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TOPOLOGICALLY MASSIVE PLANAR UNIVERSE*

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I. Vuorio

*Center for Theoretical Physics
Laboratory for Nuclear Science and
Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.*

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ABSTRACT

A stationary, radially symmetric vacuum solution to 3-dimensional, topologically massive gravity is presented. All four Killing vectors of the space-time are identified and the geodetic equation is solved. An analogy with “Gödel’s Universe” is discussed.

I. INTRODUCTION

Interest has recently increased in three-dimensional theories of gravity. The motivation has been mainly pedagogical, but there are some situations where these models might become physically relevant.¹ Particular attention is drawn to the topologically massive theory with a Chern-Simons action leading to the “three-dimensional Weyl tensor” in the field equations.² This term can be generated by fermions. In our paper, we present a stationary, radially symmetric solution to this model.

II. SOLUTION TO THE FIELD EQUATIONS

In the vacuum equation, the usual Einstein tensor is supplemented by a contribution involving a mass m :

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{m}C_{\mu\nu} = 0 \quad . \quad (2.1)$$

Here, $R_{\mu\nu}$ is the Ricci tensor, $R = g_{\mu\nu}R^{\mu\nu}$, and $C_{\mu\nu}$, the *three-dimensional Weyl tensor* is defined by

$$C_{\mu\nu} = \frac{1}{2\sqrt{\det g_{\delta\sigma}}} [\epsilon_{\mu}{}^{\alpha\beta} D_{\alpha} R_{\nu\beta} + \epsilon_{\nu}{}^{\alpha\beta} D_{\alpha} R_{\mu\beta}] \quad . \quad (2.2)$$

Note that $C^\mu{}_\mu = 0$ identically, so the scalar curvature vanishes. If $R_{\mu\nu} = 0$, then $C_{\mu\nu} = 0$ as well, and therefore, flat space-time solves (2.1). We seek a more interesting solution.

We simplify eq. (2.1) by a rotationally symmetric, stationary *Ansatz* for the metric. The most general metric with these symmetries is, up to coordinate redefinition,

$$ds^2 = h^2 \left[\left(dt + \psi \frac{\epsilon^{ij} x^i dx^j}{r^2} \right)^2 - g^2 \delta^{ij} dx^i dx^j \right] , \quad (2.3)$$

where h , ψ and g are functions of the radius $r = \sqrt{(x^1)^2 + (x^2)^2}$ only.

To compute the Ricci and Weyl tensors, Cartan's formalism of connection and curvature forms³ is used. By choosing a non-coordinate basis

$$\omega^{\hat{0}} = h \left(dt + \psi \frac{\epsilon^{ij} x^i dx^j}{r^2} \right), \quad \omega^{\hat{i}} = h g dx^i \quad (i = 1, 2) \quad (2.4)$$

the line element becomes $ds^2 = \eta_{\hat{\mu}\hat{\nu}} \omega^{\hat{\mu}} \omega^{\hat{\nu}}$.

Any tensor $M_{\hat{\alpha}\hat{\beta}}$ computed from $g_{\mu\nu}$ must respect the symmetries of the metric.

Consequently, if $M_{\hat{\alpha}\hat{\beta}} = M_{\hat{\beta}\hat{\alpha}}$, it can be written in the form

$$\begin{aligned} M_{\hat{0}\hat{0}} &= a, \quad M_{\hat{0}\hat{i}} = x^i b + \epsilon^{ij} x^j c, \\ M_{\hat{i}\hat{j}} &= x^i x^j d + \delta^{ij} e + (x^i \epsilon^{jk} + x^j \epsilon^{ik}) x^k f, \end{aligned} \quad (2.5)$$

and a , b , c , d , e , f are functions of the radius r only. For the Ricci tensor, the non-vanishing functions are

$$a_R = \frac{1}{2} (hk)^2 + \frac{\nabla^2 h}{h^3 g^2} \quad (2.6a)$$

$$c_R = -\frac{1}{2h^3 g r} (h^3 k)' , \quad (2.6b)$$

$$d_R = \frac{1}{h^3 g^2 r^2} \left(2 \frac{h' g'}{g} + 2 \frac{(h')^2}{h} - h'' + \frac{h'}{r} \right) , \quad (2.6c)$$

$$e_R = \frac{1}{2}(hk)^2 - \frac{1}{h^2 g^2} \nabla^2 \ln hg - \frac{h'}{h^3 g^2} \left(\frac{g'}{g} + \frac{h'}{h} + \frac{1}{r} \right) , \quad (2.6d)$$

where

$$k \equiv \frac{\psi'}{g^2 r} . \quad (2.7)$$

The scalar curvature is

$$R = -\frac{1}{2}(hk)^2 + \frac{2}{h^2 g^2} \left(\nabla^2 \ln hg + \frac{\nabla^2 h}{h} \right) \quad (2.8)$$

To simplify the calculation of the Weyl tensor, it is possible to use its conformal invariance and rescale the metric by h^2 : $C_{\mu\nu}(g_{\gamma\delta}) = h C_{\mu\nu}(h^{-2} g_{\gamma\delta})$. If we use a basis like (2.4), then changing the metric also changes the basis, therefore

$$C_{\hat{\mu}\hat{\nu}}(g_{\gamma\delta}) = h^3 C_{\hat{\mu}\hat{\nu}}(h^{-2} g_{\gamma\delta}) \quad (2.9)$$

The Weyl tensor can now be calculated from the simpler rescaled metric, and the non-vanishing invariant functions are

$$a_C = h^3 \left[\frac{1}{2g^2} \nabla^2 k + k \left(k^2 - \frac{1}{g^2} \nabla^2 \ln g \right) \right] , \quad (2.10a)$$

$$c_C = \frac{-h^3}{2r} \frac{d}{dr} \left[\frac{3k^2}{2} - \frac{1}{g^2} \nabla^2 \ln g \right] , \quad (2.10b)$$

$$d_C = \frac{h^3}{2g^2 r^2} \left(\frac{2k'g'}{g} + \frac{k'}{r} - k'' \right) , \quad (2.10c)$$

$$e_C = h^3 \left[\frac{k}{2} \left(k^2 - \frac{1}{g^2} \nabla^2 \ln g \right) + \frac{1}{2g^2} \left(k'' - \frac{k'g'}{g} \right) \right] , \quad (2.10d)$$

Combining (2.6) and (2.10) gives the equations which must be solved. However, they remain complicated, so we make a simplification and seek a solution with constant h , which by rescaling may be set to unity, $h = 1$.

Vanishing scalar curvature implies

$$k^2 = \frac{4}{g^2} \nabla^2 \ln g \quad (2.11)$$

and use of the above in (2.6) and (2.10) gives the equations

$$\left(1 + \frac{3k}{2m}\right) \nabla^2 \ln g + \frac{1}{4m} \left(k'' + \frac{k'}{r}\right) = 0 \quad (2.12a)$$

$$k' + \frac{5g}{m} \left(\frac{1}{g^2} \nabla^2 \ln g\right)' = 0 \quad (2.12b)$$

$$\left(\frac{k'}{rg^2}\right)' = 0 \quad (2.12c)$$

The equation which follows from (2.6d) and (2.10d) is a linear combination of (2.12a) and (2.12c).

From (2.11) and (2.12b) it follows that

$$k' \left(\frac{1}{g} + \frac{5}{2m}k\right) = 0, \quad (2.13)$$

and (2.12c) implies that k' is proportional to rg^2 . If the proportionality constant $c \neq 0$ so that k' is non-vanishing, then (2.13) implies

$$g = \left(\frac{15c}{4m}r^2 + \frac{1}{g_0^3}\right)^{-1/3}, \quad g_0 = \text{constant}, \quad k = -\frac{2m}{5g},$$

but this contradicts the remaining equations. Hence, k' vanishes, so that $k^2/4$ and $g^{-2} \nabla^2 \ln g$ are constant and equal to each other, and the field equations reduce to (2.11) and

$$\left(1 + \frac{3k}{2m}\right) \frac{1}{g^2} \nabla^2 \ln g = 0, \quad (2.14)$$

which have two solutions: the trivial, flat space-time, $k = 0$, $g^{-2} \nabla^2 \ln g = 0$, and

$$k = -\frac{2}{3}m, \quad (2.15)$$

$$\frac{1}{g^2} \nabla^2 \ln g = \frac{m^2}{9} . \quad (2.16)$$

We recognize (2.16) as the *Liouville equation* for a surface with constant negative curvature. The most general rotationally symmetric solution is

$$g = \frac{2wn\left(\frac{r}{r_0}\right)^n \frac{1}{r}}{1 - \left(\frac{r}{r_0}\right)^{2n}} , \quad (2.17)$$

where n and r_0 are arbitrary and $w = 3/m$. Substituting this in (2.15), we obtain from (2.7)

$$\psi = -4wn \frac{\left(\frac{r}{r_0}\right)^{2n}}{1 - \left(\frac{r}{r_0}\right)^{2n}} , \quad (2.18)$$

where an arbitrary integration constant has been chosen so that the solution is nonsingular; $\psi(0) = 0$.

Now define new variables by $\tilde{r} = 2(r/r_0)^n$, $\tilde{\theta} = n\theta$, $\tilde{t} = t/w$. Then, w^2 appears only as an overall scale, which we set to unity and

$$ds^2 = \left[dt - \frac{r^2}{1 - (r/2)^2} d\theta \right]^2 - \left[1 - (r/2)^2 \right]^{-2} (dr^2 + r^2 d\theta^2) . \quad (2.19)$$

The parameters n and r_0 have disappeared. However, if the original angular range is $[0, 2\pi]$, then the new range is $[0, 2\pi n]$, i.e., the points with $\theta = \theta_0$ and $\theta = \theta_0 + 2\pi n$ are identified. The metric is only non-singular if $n = 1$; in other words, if there is no “deficit angle”.

In (2.19), the spatial interval is in “conformal” form, but r only ranges from 0 to 2. The change of variable $\tilde{r} = r[1 - (r/2)^2]^{-1}$ gives

$$ds^2 = [dt - 2(\sqrt{1 + r^2} - 1)d\theta]^2 - \left(\frac{dr^2}{1 + r^2} + r^2 d\theta^2 \right) , \quad (2.20)$$

where $-\infty < t < \infty$, $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$. A final, “hyperbolic” formula is gotten by defining $r = \sinh \sigma$;

$$ds^2 = [dt - 2(\cosh \sigma - 1)d\theta]^2 - [d\sigma^2 + \sinh^2 \sigma d\theta^2] \quad (2.21)$$

III. CURVATURE SCALARS, KILLING VECTORS

Next, we calculate the curvature scalars.⁴ There are three of them, and they can be described by the eigenvalues λ of the Ricci tensor:

$$R^\mu{}_\nu X^\nu = \lambda X^\mu \quad (3.1)$$

There are two solutions: one non-degenerate

$$\lambda = 2, \quad X^\mu \partial_\mu = \partial_t ; \quad (3.2a)$$

the other doubly degenerate

$$\lambda = -1, \quad X^\mu \partial_\mu = -\psi \partial_t + \partial_\theta \quad \text{and} \quad X^\mu \partial_\mu = \partial_r . \quad (3.2b)$$

The fact that the curvature scalars are constant suggests that our spacetime M should be homogeneous, *i.e.*, for any $x, y \in M$ there should exist an isometry ϕ_{xy} such that $\phi_{xy}(x) = y$.

We seek the Killing vectors ξ corresponding to these isometries. Two of them are known already: ∂_t and ∂_θ correspond to the time translation and rotation invariances, respectively. One verifies two more are given by

$$\xi_{(i)} = x^0 \partial_i + \frac{2\epsilon^{ij} x^j}{x^0 + 1} \partial_t . \quad (3.3)$$

where $x^0 \equiv \sqrt{1+r^2}$.

Define

$$\xi_{(t)} = \partial_t, \quad \xi_{(0)} = -2\partial_t + \partial_\theta. \quad (3.4)$$

Then $\xi_{(t)}$ commutes with all the others and

$$\begin{aligned} [\xi_{(0)}, \xi_{(i)}] &= -\epsilon^{ij} \xi_{(j)} \\ [\xi_{(1)}, \xi_{(2)}] &= \xi_{(0)}. \end{aligned} \quad (3.5)$$

The $\xi_{(0)}, \xi_{(1)}, \xi_{(2)}$ form a representation of $so(2,1)$. However, the corresponding symmetry group is *not* $SO(2,1)$. Indeed, if the group were $SO(2,1)$, a rotation by $2\pi n$ would be just the identity, but here $\exp(2\pi n \xi_{(0)})(t, x^1, x^2) = (t - 4\pi n, x^1, x^2)$. This time shift reminds us of the “helical” structure encountered^{1,5} in the case of the spinning point mass. However, here the metric is nonsingular.

From (3.2) it follows that $[\xi, \xi_{(t)}] = 0$ and consequently $\partial_t \xi^\mu = 0$ for *any* Killing vector ξ . This result and the fact⁶ that ξ is determined by $\xi^\mu(x)$ and $\partial_\mu \xi^\mu(x)$ at a single point $x \in M$ can be used to prove that our list $\xi_{(t)}, \xi_{(0)}, \xi_{(1)}, \xi_{(2)}$ is complete. With the above Killing vectors, it is also easy to see that the space-time is homogeneous, as anticipated.

IV. GEODESICS

We now solve the geodesic equation

$$\ddot{X}^\mu + \Gamma_{\alpha\beta}^\mu \dot{X}^\alpha \dot{X}^\beta = 0. \quad (4.1)$$

We know that $\dot{X}_\mu \xi_{(i)}^\mu \equiv D_{(i)}$ is constant along any geodesic and for any Killing vector $\xi_{(i)}$. For $i = t, 1, 2$, the 3×3 matrix $\xi_{(i)}^\mu$ is nonsingular. Call the inverse

$Y_\mu^{(i)}: Y_\mu^{(i)} \xi_{(j)}^\mu = \delta_j^i$, so that

$$\dot{X}^\mu = \sum_{i=t,1,2} D_{(i)} Y_{(i)}^\mu . \quad (4.2)$$

In component form, the Y 's are:

$$\begin{aligned} Y_{(t)}^\mu &= \frac{1}{x^0} (-3x^0 + 4, 2x^2, -2x^1) , \\ Y_{(1)}^\mu &= \frac{1}{x^0} \left(\frac{2x^2}{x^0 + 1}, -1 - (x^1)^2, -x^1 x^2 \right) , \\ Y_{(2)}^\mu &= \left(\frac{-2x^1}{x^0 + 1}, x^1 x^2, -1 - (x^2)^2 \right) . \end{aligned} \quad (4.3)$$

The first order differential equation (4.2) can easily be solved for some judicious choices of $D_{(i)}$:

(A) Set $D_{(t)} = \tilde{x}^0/2$, $\tilde{x}^0 \equiv \sqrt{1 + \tilde{r}^2}$, $D_{(1)} = D_{(2)} = 0$. We get helix-like curves with constant $r(\tau) \equiv \tilde{r}$ and

$$t(\tau) = \frac{1}{2}(4 - 3x^0)\tau + t_0, \quad \theta(\tau) = -\tau - \pi/2 . \quad (4.4)$$

(The $\pi/2$ in the expression for θ is inserted so that the curve starts on the negative x^2 -axis. The reason for this will be explained shortly.) These curves are timelike and future-pointing for $r < 1/\sqrt{3}$, lightlike for $r = 1/\sqrt{3}$ and space-like for $r > 1/\sqrt{3}$. For $r = \sqrt{7}/3$, it is closed ($t = \text{constant}$). The projection in the (x^1, x^2) -plane is a circle centered at the origin.

(B) $D_{(t)} = -\tilde{x}^2/2$, $D_{(1)} = 1$, $D_{(2)} = 0$. We get a geodesic with constant $x^2(\tau) \equiv \tilde{x}^2$ and

$$\begin{aligned} x^1(\tau) &= -\sqrt{1 + (x^2)^2} \sinh \tau , \\ t(\tau) &= \frac{3}{2} x^2 \tau - 4 \tan^{-1} \left[\frac{x^2}{x^0 + 1} \tanh \tau/2 \right] + t_0 . \end{aligned} \quad (4.5)$$

For each $\tilde{x}^2$, this is spacelike and the (x^1, x^2) projection is a straight line.

We get more geodesics by applying isometries to the ones we now have. In fact, almost all of them can be obtained in this fashion. By choosing t_0 and λ suitably and applying $\exp(\lambda \xi_{(2)})$ to any of these curves, it can be made to begin at the origin [we needed the $\pi/2$ in (4.4) to achieve this.] The circles become ellipses and the straight lines turn into hyperbolae. Between the ellipses and hyperbolae there is a parabola which cannot be obtained directly from (4.4) or (4.5) by an isometry, but only by a limiting process; it corresponds to a circle with $r \rightarrow \infty$ or a straight line with $\tilde{x}^2 \rightarrow -\infty$. It is not difficult to prove that all the geodesics starting at the origin are obtained from those described above by a rotation around the t -axis (modulo trivial reparametrizations).

In general, the (x^1, x^2) projections of *all* geodesics are conical sections. The timelike and lightlike geodesics are periodic and the projections are ellipses. If a is the minor axis, then the period is

$$\Delta t = (4 - 3\sqrt{a^2 + 1})\pi . \quad (4.6)$$

The situation resembles the motion of a charged particle in a plane perpendicular to a uniform magnetic field.

V. THREE - DIMENSIONAL GÖDEL UNIVERSES

Let us now consider a 3-dimensional manifold M with the metric

$$ds^2 = [dt - \lambda(\cosh \sigma - 1)d\theta]^2 - [d\sigma^2 + \sinh^2 \sigma d\theta^2] \quad (5.1)$$

$\lambda = \text{constant}$. It is equal to (2.23) for $\lambda = 2$. If we set $\lambda = \sqrt{2}$, it is easy to verify

that the product space $M \times R$ (the real line R having a negative definite metric) is isometric to “Gödel’s Universe”⁷, up to overall scale.

Choosing a dreibein basis $\omega^t = dt - \lambda(\cosh \sigma - 1)d\theta$, $\omega^\sigma = d\sigma$, $\omega^\theta = \sinh \sigma d\theta$ the Ricci tensor becomes diagonal:

$$R_{\mu\nu} = \text{diag}(\lambda^2/2, \lambda^2/2 - 1, \lambda^2/2 - 1) \quad (5.2)$$

and $R = 2 - \lambda^2/2$.

For each λ , there is a 3-dimensional Gödel-type solution of the Einstein equation with a cosmological constant:

$$R_{\mu\nu} + \left(\Lambda - \frac{1}{2}R\right)\eta_{\mu\nu} = 8\pi GT_{\mu\nu} \quad (5.3)$$

where $\Lambda = \lambda^2/4$ and $T^{\mu\nu} = \rho v^\mu v^\nu$; $v^\mu = (1, 0, 0)$; $\rho = (\lambda^2 - 1)/8\pi G$. $T^{\mu\nu}$ describes a stationary, homogeneous, pressureless dust.

Gödel⁷ enumerates several properties of his space-time, which are also true of (5.1) for any non-zero value of λ . The angular velocity of the rotation described by Gödel is now λ -dependent. We consider the Fermi-Walker parallel transportation of vectors (or of “the compass of inertia” or of “gyroscope axes”): If $Y^\mu(\tau)$ is a vector field along the dust world line $\tau \rightarrow (\tau, 0, 0)$, (which is also a geodesic), then $Y^\mu(\tau)$ is parallel (i.e. $\dot{Y}^\mu + \Gamma_{t\lambda}^\mu Y^\lambda = 0$) if it has the form $Y^\mu(\tau) = (\alpha, \beta \cos \omega(\tau + \tau_0), \beta \sin \omega(\tau + \tau_0))$, α , β and τ_0 are constants and $\omega = -\lambda/2$. Therefore, the compass of inertia rotates with $\omega = -\lambda/2$ with respect to the dust world lines; hence, the dust rotates with angular velocity $\omega = \lambda/2$ with respect to an observer’s compass of inertia, if the observer moves along with the dust.

We have found a 1-parameter family of 3-dimensional "Gödel Universes". The reason that in four dimensions there is only one of them, is that the introduction of a fourth dimension x^3 gives the condition $R_{11} = R_{33} \equiv 0$, and according to (5.2), this fixes λ to the value of $\lambda = \sqrt{2}$.

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