tensor representations of $SO(12)$ with dimension 12, 220 and 792, respectively. At most, three additional independent Yukawa couplings are possible in our six-dimensional model. The components of these fields with possible Yukawa couplings to the four-dimensional chiral quarks and leptons (not mirrors) have the following $SO(10) \times U(1)_q$ transformation properties:

$$\begin{aligned} 12 &> (10, 0) \\ 220 &> (120, 0) + (10, 0) \\ 792 &> (126 + \bar{126}, 0) + (120, 0) \end{aligned} \tag{63}$$

All three fields have $U(1)_q$ charge $q = 0$. Comparison with Table 3 shows that Yukawa couplings are only possible to the bilinears $t'c^c, c't^c, t'u^c, u't^c, s's^c, d'd^c, s'b^c, d'b^c, s'd^c, d's^c, \mu'\mu^c, e'e^c, \tau'\mu^c, \tau'e^c, \mu'e^c$ and $e'\mu^c$. These additional terms in the mass matrices can now induce masses for all quarks and leptons.

Let us restrict our discussion to the scalar in the 792-dimensional representation. This is the minimal setting where the gauge group can be broken to $SU(3)_C \times SU(2)_L \times U(1)_Y$ for the approximate ground state. We add a scalar piece to the six-dimensional action⁴⁾:

$$S_\phi = -\int d^6x \hat{g}_6^{1/2} \left\{ -\frac{1}{2} D_\mu \phi D^\mu \phi + (\hat{f} \bar{\psi}_1 \phi \psi_2 + h.c.) + V(\phi) \right\} \tag{64}$$

with ϕ in the 792-dimensional representation, the Yukawa coupling $\hat{f}$ a new free parameter and the potential $V(\phi)$ so far unspecified. With respect to $SO(10) \times U(1)_q$, 792 decomposes into

$$792 \rightarrow (126 + \overline{126}, 0) + (210, \pm 1) + (120, 0) \quad (65)$$

The 210 contains three $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlets transforming as (1,1,1), (15,1,1) and (15,1,3) under $SU(4)_C \times SU(2)_L \times SU(2)_R$ or 1, 24, 75 under $SU(5)$. The $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet in 126 transforms as (10,1,3) under $SU(4)_C \times SU(2)_L \times SU(2)_R$ and as a singlet under $SU(5)$. These are the fields whose vacuum expectation values can induce the spontaneous symmetry breaking

$$SU(5) \times \tilde{U}(1) \times SU(2)_G \times U(1)_q \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \quad (66)$$

We assume that the approximate ground state is in the same $SU(3)_C \times SU(2)_L \times U(1)_Y$ deformation class as the monopole solution with $n = 3$, $m = p = 1$. We can then use harmonic expansion on the monopole solution on S^2 to gain some insight into the qualitative structure of contributions to the mass matrices induced by the Yukawa coupling $\hat{f}$. The 126 contains two colour singlet $SU(2)_L$ doublets belonging to (15,2,2) of $SU(4)_C \times SU(2)_L \times SU(2)_R$ whose electrically neutral components are denoted by h_1^{126} and h_2^{126} . The 120 contains a (15,2,2) with electrically neutral colour singlets h_1^{120} and $h_2^{120} = (h_1^{120})^*$ and a (1,2,2) with neutral fields $\tilde{h}_1^{120}$ and $\tilde{h}_2^{120} = (\tilde{h}_1^{120})^*$. We give various quantum numbers of these fields in Table 2. The helicity for scalars in a monopole configuration is

$$\lambda = -\frac{1}{2} N \quad (67)$$

The harmonic expansion for all these fields contains $SU(2)_G$ angular momentum $\lambda = 1/2, 3/2, 5/2 \dots$. Comparing with the quantum numbers of fermion bilinears in Table 3 we find that only the doublets and quartets can have non-vanishing Yukawa couplings with the chiral quarks and leptons. The $SU(2)_G \times U(1)_q$ quantum numbers of all scalars with Yukawa couplings are listed in Table 4. Comparison with Table 3 together with I_{3R} conservation tells us the allowed Yukawa couplings for the various fields.

At this stage, predictions on the fermion mass matrices appear to be rather hopeless unless we have some information on the mixing coefficients γ . These coefficients are determined to a large extent by the vacuum expectation values of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet fields in 210 and 126. Let us assume

that the symmetry breaking (66) takes place in two steps. The first step, mediated by a vacuum expectation value of 210 with definite $SU(2)_G$ spin I, breaks

$$SU(5) \times \tilde{U}(1) \times SU(2)_G \times U(1)_q \rightarrow SU(3)_C \times SU(2)_L \times U(1)_R \times U(1)_{B-L} \times U(1)_{\tilde{q}} \quad (68)$$

The symmetry $U(1)_R \times U(1)_{\tilde{q}}$ will then restrict possible mixings of scalars. The $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet fields in 210 have all $q = \pm 1$, $I_{3R} = 0$, $Y_{B-L} = 0$, $N = \pm 3$, $\lambda = \mp 3/2$. $SU(2)_G$ angular momentum ℓ is therefore half integer and starts with $\ell = 3/2$. Assume the vacuum expectation value responsible for (68) has $q = 1$, $I = -1/2$. The charge

$$\tilde{q} = I + \frac{1}{2} q \quad (69)$$

is then conserved. The symmetry Γ transforms $\phi \rightarrow -\phi$ and is broken by this vacuum expectation value. The monopole solutions also have a conserved CP symmetry⁵⁾. This is broken by $\langle 210 \rangle^* \neq \langle 210 \rangle$ and CP violating phases are expected in the mass matrices.

The symmetries $U(1)_R \times U(1)_{\tilde{q}}$ imply that the only Higgs fields which can mix at this stage with H_1 must have $I_{3R} = -1/2$, $I + 1/2q = -1/2$. These fields are H_2^- , h_2 ($I = -1/2$) and h_1 ($I = 1/2$)*. In this approximation the mass matrices have the form

$$M_U = \begin{pmatrix} a_{33} \langle H_1 \rangle & a_{32} \langle h \rangle & 0 \\ a_{23} \langle h \rangle & a_{22} \langle H_2^- \rangle & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (70)$$

$$M_D = \begin{pmatrix} b_{33} \langle H_2^- \rangle & 0 & b_{31} \langle h \rangle^* \\ 0 & b_{22} \langle h \rangle^* & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (71)$$

$$M_E = \begin{pmatrix} c_{33} \langle H_2^- \rangle & 0 & 0 \\ 0 & c_{22} \langle h \rangle^* & 0 \\ c_{13} \langle h \rangle^* & 0 & 0 \end{pmatrix} \quad (72)$$

Here h is an appropriate combination of fields h_1 and h_2^* . Once the approximate ground state and the corresponding harmonics for chiral fermions and scalar fields are known, the coefficients a_{33} etc., can be calculated. The scale of the coefficients in front of $\langle h \rangle$ is given by the six-dimensional Yukawa coupling $\hat{f}$ and is therefore a free parameter. In the limit of unbroken $U(1)_R \times U(1)_q$ symmetry, one finds

$$m_d = m_u = m_e = 0 \quad (73)$$

In addition to the masses discussed in the previous section, we now have also non-vanishing m_s and m_μ . It seems not to be difficult to arrange for a scale of a few hundred MeV for these masses using the free parameter $\hat{f}$. We also note the appearance of non-vanishing mixing angles from the diagonalization of M_U and M_D .

Finally, the symmetry $U(1)_R \times U(1)_{B-L} \times U(1)_q$ must be broken to $U(1)_Y$. This can be done by fields in 210 with $q = 1$, $I \neq -1/2$ and by the $SU(5)$ singlet in 126. The scale of this breaking should not be too low, otherwise the masses m_μ , m_d and m_e would come out too small and masses for left-handed neutrinos²³⁾ would be too large.

6. CONCLUSIONS

We have discussed the general method how to calculate quark and lepton masses in higher dimensional theories. This method was illustrated in a six-dimensional model with gauge group $SO(12)$. In this model we have calculated the fermion mass matrices for all $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric monopole solutions on an internal sphere S^2 . We expect the qualitative features of these mass matrices to be valid for a more realistic $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric approximate ground state. Already at this qualitative level this model is rather predictive. Most of the monopole solutions lead to a qualitatively unac-

ceptable fermion mass spectrum and should be discarded. Only a few solutions remain in agreement with the structure of fermion mass matrices. This shows how the observed hierarchy of fermion masses becomes an important restriction for realistic higher dimensional models.

We have proposed a semi-realistic six-dimensional $SO(12)$ model with scalars in the fifth rank antisymmetric tensor representation of $SO(12)$. It includes fields with appropriate quantum numbers for a realistic spontaneous symmetry breaking. Interesting fermion mass matrices are obtained for solutions in the same $SU(3)_C \times SU(2)_L \times U(1)_Y$ -deformation class as the $SU(5)$ symmetric monopole solution with $n = 3$, $m = p = 1$. These solutions lead to three generations of quarks and leptons. In the limit of unbroken $U(1)_R \times U(1)_q$ symmetry, three types of scalar fields couple to the chiral fermions. The scalar H_1 only couples to the top quark, H_2^- contributes to m_b , m_c and m_τ whereas h gives masses to s and μ and induces non-zero mixing angles. In this limit the up and down quark and the electron remain massless. The first generation will finally get a mass from effects breaking $U(1)_R \times U(1)_q$.

The scalars H_1 , H_2^- and h have different quantum numbers with respect to the continuous local generation group $SU(2)_G \times U(1)_q$ and discrete symmetry Γ . All mixings between these scalars are suppressed by powers of the ratio between the scale of symmetry breaking of $SU(2)_G \times U(1)_q \times \Gamma$ and the compactification scale. This gives us hope that vacuum expectation values $\langle H_1 \rangle \gg \langle H_2^- \rangle \gg \langle h \rangle$ can be obtained, leading to the mass pattern $m_t \gg m_b, m_c, m_\tau \gg m_s, m_\mu \gg m_d, m_u, m_e$.

In conclusion, we propose that differences in symmetry breaking scales around the compactification scale lead to a suppression of mixings between different scalar doublets. This in turn implies small ratios between fermion masses. In other words, the fine structure of scales for spontaneous compactification is responsible for the observed structure in the fermion mass matrices. If this program succeeds, the particular chemistry of our world, which crucially depends on the masses m_u , m_d and m_e , would finally be traced back to some fine structure of scales around 10^{17} GeV.

ACKNOWLEDGEMENTS

The author would like to thank M. Gell Mann and the Theory Group at CALTECH, where part of this work was performed, for their hospitality.

APPENDIX A

SO(12) spinor representations

The Clifford algebra in twelve dimensions is obtained from the twelve anti-commuting Dirac matrices Γ_A :

$$\{\Gamma_A, \Gamma_B\} = 2\gamma_{AB} = -2\delta_{AB} \quad ; \quad A, B = 1 \dots 12 \quad (A1)$$

The Γ_A are 64×64 matrices and their commutators form the generators Σ_{AB} of SO(12) in the reducible Dirac representation

$$\Sigma_{AB} = -\frac{1}{4} [\Gamma_A, \Gamma_B] \quad (A2)$$

We will use Hermitian generators

$$T_{AB} = -i \Sigma_{AB} \quad (A3)$$

which obey the usual SO(12) commutation relations

$$[T_{AB}, T_{CD}] = i\delta_{AC} T_{BD} - i\delta_{AD} T_{BC} + i\delta_{BD} T_{AC} - i\delta_{BC} T_{AD} \quad (A4)$$

The two inequivalent irreducible pseudo-real spinor representations of SO(12) have dimension 32 and are obtained from the Dirac representation ψ , using the standard Weyl projection operators:

$$\psi_1 = \psi(32_1) = \frac{1}{2} (1 + \bar{\Gamma}_{12}) \psi \quad (A5)$$

$$\psi_2 = \psi(32_2) = \frac{1}{2} (1 - \bar{\Gamma}_{12}) \psi$$

$$\bar{\Gamma}_{12} = -\Gamma_1 \Gamma_2 \dots \Gamma_{12} \quad (A6)$$

$$\bar{\Gamma}_{12}^2 = 1 \quad , \quad \{\bar{\Gamma}_{12}, \Gamma_A\} = 0$$

In our conventions, the Dirac matrices are anti-Hermitian and $\bar{\Gamma}_{12}$ is Hermitian

$$\Gamma_A^\dagger = -\Gamma_A, \quad \bar{\Gamma}_{12}^\dagger = \bar{\Gamma}_{12} \quad (A7)$$

We next give an explicit representation for the matrices Γ_A in a basis which is convenient for the descriptions of quarks and leptons. With respect to the subgroup $SO(10) \times U(1)_G$, the $SO(12)$ spinor representations decompose

$$\begin{aligned} 32_1 &\rightarrow 16_{1/2} + \bar{16}_{-1/2} \\ 32_2 &\rightarrow 16_{-1/2} + \bar{16}_{1/2} \end{aligned} \quad (A8)$$

The subscript indicates the Abelian charge q . We write the 64 component Dirac spinor ψ in the form

$$\psi = \begin{pmatrix} \psi(32_1) \\ \psi(32_2) \end{pmatrix} = \begin{pmatrix} \psi(16_{1/2}) \\ \psi(\bar{16}_{-1/2}) \\ \psi(16_{-1/2}) \\ \psi(\bar{16}_{1/2}) \end{pmatrix} \quad (A9)$$

with

$$\begin{aligned} \psi(16)^T &= (u_1, u_2, u_3, v, d_1, d_2, d_3, e, \bar{u}_1^c, \bar{u}_2^c, \bar{u}_3^c, \bar{v}^c, \bar{d}_1^c, \bar{d}_2^c, \bar{d}_3^c, \bar{e}^c) \\ \psi(\bar{16})^T &= (\bar{\bar{u}}_1^c, \bar{\bar{u}}_2^c, \bar{\bar{u}}_3^c, \bar{\bar{v}}^c, \bar{\bar{d}}_1^c, \bar{\bar{d}}_2^c, \bar{\bar{d}}_3^c, \bar{\bar{e}}^c, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{v}, \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{e}) \end{aligned} \quad (A10)$$

The fields in $\bar{16}$ have the quantum numbers of mirror particles. In the basis (A9) and (A10) the Dirac matrices are given in terms of direct products of Pauli matrices

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (A11)$$

as

$$\begin{aligned}
\Gamma_1 &= i \tau_0 \otimes \tau_0 \otimes \tau_2 \otimes \tau_3 \otimes \tau_1 \otimes \tau_1 \\
\Gamma_2 &= i \tau_0 \otimes \tau_0 \otimes \tau_1 \otimes \tau_0 \otimes \tau_1 \otimes \tau_1 \\
\Gamma_3 &= i \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_0 \otimes \tau_1 \otimes \tau_1 \\
\Gamma_4 &= i \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_2 \otimes \tau_1 \\
\Gamma_5 &= i \tau_2 \otimes \tau_1 \otimes \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes \tau_1 \\
\Gamma_6 &= i \tau_1 \otimes \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes \tau_1 \otimes \tau_1 \\
\Gamma_7 &= i \tau_3 \otimes \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes \tau_1 \otimes \tau_1 \\
\Gamma_8 &= i \tau_0 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes \tau_1 \\
\Gamma_9 &= i \tau_2 \otimes \tau_0 \otimes \tau_2 \otimes \tau_1 \otimes \tau_1 \otimes \tau_1 \\
\Gamma_{10} &= i \tau_2 \otimes \tau_3 \otimes \tau_2 \otimes \tau_2 \otimes \tau_1 \otimes \tau_1 \\
\Gamma_{11} &= i \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_1 \\
\Gamma_{12} &= i \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_2
\end{aligned} \tag{A12}$$

The six generators of the Cartan subalgebra of $SO(12)$ read

$$\begin{aligned}
T_{12} &= -\frac{1}{2} \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_3 \otimes \tau_0 \otimes \tau_0 \\
T_{34} &= \frac{1}{2} \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_0 \otimes \tau_3 \otimes \tau_0 \\
T_{56} &= -\frac{1}{2} \tau_3 \otimes \tau_3 \otimes \tau_0 \otimes \tau_3 \otimes \tau_0 \otimes \tau_0 \\
T_{78} &= \frac{1}{2} \tau_3 \otimes \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_0 \otimes \tau_0 \\
T_{9,10} &= \frac{1}{2} \tau_0 \otimes \tau_3 \otimes \tau_0 \otimes \tau_3 \otimes \tau_0 \otimes \tau_0 \\
T_{11,12} &= \frac{1}{2} \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_3
\end{aligned} \tag{A13}$$

In our basis the standard assignment of left-handed and right-handed weak isospin and of the B-L charge is given by

$$\begin{aligned}
 I_{3L} &= -\frac{1}{2} (T_{12} - T_{34}) \\
 I_{3R} &= -\frac{1}{2} (T_{12} + T_{34}) \\
 Y_{B-L} &= \frac{2}{3} (T_{56} + T_{78} + T_{9,10})
 \end{aligned}
 \tag{A14}$$

The Abelian quantum numbers of the representation 32_1 are shown in Table 1. For the representation 32_2 , the sign of $q = T_{11,12}$ has to be reversed. Our basis is adapted to the subgroup $SO(6) \times SO(4) \equiv SU(4)_C \times SU(2)_L \times SU(2)_R$ of $SO(10)$. The generators of $SO(10)$ are given by T_{MN} ; $M, N = 1 \dots 10$, the $SU(2)_L \times SU(2)_R$ generators are T_{MN} ; $M, N = 1 \dots 4$ and $SU(4)_C$ is obtained for T_{MN} ; $M, N = 5 \dots 10$. The projection operators for representations of $SU(2)_L \times SU(2)_R$, $SU(4)_C$, $SO(10)$ and $U(1)_C$ respectively are given similar to (A5) by

$$\begin{aligned}
 \bar{\Gamma}_4 &= -\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 = -\tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_3 \otimes \tau_0 \\
 \bar{\Gamma}_6 &= -i \Gamma_5 \Gamma_6 \dots \Gamma_{10} = -\tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_0 \otimes \tau_0 \\
 \bar{\Gamma}_{10} &= \bar{\Gamma}_4 \bar{\Gamma}_6 = \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_0 \\
 \bar{\Gamma}_2 &= i \Gamma_{11} \Gamma_{12} = \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_3 \otimes \tau_3 \\
 \bar{\Gamma}_{12} &= \bar{\Gamma}_{10} \bar{\Gamma}_2 = \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_3
 \end{aligned}
 \tag{A15}$$

We finally define the $SO(12)$ charge conjugation matrix B_{12} by

$$(\Gamma_A)^* = -B_{12} \Gamma_A B_{12}^{-1}
 \tag{A16}$$

$$B_{12}^* B_{12} = -1$$

In our convention we also have

$$(\Gamma_A)^T = B_{12} \Gamma_A B_{12}^{-1}
 \tag{A17}$$

and

$$B_{12} = B_{12}^* = -B_{12}^{-1} = -B_{12}^T = -B_{12}^T
 \tag{A18}$$

One finds

$$\begin{aligned} B_{12} &= \Gamma_1 \Gamma_4 \Gamma_5 \Gamma_8 \Gamma_{10} \Gamma_{12} \\ &= i \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_1 \otimes \tau_2 \otimes \tau_3 \end{aligned} \tag{A19}$$

APPENDIX B

Generalized spherical harmonics

In this Appendix we collect the formulae for generalized spherical harmonics used in this paper. We have adapted our conventions to the book of Edmonds²⁴⁾.

The harmonic expansion of a field with helicity λ on a sphere is

$$\psi^{(\lambda)}(\vartheta, \varphi) = \sum_{\ell k} a_{\ell k} D_{\ell k}^{(\lambda)}(\vartheta, \varphi) \quad (B1)$$

with $D_{\ell k}^{(\lambda)}$ defined by

$$-i \partial_{\varphi} D_{\ell k}^{(\lambda)} = \bar{m} D_{\ell k}^{(\lambda)} = (k - \lambda) D_{\ell k}^{(\lambda)} \quad (B2)$$

$$\left\{ -\frac{1}{\sin^2 \vartheta} \partial_{\vartheta} (\sin^2 \vartheta \partial_{\vartheta}) + \frac{\bar{m}^2}{\sin^2 \vartheta} + 2\lambda(\lambda + \bar{m}) \frac{(1 - \cos \vartheta)}{\sin^2 \vartheta} \right\} D_{\ell k}^{(\lambda)} \\ = \ell(\ell + 1) D_{\ell k}^{(\lambda)} \quad (B3)$$

The quantum numbers ℓ , $\bar{m}$ and k correspond to total angular momentum, the third component of "orbital" angular momentum and the third component of total angular momentum. They have the range

$$\ell = |\lambda|, |\lambda| + 1, \dots \quad (B4)$$

$$k = \lambda + \bar{m} = -\ell, -\ell + 1, \dots, \ell$$

We normalize the functions $D_{\ell k}^{(\lambda)}$ according to

$$\int d^2 y g_2^{\frac{1}{2}} (D_{\ell k}^{(\lambda)})^* D_{\ell' k'}^{(\lambda)} = \delta_{\ell \ell'} \delta_{k k'} \quad (B5)$$

with $d^2 y g_2^{\frac{1}{2}} = L_0^2 d(\cos \vartheta) d\varphi$ the volume element on the sphere with radius L_0 .

The harmonics $D_{\lambda k}^{(\lambda)}$ are related to the reduced matrix elements of rotations²⁴⁾

$$D_{\ell k}^{(\lambda)}(\vartheta, \varphi) = \left(\frac{2\ell+1}{4\pi L_0^2} \right)^{1/2} d_{k\lambda}^{\ell}(\vartheta) \exp i m \varphi \quad (B6)$$

The real functions $d_{k\lambda}^{\ell}(\theta)$ have the symmetries

$$d_{k\lambda}^{\ell}(\vartheta) = d_{-\lambda, -k}^{\ell}(\vartheta) = (-1)^{k-\lambda} d_{\lambda k}^{\ell}(\vartheta) = (-1)^{k-\lambda} d_{-\lambda, -k}^{\ell}(\vartheta) \quad (B7)$$

and are normalized (without summation over k and λ)

$$\int_{-1}^1 d(\cos \vartheta) d_{k\lambda}^{\ell} d_{k'\lambda'}^{\ell'} = \frac{2}{2\ell+1} \delta_{\ell\ell'} \quad (B8)$$

For $\lambda = 0$ one recovers the Legendre functions

$$d_{k0}^{\ell}(\vartheta) = \left(\frac{(\ell-k)!}{(\ell+k)!} \right)^{1/2} P_k^{\ell}(\vartheta) \quad (B9)$$

and $d_{00}^0(\theta) = 1$. We also note that the symmetries (B7) imply

$$(D_{\ell k}^{(\lambda)})^* = (-1)^{k-\lambda} D_{\ell, -k}^{(-\lambda)} \quad (B10)$$

The integral over three generalized spherical harmonics can be expressed in terms of the Wigner 3j symbols:

$$\begin{aligned} & \int d^2 y g_2^{1/2} D_{\ell_1 k_1}^{(\lambda_1)} D_{\ell_2 k_2}^{(\lambda_2)} D_{\ell_3 k_3}^{(\lambda_3)} \\ &= \left(\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi L_0^2} \right)^{1/2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \end{aligned} \quad (B11)$$

This integral vanishes unless

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{B12}$$

$$k_1 + k_2 + k_3 = 0$$

and total angular momenta obey the restrictions

$$|\ell_1 - \ell_2| \leq \ell_3 \leq |\ell_1 + \ell_2| \tag{B13}$$

(similar for all other combinations).

TABLE 1: Abelian quantum numbers for SO(12) spinor representation 32_1 *)

32_1 *)	T_{12}	T_{34}	T_{56}	T_{78}	$T_{9,10}$	$T_{11,12}$ *)	I_{3L}	I_{3R}	Y_{B-L}	N *)
u_1	-1/2	1/2	-1/2	1/2	1/2	1/2	1/2	0	1/3	$n/2+p/2$
u_2	-1/2	1/2	1/2	-1/2	1/2	1/2	1/2	0	1/3	$n/2+p/2$
u_3	-1/2	1/2	1/2	1/2	-1/2	1/2	1/2	0	1/3	$n/2+p/2$
v	-1/2	1/2	-1/2	-1/2	-1/2	1/2	1/2	0	-1	$n/2-3p/2$
d_1	1/2	-1/2	-1/2	1/2	1/2	1/2	-1/2	0	1/3	$n/2+p/2$
d_2	1/2	-1/2	1/2	-1/2	1/2	1/2	-1/2	0	1/3	$n/2+p/2$
d_3	1/2	-1/2	1/2	1/2	-1/2	1/2	-1/2	0	1/3	$n/2+p/2$
e	1/2	-1/2	-1/2	-1/2	-1/2	1/2	-1/2	0	-1	$n/2-3p/2$
<hr/>										
16										
u_1^c	1/2	1/2	1/2	-1/2	-1/2	1/2	0	-1/2	-1/3	$n/2-p/2+m$
u_2^c	1/2	1/2	-1/2	1/2	-1/2	1/2	0	-1/2	-1/3	$n/2-p/2+m$
u_3^c	1/2	1/2	-1/2	-1/2	1/2	1/2	0	-1/2	-1/3	$n/2-p/2+m$
v^c	1/2	1/2	1/2	1/2	1/2	1/2	0	-1/2	1	$n/2+3p/2+m$
d_1^c	-1/2	-1/2	1/2	-1/2	-1/2	1/2	0	1/2	-1/3	$n/2-p/2-m$
d_2^c	-1/2	-1/2	-1/2	1/2	-1/2	1/2	0	1/2	-1/3	$n/2-p/2-m$
d_3^c	-1/2	-1/2	-1/2	-1/2	1/2	1/2	0	1/2	-1/3	$n/2-p/2-m$
e^c	-1/2	-1/2	1/2	1/2	1/2	1/2	0	1/2	1	$n/2+3p/2-m$
<hr/>										
16										
u_1^c	-1/2	-1/2	-1/2	1/2	1/2	-1/2	0	1/2	1/3	$-n/2+p/2-m$
u_2^c	-1/2	-1/2	1/2	-1/2	1/2	-1/2	0	1/2	1/3	$-n/2+p/2-m$
u_3^c	-1/2	-1/2	1/2	1/2	-1/2	-1/2	0	1/2	1/3	$-n/2+p/2-m$
v^c	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	0	1/2	-1	$-n/2-3p/2-m$
d_1^c	1/2	1/2	-1/2	1/2	1/2	-1/2	0	-1/2	1/3	$-n/2+p/2+m$
d_2^c	1/2	1/2	1/2	-1/2	1/2	-1/2	0	-1/2	1/3	$-n/2+p/2+m$
d_3^c	1/2	1/2	1/2	1/2	-1/2	-1/2	0	-1/2	1/3	$-n/2+p/2+m$
e^c	1/2	1/2	-1/2	-1/2	-1/2	-1/2	0	-1/2	-1	$-n/2-3p/2+m$
<hr/>										
16										
u_1	1/2	-1/2	1/2	-1/2	-1/2	-1/2	-1/2	0	-1/3	$-n/2-p/2$
u_2	1/2	-1/2	-1/2	1/2	-1/2	-1/2	-1/2	0	-1/3	$-n/2-p/2$
u_3	1/2	-1/2	-1/2	-1/2	1/2	-1/2	-1/2	0	-1/3	$-n/2-p/2$
v	1/2	-1/2	1/2	1/2	1/2	-1/2	-1/2	0	1	$-n/2+3p/2$
d_1	-1/2	1/2	1/2	-1/2	-1/2	-1/2	1/2	0	-1/3	$-n/2-p/2$
d_2	-1/2	1/2	-1/2	1/2	-1/2	-1/2	1/2	0	-1/3	$-n/2-p/2$
d_3	-1/2	1/2	-1/2	-1/2	1/2	-1/2	1/2	0	-1/3	$-n/2-p/2$
e	-1/2	1/2	1/2	1/2	1/2	-1/2	1/2	0	1	$-n/2+3p/2$

*) For the representation 32_2 : reverse sign of all eigenvalues of $T_{11,12}$ and of n in the column for N .

TABLE 2 : SO(12) Quantum numbers for neutral components of Higgs doublets

	I_{3L}	I_{3R}	q	Y_{B-L}	$SU(4)_C \times SU(2)_L$ $\times SU(2)_R$ rep	SU(5) rep	SO(10) rep
H_1	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	(1,2,2)	$\bar{5}$	10
H_2	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	(1,2,2)	$\bar{5}$	10
h_1^{126}	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	(15,2,2)	45	126
h_2^{126}	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(15,2,2)	$\bar{5}$	126
h_1^{120}	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	(15,2,2)	5,45	120
$(h_1^{120})^* = h_2^{120}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(15,2,2)	$\bar{5}, \bar{45}$	120
$\tilde{h}_1^{120}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	(1,2,2)	5,45	120
$(\tilde{h}_1^{120})^* = \tilde{h}_2^{120}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,2,2)	$\bar{5}, \bar{45}$	120

TABLE 3 : $SU(2)_G \times U(1)_q$ quantum numbers for chiral fermions ($n=3, p=m=1$)

	t'	$t'^{C'}$	c'	$c'^{C'}$	u'	$u'^{C'}$	b'	$b'^{C'}$	s'	$s'^{C'}$	d'	$d'^{C'}$	τ'	$\tau'^{C'}$	μ'	$\mu'^{C'}$	e'	$e'^{C'}$
$2\ell+1$	1	1	2	2	2	2	1	3	2	3	2	3	3	1	3	2	3	2
I	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	1	0	0	$\frac{1}{2}$	-1	$-\frac{1}{2}$
q	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

	$t't^{C'}$	$c'c^{C'}$	$u'u^{C'}$	$t'c^{C'}$	$c't^{C'}$	$t'u^{C'}$	$u't^{C'}$	$c'u^{C'}$	$u'c^{C'}$
$2\ell+1$	1	3	3	2	2	2	2	3,1	3,1
I	0	-1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
q	-1	1	1	0	0	0	0	1	1

	$b'b^{C'}$	$s's^{C'}$	$d'd^{C'}$	$b's^{C'}$	$s'b^{C'}$	$b'd^{C'}$	$d'b^{C'}$	$s'd^{C'}$	$d's^{C'}$
$2\ell+1$	3	4,2	4	3	4	3	4,2	4,2	4,2
I	1	$\frac{1}{2}$	$-3/2$	0	$3/2$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
q	-1	0	0	-1	0	-1	0	0	0

	$\tau'\tau^{C'}$	$\mu'\mu^{C'}$	$e'e^{C'}$	$\tau'\mu^{C'}$	$\mu'\tau^{C'}$	$\tau'e^{C'}$	$e'\tau^{C'}$	$\mu'e^{C'}$	$e'\mu^{C'}$
$2\ell+1$	3	4,2	4	4	3	4,2	3	4,2	4,2
I	1	$\frac{1}{2}$	$-3/2$	$3/2$	0	$\frac{1}{2}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
q	-1	0	0	0	-1	0	-1	0	0

TABLE 4 : $SU(2)_G \times U(1)_q$ quantum numbers for scalars with Yukawa couplings
($n = 3, p = m = 1$)

	H_1^*	H_2^+	H_2^0	H_2^-
$2\ell+1$	1	3	3	3
I	0	1	0	-1
q	-1	1	1	1
$I+\frac{1}{2}q$	$-\frac{1}{2}$	$3/2$	$\frac{1}{2}$	$-\frac{1}{2}$

	$h_1^*(2, \frac{1}{2})$	$h_1^*(2, -\frac{1}{2})$	$h_2(2, \frac{1}{2})$	$h_2(2, -\frac{1}{2})$
$2\ell+1$	2	2	2	2
I	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
q	0	0	0	0
$I+\frac{1}{2}q$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

	$h_1^*(4, 3/2)$	$h_1^*(4, \frac{1}{2})$	$h_1^*(4, -\frac{1}{2})$	$h_1^*(4, -3/2)$
$2\ell+1$	4	4	4	4
I	$-3/2$	$-\frac{1}{2}$	$\frac{1}{2}$	$3/2$
q	0	0	0	0
$I+\frac{1}{2}q$	$-3/2$	$-\frac{1}{2}$	$\frac{1}{2}$	$3/2$

	$h_2(4, 3/2)$	$h_2(4, \frac{1}{2})$	$h_2(4, -\frac{1}{2})$	$h_2(4, -3/2)$
$2\ell+1$	4	4	4	4
I	$3/2$	$\frac{1}{2}$	$-\frac{1}{2}$	$-3/2$
q	0	0	0	0
$I+\frac{1}{2}q$	$3/2$	$\frac{1}{2}$	$-\frac{1}{2}$	$-3/2$

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