EUROPEAN ORGANISATION FOR NUCLEAR RESEARCH

CERN LEP-TH/85-8

CERN LIBRARIES, GENEVA

CM-P00064423

AN ANALYTIC APPROACH TO THE OFF-MOMENTUM CLOSED ORBIT IN STORAGE RINGS

J. Hage1

CERN, Geneva, Switzerland

Presented at the 1985 Particle Accelerator Conference
13-16 May 1985, Vancouver, British Columbia, Canada

G E N E V A
May 1985

AN ANALYTIC APPROACH TO THE OFF-MOMENTUM CLOSED ORBIT IN STORAGE RINGS

J. Hagel
LEP Division, CERN,
Geneva, Switzerland

1. Summary and Conclusions

The non-linear equation for the off-momentum closed orbit is solved analytically by using two independent approaches. Firstly expanding the solution with respect to the small parameter $\delta = \Delta p/p_0$ up to the first order in δ and secondly by using a new type of linearization procedure. The structure is made of LEP-type octants containing 32 regular cells ($\mu=90^\circ$) with bending magnets, quadrupoles and four sextupole families, as well as two dispersion suppressors at both lattice ends and straight sections where the on-momentum η -function vanishes (see Fig.1). For the computations we replace the actual lattice elements of the structure by thin lenses with superimposed quadrupoles and sextupoles, which is a good approximation for LEP. All our results are compared with numerical computations, mainly with MAD [1]. While the first order theory works well up to $\delta = 1\%$, the linearization results are very accurate even at $\delta = 1.8\%$.

Fig.1 - Model structure with two insertions, two suppressors and 32 cells with sextupoles.

2. Differential Equation

The equation is taken from Refs.[2] and [3] :

$$\eta'' - k(s)(1 - \delta + \delta^2 \dots) \eta - \frac{1}{2} m(s)(\delta - \delta^2 \dots) \eta^2 = h(s)(1 - \delta + \delta^2 \dots) \quad (1)$$

Here η is defined as x_c/δ , $x_c(s)$ being the off-momentum closed orbit as a function of the longitudinal coordinate s , while $k(s)$, $m(s)$ and $h(s)$ are the quadrupole strength, sextupole strength and inverse bending radius, respectively. We look for a solution satisfying the boundary conditions :

$$\eta'(s_1, \delta) = \eta'(s_2, \delta) = 0, \quad (2)$$

where s_1 and s_2 are the positions of the interaction points in the straight sections.

3. First Order Perturbation Theory

We insert $\eta(s, \delta) = \eta^{(0)}(s) + \delta \eta^{(1)}(s)$ into Eq.(1) and compare for equal powers in δ up to first order. Then we find two linear, inhomogeneous equations for $\eta^{(0)}$ and $\eta^{(1)}$:

$$\eta^{(0)''} - k(s)\eta^{(0)} = h(s), \quad (3)$$

$$\eta^{(1)''} - k(s)\eta^{(1)} = -h(s) - k(s)\eta^{(0)} + \frac{1}{2} m(s)\eta^{(0)2}. \quad (4)$$

This method has also been used in Refs.[2] and [5]. The function $\eta^{(0)}$ vanishes in the straight sections which are preceded by dispersion suppressors and do not contain dipoles. At the position of the sextupoles, located close to the quadrupoles, we then find the values for $\eta_F^{(0)}$ and $\eta_D^{(0)}$ from Ref.[4] ($\mu=90^\circ$) :

$$\eta_{F,D}^{(0)} = \frac{L^2 h}{8} (4 \pm \sqrt{2}). \quad (5)$$

For the suppressors which are of missing dipole type and consist of two regular FODO-cells [4] the results are derived and given in Ref.[3].

The contribution $\eta^{(0)}$ does not vanish in the straight sections, since the suppressors only work exactly at $\delta=0$. However, from Eq.(4) it is obvious that $\eta^{(1)}$ just fulfills Hill's homogeneous equation since $\eta^{(0)}(s) = h(s) = m(s) = 0$ in the straight sections. So in that part of the structure we may use the standard matrix-formalism for the linear betatron motion [4]. In the dispersion suppressors and the regular lattice we then may deduce exactly a linear vector recurrence from Eq.(4) connecting the displacements and slopes of $\eta^{(0)}$ at two successive quadrupoles (piecewise integration of Eq.(4)) :

$$\eta_{j+1}^{(1)} = -\frac{L^2 h_j}{8} + \eta_j^{(1)'} + \eta_j^{(1)} \quad (6)$$

$$\eta_{j+1}^{(1)'} = -\frac{L^2 h_j}{4} + \eta_j^{(1)'} + (-1)^{j+1/2} \eta_{j+1}^{(1)} + (-1)^{j+1} \sqrt{2} \eta_{j+1}^{(0)} (1 + \lambda_{j+1} \eta_{j+1}^{(0)}) \quad (7)$$

where j is the occurrence number of the quadrupole (sextupole) in the lattice and h_j the bending radius between the j^{th} and $(j+1)^{\text{st}}$ lens (constant in the lattice but non-constant in the suppressors [4]). The quantity λ_j is defined as $\lambda_j = (m_j L)/(4\sqrt{2})$, m_j being the integrated strength of the j^{th} sextupole. The recurrence system [6]-[7] was solved by means of the symbolic language processor LISP/REDUCE. We solved the recurrence starting from both suppressor entrances and executing the mapping in opposite directions up to a "meeting point" (which we defined as being the first quadrupole of the regular lattice near the interaction point 1 (see Fig.1). There we required the result $(\eta_j^{(1)}, \eta_j^{(1)'})$ to be the same for both mappings (closed solution !) which lead us to definite values for $\eta_j^{(1)}$ at the interaction points. This method is called "shooting method" and is described in Ref.[6]. All the details can be obtained from Ref.[3]. For the values of $\eta_j^{(1)}$ at the interaction points we found :

$$\eta_{1,2}^{(1)} = \delta \frac{L^2 h (3 - \sqrt{2})}{2(\psi_{1,2} + \sqrt{2} \phi_{1,2} + \psi_{2,1} \frac{\phi_{1,2}}{\phi_{2,1}})} \quad (8)$$

Here $\phi_{1,2}$ and $\psi_{1,2}$ are the two components of the two vectors $\vec{\phi}_1$ and $\vec{\phi}_2$ being defined as :

$$\vec{\phi}_{1,2} = M_{str,1,2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (9)$$

where the vector $(1,0)$ is due to the boundary conditions and $M_{str,1,2}$ are the transfer matrices for the two straight sections including the first QF of the suppressor. We see as an important result, that to this order of approximation, the η -values at the interaction points and thus all over the straight sections do not depend on any of the sextupole strengths in the lattice. From the general theory applicable to the recurrences (6) and (7) one can prove that this is true for any lattice structure with phase advance $\mu = \pi/\kappa$, $\kappa \in \mathbb{Z}$ and with $2 \times 2\kappa$ sextupoles [3]. The lattice results, given in Ref.[3], fulfill a periodicity over four cells. For a general lattice phase advance of $\mu = \pi/\kappa$ we

may prove [3] that the periodicity covers 2π cells. The η -function ($\eta = \eta^{(0)} + \delta\eta^{(1)}$) at the first QF and QD nearby interaction point 1 is shown in Fig.2 as function of δ . The results, by definition represented by straight lines, are compared with MAD results. In this example we used $\beta_{1,2}$ of 7 cm and 17.5 cm, which are the values of LEP. As we see, the results agree quite well up to $\delta = 1\%$ but then the curvature of the exact solution becomes stronger and the accuracy of the first order results decreases. As is demonstrated in Ref.[3] the deviations of the vertical machine tune Q_V produced by that discrepancy is tolerable up to about $\delta = 1\%$. When computing analytically the dispersion for $\delta > 1\%$ we have either to push the perturbation theory to higher order or to use alternative methods.

Fig.2 - Comparison of exact, MAD and perturbation results.

The Successive Linearization Method

This method, due to the author was first presented in Ref.[3] and later in Refs.[7] and [8] for different applications. It can be used to calculate the η -function with very good accuracy up to $\delta = 2\%$.

Instead of taking as the unperturbed solution of Eq.(1) the solution for $\delta=0$, we now choose for $\eta^{(0)}$ the solution for any δ but without the sextupole fields. This is equivalent to linearizing Eq.(1) and we call this step **first linearization**. Then we write the exact solution as $\eta = \eta^{(0)} + u$, u being called the correction function and insert η into Eq.(1). This leads to a non-linear exact equation for u which we then again linearize, calling that step **second linearization**:

$$u^{(0)''} - [k(s)(1-\Delta) + m(s)\Delta\eta^{(0)}(s)]u^{(0)} = \frac{1}{2}m(s)\Delta\eta^{(0)2}(s) \quad (10)$$

where $\Delta = \delta/(1+\delta)$. Then :

$$\eta = \eta^{(0)} + u^{(0)} \quad (11)$$

is an approximation to the solution of Eq.(1). The theory of the successive linearization method containing accuracy estimates is given in Ref.[3]. It is also demonstrated there that replacing $\eta^{(0)}(s)$ by its average value in the coefficient function of Eq.(10) does not affect significantly the accuracy of the method.

Inspecting Eq.(10) for $u^{(0)}$ and the linear part of Eq.(1) for $\eta^{(0)}$ shows that in the straight sections both equations become again of the linear betatron motion type. So we may for this part use exactly the transfer matrices of the straight sections, but in this case the ones for $\delta \neq 0$. In the suppressors and lattice part of the structure we may again reduce the equation for $\eta^{(0)}$ to a linear vector recurrence :

$$\begin{pmatrix} \eta_{n+1}^{(0)} \\ \eta_n^{(0)} \end{pmatrix} = \begin{pmatrix} 1+\alpha & 2+\alpha \\ -\alpha^2 & 1-\alpha-\alpha^2 \end{pmatrix} \begin{pmatrix} \eta_n^{(0)} \\ \eta_{n-1}^{(0)} \end{pmatrix} + \begin{Bmatrix} \frac{L^2 h}{16\sqrt{2}} \\ \frac{L^2 h}{8\sqrt{2}} \end{Bmatrix} \begin{pmatrix} 4+\alpha \\ 4-3\alpha-\alpha^2 \end{pmatrix} \quad (12)$$

with $\alpha = \sqrt{2}(1-\Delta)$ and n numbering the focussing quadrupoles. The upper factor in the brackets belongs to the suppressor and the lower one to the lattice. The theory of the above recurrence is treated extensively in Refs.[3] and [6]. Its solution in the lattice at quadrupole n can be written in closed form :

$$\begin{pmatrix} \eta_n^{(0)} \\ \eta_{n-1}^{(0)} \end{pmatrix} = M^n \begin{pmatrix} \eta_0^{(0)} \\ \eta_{-1}^{(0)} \end{pmatrix} + \frac{L^2 h}{8\sqrt{2}} (4+\alpha) \begin{pmatrix} 2 \\ -1 \end{pmatrix} \alpha^n \quad (13)$$

where M is the matrix occurring in Eq.(12). Applying again the "shooting method" to obtain a closed solution for $\eta^{(0)}$ and satisfying the boundary conditions at the interaction points 1 and 2 leads to the $\eta^{(0)}$ values at these points. The value for $\eta^{(0)}$ at interaction point 2 is (N ... number of cells) :

$$\eta_2^{(0)} = \frac{1}{2} \frac{\vec{\phi}_1(n) A [M^{2N} \vec{F} + \vec{F}] - (M^{2N} \vec{F} + \vec{F})}{\vec{\phi}_1(n) A M^{2N} \vec{F} + \vec{F}} \quad (14)$$

where $A = \begin{pmatrix} 1 & 0 \\ -\alpha & -1 \end{pmatrix}$ and $\vec{F}$ is the inhomogeneous part of Eq.(13), while $\vec{\phi}_{1,2}$ are the vectors defined in Eq.(9) but for $\delta \neq 0$. For the lattice values of $\eta^{(0)}$ we find :

$$\eta_n^{(0)} F_{,D} = A_{F,D} \cos(n+1)\mu + B_{F,D} \sin(n+1)\mu + C_{F,D} \quad (15)$$

Here μ is the off-momentum phase advance per cell in the lattice ($\cos\mu = 1/2 \text{Tr} M = 2\Delta - \Delta^2$). The coefficients $A_{F,D}$, $B_{F,D}$ and $C_{F,D}$ are given explicitly in Ref.[3]. The coefficients $C_{F,D}$

$$C_{F,D} = \frac{L^2 h (4 \pm \alpha)}{4\alpha\sqrt{2}} \quad (16)$$

are the off-momentum values for $\eta_{F,D}$ in a FODO-lattice, derived in Ref.[9]. So we see that $\eta^{(0)}$ oscillates sinusoidally around these values due to the influence of the straight-section quadrupoles.

For the correction function $u^{(0)}$ the same method may again be used. Having replaced $\eta^{(0)}(s)$ in the coefficient of Eq.(10) as mentioned above, we see that the period of the focussing function is the one of the sextupole distribution in the lattice. So the elementary transfer matrix R related to the homogeneous part of Eq.(10) covers two FODO-cells in our case. The matrix R is given explicitly in Ref.[3]. Given the form for $\eta_n^{(0)}$ in Eq.(15) it becomes clear that the recurrence associated to Eq.(10) is :

$$\vec{U}_{v+1} = R \vec{U}_v + \vec{A} \cos 2v\mu + \vec{B} \sin 2v\mu + \vec{C} \quad (17)$$

where v numbers the magnetic periods of the lattice (2 cells). The exact form of $\vec{A}$, $\vec{B}$, $\vec{C}$ can also be found in Ref.[3]. As expected R , $\vec{A}$, $\vec{B}$ and $\vec{C}$ do not only depend on the linear lattice parameters μ , δ , but also on the sextupole strengths in the lattice. The theory of the recurrence (17), Ref.[3], states that we may write its solution as :

$$\vec{U}_v = R^v (\vec{U}_0 - \vec{\alpha} - \vec{\gamma}) + \vec{\alpha} \cos 2v\mu + \vec{\beta} \sin 2v\mu + \vec{\gamma} \quad (18)$$

where $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ are given by :

$$\begin{Bmatrix} \vec{\alpha} \\ \vec{\beta} \end{Bmatrix} = (R^2 + I - 2R \cos \mu)^{-1} \begin{Bmatrix} \vec{B} \\ \vec{A} \end{Bmatrix} \sin 2\mu \pm (R - I \cos 2\mu) \begin{Bmatrix} \vec{A} \\ \vec{B} \end{Bmatrix}$$

$$\vec{\gamma} = -(R - I)^{-1} \vec{C} \quad (19)$$

It is again possible to apply the "shooting method" to satisfy the boundary conditions and the final results can be written in the following form for $\eta_v = \eta_v^{(0)} + u_v^{(0)}$:

$$\eta_v = [A_F + \{\vec{\alpha}\}_1] \cos 2v\mu + [B_F + \{\vec{\beta}\}_1] \sin 2v\mu + C_F + \{\vec{\gamma}\}_1 + R_V (\vec{U}_0 - \vec{\alpha} - \vec{\gamma}) \quad (20)$$

where $\{\}_1$ means the first component of the vector enclosed in the brackets. $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ are given in Eq.(19), while $A_{F,D}$, $B_{F,D}$ and $C_{F,D}$ come from Eq.(15). For $\vec{U}_0$ the expression :

$$\vec{U}_0 = \frac{\vec{\phi}_1(n) AM^2 [RN/2(\vec{\alpha} + \vec{\gamma}) - \vec{\alpha} \cos N\mu - \vec{\beta} \sin N\mu - \vec{\gamma}]}{\vec{\phi}_1(n) AM^2 RN/2 M^2 \vec{\phi}_2} \vec{\phi}_2 \quad (21)$$

holds.

In Fig.3 the results of the successive linearization method are compared to the perturbation result for the η -function at the 4th ($v=2$), 16th ($v=8$) and 32rd ($v=16$) focussing quadrupole of the regular lattice. In addition the exact results being due to MAD calculations are represented. We see that the results of the linearization method agree extremely well from $\delta = -1.6\%$ to $\delta = +2\%$ with the numerical ones. There exists a small interval of δ around $\delta = +1.5\%$ where the linearization results cannot be calculated since the denominators in Eqs.(14) and (21) become small. In fact as we approach that critical region we observe $\eta^{(0)}$ and $u^{(0)}$ growing in opposite directions tending to cancel the physically non-existent singularity. But since $\eta^{(0)} + u^{(0)}$ is an approximate solution, this cancellation is not exact so that a small interval is left where the solution cannot be calculated. In fact we see that mainly for positive δ the perturbation method fails above $\delta = 1\%$. We may state that with our new method it is possible to obtain very precise results for non-linear lattice parameters. In Ref.[7], [8] the linearization method is used to obtain invariant curves for the betatron motion, also giving highly precise values. In addition it turned out to be possible even to give an estimate for the dynamic aperture of simple non-linear structures.

Fig.3 - Results for $\eta(\delta)$ at different positions in the lattice.

REFERENCES

- [1] C. Iselin, The MAD Program, Reference manual (To be published).
- [2] H. Wiedemann, PEP Note 220 (September 1976).
- [3] J. Hagel, Analytic calculation of the off-momentum closed orbit in storage rings with insertions and sextupoles, CERN/LEP-TH/84-14, 3 October 1984.
- [4] Proc. First Course of Int. School of Particle Physics : Theoretical aspects of the behaviour of beams in accelerators and storage rings, Erice, 1976 (CERN 77-13, Geneva 1977).
- [5] J. Jäger, J.P. Delahaye, First order variation of the dispersion function with particle energy deviation, SLAC/AP-40, December 1984 (AP).
- [6] L. Berg, Differenzengleichungen zweiten Ordnung mit Anwendungen, UTB 906, 1979.
- [7] G. Guignard, J. Hagel, Sextupole correction and dynamic aperture, numerical and analytical tools, CERN/LEP-TH/85-3 (1985).
- [8] G. Guignard, J. Hagel, Dynamic aperture, analytical derivation and sextupole algorithms, Proc. Top. Course on Non-Linear Dynamics 1985 (to be published).
- [9] J. Hagel, Single particle theory of AG-synchrotrons taking into account momentum dependence and non-linearities of the focusing force, Thesis submitted to the Faculty of Sciences of the Technical University of Graz, 1984 (unpublished).