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EFFECTIVE LAGRANGIAN ANALYSIS OF THE CHIRAL PHASE TRANSITION
AT FINITE DENSITY

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A B S T R A C T

Introducing a finite chemical potential μ for the quark number density $\phi^\dagger\psi$, we study analytically the restoration of chiral symmetry as μ is varied. In the strong coupling limit, the effective Lagrangian for SU(N) gauge theories coupled to fermion fields in d dimensions is derived for all N. In the case of SU(2) we predict a second order chiral symmetry restoration phase transition, whereas for all $N > 3$ the transition is first order. Predictions are given for the critical values of the chemical potential μ . II †

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When the average density of ordinary matter approaches that of the nucleons themselves, a transition to a new state of matter is expected to occur. This transition may involve "deconfinement", chiral symmetry restoration or both. The study of matter at such extreme density should not only be of interest for heavy ion collision phenomenology, but is believed to play an important role in astrophysical contexts as well. The expectation of a chiral phase transition at finite density has already been confirmed by Monte Carlo calculations of lattice gauge theories in the quenched approximation¹⁾.

It has recently been observed that at finite temperature, where similar transitions occur, the effective Lagrangian methods of lattice gauge theories^{2),3)} can be used to provide both a qualitative and quantitative understanding of the phase transitions⁴⁾⁻⁶⁾.

In this letter we shall show how similar techniques can be used to study the phase transition associated with the restoration of chiral symmetry at finite density. We consider SU(N) lattice gauge theories coupled to dynamical fermions in d space-time dimensions, and go to the strong coupling region. In a "staggered" fermion formalism our action, including a finite chemical potential μ , takes the form

$$S = - \sum_x \bar{\chi}(x) M \chi(x) - \frac{1}{2} \sum_x \sum_j \gamma_j(x) \{ \bar{\chi}(x) U_j(x) \chi(x+j) - \bar{\chi}(x+j) U_j^\dagger(x) \chi(x) \} \\ - \frac{1}{2} \sum_x \{ \bar{\chi}(x) e^\mu U_0(x) \chi(x+\hat{0}) - \bar{\chi}(x+\hat{0}) U_0^\dagger(x) e^{-\mu} \chi(x) \} \quad (1)$$

where

$$\gamma_1(x) = (-1)^{x_0}, \quad \gamma_2(x) = (-1)^{x_0+x_1}, \quad \gamma_3(x) = (-1)^{x_0+x_1+x_2}, \dots \quad (2)$$

In the continuum the chemical potential μ couples to the fermion number density $\psi^\dagger \psi: \mu \psi^\dagger \psi = \mu \psi \gamma^0 \psi$, i.e., acts as the zero component of an (imaginary) external gauge field. As can be seen from Eq. (1) this analogy $\mu \rightarrow \{\text{time-component of external imaginary gauge field}\}$ holds on the lattice as well^{1),7)}.

Letting $M = 0$, the action (1) has a continuous $U(N_f) \times U(N_f)$ symmetry, with N_f denoting the number of flavours in the strong coupling region. When $\mu = 0$ this symmetry, identified as "chiral symmetry" on the lattice, is spontaneously broken^{2),3)}: $U(N_f) \times U(N_f) \rightarrow U(N_f)$. In the following, consider the case $N_f = 1$.

Defining

$$\begin{aligned} A_j^{ab}(x) &= -\frac{1}{2} \gamma_j(x) \bar{\chi}^a(x) \chi^b(x+j) \\ \bar{A}_j^{ab}(x) &= \frac{1}{2} \gamma_j(x) \bar{\chi}^a(x+j) \chi^b(x) \end{aligned} \quad (3)$$

and using such $SU(N)$ relations as

$$\begin{aligned} \int dU U_{ij} U_{kl}^\dagger &= \frac{1}{N} \delta_{il} \delta_{jk} \\ \int dU U_{i_1 j_1} U_{i_2 j_2} \dots U_{i_N j_N} &= \frac{1}{N!} \epsilon_{i_1 \dots i_N} \epsilon_{j_1 \dots j_N} \end{aligned} \quad (4)$$

we can explicitly integrate out all spatial link variables:

$$\begin{aligned} \int dU_j e^{-\frac{1}{2} \sum_j \gamma_j(x) \{ \bar{\chi}(x) U_j(x) \chi(x+j) - \bar{\chi}(x+j) U_j^\dagger(x) \chi(x) \}} \\ = e^{\sum_j \{ \frac{1}{4N} \chi^a(x) \bar{\chi}^a(x) \chi^b(x+j) \bar{\chi}^b(x+j) + \frac{1}{N!} (\det A_j + \det \bar{A}_j) + \dots \}} \end{aligned} \quad (5)$$

where

$$\det A_j = \frac{1}{N!} \left(\frac{1}{2} \right)^N [\gamma_j(x)]^N \epsilon_{i_1 \dots i_N} \epsilon_{j_1 \dots j_N} \bar{\chi}^{i_1}(x) \chi^{j_1}(x+j) \dots \bar{\chi}^{i_N}(x) \chi^{j_N}(x+j) \quad (6)$$

Similarly, the timelike integrations give

$$\begin{aligned} \int dU_0 e^{-\frac{1}{2} \{ \bar{\chi}(x) U_0(x) e^\mu \chi(x+\delta) - \bar{\chi}(x+\delta) e^{-\mu} U_0^\dagger(x) \chi(x) \}} \\ = e^{\{ \frac{1}{4N} \chi^a(x) \bar{\chi}^a(x) \chi^b(x+j) \bar{\chi}^b(x+j) + \frac{1}{N!} (e^{N\mu} \det A_0 + e^{-N\mu} \det \bar{A}_0) + \dots \}} \end{aligned} \quad (7)$$

Higher order terms in the expansions (5) and (7) can be kept, but we have checked that these higher order terms do not change the conclusions to be presented below. The expansions (5) and (7) correspond to systematic $1/d$ expansions³⁾. Now introduce a suitably normalized baryon field $B(x)$:

$$B(x) = d^{N/4} \frac{1}{N!} \epsilon_{i_1 \dots i_N} \chi^{i_1}(x) \dots \chi^{i_N}(x) \quad (8)$$

with the help of which we can write

$$\det A_j = (-1)^{N(N+1)/2} N! \left(\frac{1}{2}\right)^N \left(\frac{1}{d}\right)^{N/2} [\eta_j(x)]^N \bar{B}(x) B(x+j) \quad (9)$$

and

$$\det \bar{A}_j = (-1)^{N(N+1)/2} N! \left(-\frac{1}{2}\right)^N \left(\frac{1}{d}\right)^{N/2} [\eta_j(x)]^N \bar{B}(x+j) B(x) \quad (10)$$

Collecting all baryon terms, this gives a contribution

$$\sum_{x, x'} \bar{B}(x) V_B(x, x') B(x')$$

to the effective action, with a baryon potential

$$V_B(x, x') = \frac{(-1)^{N(N+1)/2}}{2^N d^{N/2}} \left[\sum_j [\eta_j(x)]^N \{ \delta_{x, x+j} + (-1)^N \delta_{x', x-j} \} \right. \\ \left. + \{ e^{N\mu} \delta_{x', x+\delta} + (-1)^N e^{-N\mu} \delta_{x', x-\delta} \} \right] \quad (11)$$

We now substitute an elementary meson field $\sigma(x)$ for the composite operator $\chi(x)\bar{\chi}(x)$: $\sigma(x) = N^{-1} \chi^a(x) \bar{\chi}^a(x)$ by means of the identity

$$e^{\frac{1}{4N} \sum_x \chi^a(x) \bar{\chi}^a(x) \chi^b(x+j) \bar{\chi}^b(x+j)} \\ = \int d[\sigma] e^{\sum_x \sum_j \left\{ -\frac{N}{4} \sigma(x) \sigma(x+j) + \frac{1}{4} [\sigma(x) \chi^a(x+j) \bar{\chi}^a(x+j) + \sigma(x+j) \chi^a(x) \bar{\chi}^a(x)] \right\}} \quad (12)$$

This leads to an effective action S_{eff} defined by

$$Z = \int d[\sigma] d[\chi] d[\bar{\chi}] e^{S_{\text{eff}}} \quad (13)$$

$$S_{\text{eff}} = - \sum_x \sum_j \left\{ \frac{N}{4} \sigma(x) \sigma(x+j) - \frac{1}{4} [\sigma(x) \chi^a(x+j) \bar{\chi}^a(x+j) + \sigma(x+j) \chi^a(x) \bar{\chi}^a(x)] \right\} \\ + \sum_{x,x'} \bar{B}(x) V_B(x,x') B(x') \quad (14)$$

with $B(x)$ expressed in term of $\chi(x)$ as given by Eq. (8).

By a Laplace transform³⁾ this can be turned into an effective action which is local in the baryon field $B(x)$. This follows from the identity

$$e^{\sum_{x,x'} \bar{B}(x) V_B(x,x') B(x')} \\ = \int d[b] d[\bar{b}] e^{- \sum_{x,x'} \bar{b}(x) V_B^{-1}(x,x') b(x') + \sum_x [\bar{b}(x) B(x) + \bar{B}(x) b(x)]} \quad (15)$$

which holds irrespective of whether N is even or odd. We now integrate out the fermion fields $\chi(x)$ and $\bar{\chi}(x)$ from the partition function to get

$$Z = \int d[\sigma] d[b] d[\bar{b}] \prod_x \left(\left(\frac{d}{2} \right)^N \sigma^N + (-1)^{N(N-1)/2} d^{N/2} \bar{b} b \right) \times \\ \exp \left\{ - \sum_x \frac{Nd}{4} \sigma^2 - \sum_x \bar{b} V_B^{-1} b \right\} \quad (16)$$

where we have inserted classical fields $\sigma = \sigma(x)$, $b = b(x)$ and $\bar{b} = \bar{b}(x)$ in order to extract the effective potential

$$V_{\text{eff}} = \frac{Nd}{4} \sigma^2 + \bar{b} V_B^{-1} b - \ln \left\{ \left(\frac{d}{2} \right)^N \sigma^N + (-1)^{N(N-1)/2} d^{N/2} \bar{b} b \right\} \quad (17)$$

Note that for N even the fields $\bar{b}$ and b are bosonic, whereas for N odd these fields are fermionic. In order to express the effective potential in terms of a single field σ , we perform the final integration over the baryon fields $\bar{b}$ and b :

$$\int d\bar{b} db \bar{b} b \exp \{ - \bar{b} V_B^{-1} b \} = (-1)^N V_B \quad (18)$$

Alternatively, for N odd one can make use of the identity

$$\ln \left\{ \left(\frac{d}{2} \right)^N \sigma^N + (-1)^{N(N-1)/2} d^{N/2} \bar{b} b \right\} = \ln \left[\left(\frac{d}{2} \right)^N \right] + N \ln[\sigma] \\ + (-1)^{N(N-1)/2} \left(\frac{2}{d} \right)^N \sigma^{-N} d^{N/2} \bar{b} b \quad (19)$$

to reduce the integration over $\bar{b}$ and b to a trivial Gaussian integration. The result agrees with what one obtains from eq. (18), and one is left with the partition function

$$Z = \int d[\sigma] e^{-\sum_x \frac{Nd}{4} \sigma^2 + \sum_x \ln \left\{ \sigma^N + \left(\frac{2}{d} \right)^N (-1)^{N(N-1)/2} d^{N/2} V_B \right\}} \quad (20)$$

for both N even and odd. This corresponds to a one-parameter effective potential

$$V_{\text{eff}} = \frac{Nd}{4} \sigma^2 - \ln \left\{ \sigma^N + d^{-N} e^{N\mu} \right\} \quad (21)$$

in the region $e^{N\mu} \gg (d-1)$. Note that in the limit where we drop all baryon terms this reduces to

$$V_{\text{eff}} \approx \frac{Nd}{4} \sigma^2 - N \ln[\sigma] \quad (22)$$

which shows that chiral symmetry is spontaneously broken by an amount $\langle \sigma \rangle = N^{-1} \langle \bar{\chi} \chi \rangle = \sqrt{2/d}$, thus reproducing the known result^{2),3),*)}. As can be seen from Eqs. (11) and (20) the baryon contribution is for $\mu = 0$ heavily suppressed by powers of d^{-N} . However, for large enough μ this $1/d$ expansion will break down, and the possibility of a chiral symmetry restoration phase transition arises. As can be seen from Eq. (21), this occurs roughly when $e^{N\mu}/d^N \approx 1$, corresponding to a critical chemical potential of the order

$$\mu_{\text{ch}} \sim \ln[d] \quad (23)$$

The logarithmic scaling law (23) is no coincidence. We would expect a phase transition at a critical μ_{ch} roughly when $N\mu_{\text{ch}} \sim M_B$ (M_B = baryon mass). Since at strong coupling the mass of the lowest lying baryon state can be computed²⁾: $M_B \approx N \ln[d]$ we recover the estimate (23).

*) Keeping higher order terms in the expansions (5) and (7) only changes this result by a very small amount^{2),3)}.

The above analysis was performed for all $N > 3$. The case $SU(2)$, having pseudoreal representations, requires special care. The effective potential, however, turns out to be given by the same equation (21).

We now turn to the actual determination of the phase transition. From Eq. (21) we have

$$\frac{\partial V_{\text{eff}}}{\partial \sigma} = N\sigma \left(\frac{d}{2} - \frac{\sigma^{N-2}}{\sigma^N + d^{-N} e^{N\mu}} \right) \quad (24)$$

This shows explicitly that only for $SU(2)$ do we have a second order phase transition. For all $N > 3$ the phase transition, if present, must be of first order. For $SU(2)$ the critical μ_{ch} can be computed analytically from (24):

$$\mu_{\text{ch}} = \frac{1}{2} \ln(2d) \quad (25)$$

[Again it is interesting to note that this result corresponds exactly to $N\mu_{\text{ch}} = M_B$, where for $SU(2)$ $M_B = \frac{1}{2}N \ln(2d)$ is the strong coupling expression for the baryon mass²⁾.] We have plotted the effective potential for $SU(2)$ in Fig. 1, to explicitly show the second order nature of the phase transition.

For $SU(N)$, $N > 3$ the phase transition points can in principle be found analytically, but we have relied on a numerical study instead. The effective potentials for $SU(3)$ and $SU(4)$ are shown in Fig. 2 and Fig. 3, respectively. A clear first order phase transition is found, and we expect this to hold for all $N > 3$.

The system of most interest is, of course, based on the $SU(3)$ gauge group. Here, in $d = 4$ dimensions, we find a clear first order phase transition at $\mu_{\text{ch}} = 0.66$. With a strong coupling lattice spacing of $a^{-1} = 440 \text{ MeV}^3$, this gives a critical chemical potential $\mu_{\text{ch}} \approx 290 \text{ MeV}$ in physical units. Notice that again $N\mu_{\text{ch}} \approx M_B$.

Since first order phase transitions generally exhibit stability towards small perturbations, we have rather strong confidence in our prediction of a first order chiral phase transition in $SU(3)$. The pattern of a second order phase

transition for $SU(2)$, but first order phase transition for $SU(3)$ also seems to be indicated in the Monte Carlo study of Ref. 1)^{*}).

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^{*}) The occurrence of a first order phase transition for $N > 3$ has also been argued to follow from a Hamiltonian analysis⁸⁾.

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FIGURE CAPTIONS

- Fig. 1 The effective potential V_{eff} for the gauge group SU(2) in four dimensions. For $\mu < \mu_{\text{ch}}$ chiral symmetry is spontaneously broken, whereas for $\mu > \mu_{\text{ch}}$ it is restored. There is a second order phase transition at $\mu_{\text{ch}} = \frac{1}{2} \ln(2d) = 1.04\dots$
- Fig. 2 The effective potential V_{eff} for SU(3) in four dimensions. There is a clear first order phase transition at $\mu_{\text{ch}} = 0.58\dots$
- Fig. 3 Same as Fig. 2, but for SU(4) in four dimensions. The phase transition is again first order, now with $\mu_{\text{ch}} = 0.66\dots$. All potentials shown in this paper have been normalized so that they vanish at the origin.

- Figure 1 -

- Figure 2 -

- Figure 3 -