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GENERATING FUNCTIONS FOR NON-CANONICAL MAPS

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A B S T R A C T

A differential representation, analogous to the generating function representation for canonical maps, is obtained for arbitrary smooth maps in even dimensional spaces. A few two-dimensional examples are worked out.

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Generating functions for volume preserving (canonical) maps are interesting tools not only as a compact description of the maps but also as a starting point for the variational methods that led to some important results and applications^{1),2)}. This raises the natural question of whether a similar description is possible for non-canonical maps.

The existence of generating functions for canonical maps is closely related to the fact that the graph of the map is a Lagrangian submanifold in the product space of the phase space M with itself. I will briefly review this standard result, phrasing it in such a way as to provide a natural introduction to the results to be proved later on. Let M be a real symplectic manifold and f a smooth map $(x,y) \xrightarrow{f} (\bar{x},\bar{y})$ where, for simplicity, one has assumed a separation of the variables into two sets such that the symplectic form is $\omega = \sum dy_i \wedge dx_i$. (By Darboux's theorem, this is always possible, at least locally.) In $M \times M$, with co-ordinates $\{(x,y), (x',y')\}$ one introduces a symplectic structure by the form $W = \sum dy_i \wedge dx_i - dy'_i \wedge dx'_i$. By definition, a map is canonical if the integral of the one-form

$$\xi_f = y \cdot dx - \bar{y} \cdot d\bar{x} \quad (1)$$

vanishes. ξ_f is the restriction of $\xi = y \cdot dx - y' \cdot dx'$ to the graph of the map f . By Stokes' theorem, $\int \xi_f = 0$ implies that the symplectic form $W = d\xi$ vanishes on restriction to the graph of f [i.e., f is a Lagrangian submanifold of $(M \times M, W)$]. Now by Poincaré's lemma $d\xi_f = 0$ implies (locally) $\xi_f = dS$, from which we obtain the generating function representation of f ,

$$y_i = \frac{\partial S(x, \bar{x})}{\partial x_i} \quad (2a)$$

$$y'_i = - \frac{\partial S(x, \bar{x})}{\partial \bar{x}_i} \quad (2b)$$

For any given map the generating function is not unique, different choices corresponding to different parametrizations of the Lagrangian submanifold³⁾. Also, if the phase space manifold M has non-trivial cohomology, one may not have a global generating function. These questions, however, will not be dealt with in this paper and in particular one always assumes to be in some well defined local chart of M .

The generating function appears therefore through a representation of the one-form ξ_f associated to the map f and the representation is particularly simple because ξ_f is closed. If one is to generalize the idea of representing

maps by derivatives of functions to the non-canonical case, the question to ask is what is the representation of ξ_f when this is not a closed form. The answer to this question leads to the following.

Theorem: Let Ω be an open set in the graph of the C^∞ map $(y,x) \xrightarrow{f} (\bar{y},\bar{x})$, which is diffeomorphic to an open set in R^N (N even). Then, there are N smooth functions $\{S, A_i: i=1, \dots, N-1\}$ and $N-1$ symplectic forms $\alpha_i = (\alpha_i)_{\mu\nu} dz^\mu \wedge dz^\nu$ such that in Ω the following representation holds:

$$Y_\mu = \frac{\partial S}{\partial z^\mu} + \sum_{i=1}^{N-1} \sum_{\nu=1}^N (\alpha_i)_{\mu\nu} \frac{\partial A_i}{\partial z^\nu} \quad (3)$$

where z and Y are vectors with components $z = \{x, \bar{x}\}$, $Y = \{y, -\bar{y}\}$.

Proof: To obtain a representation of the form ξ_f [Eq. (1)], one could construct a representation of $\xi = y \cdot dx - y' \cdot dx'$ in $M \times M$ and then project on the graph V of f .

$$V = \{(x, y)(x', y') : x' = \bar{x}(x, y), y' = \bar{y}(x, y)\} \quad (4)$$

However, because of the restriction to an open set Ω of V diffeomorphic to R^N it is simpler to work in V directly and look straight away for a representation of ξ_f .

Choose in V the set of co-ordinates $z = (x, \bar{x})$. From the Hodge decomposition theorem⁴⁾ and Poincaré's lemma one has⁵⁾

$$\xi_f = dS + \delta\eta \quad (5)$$

where S is a smooth function, η a smooth two-form and δ denotes the co-differential.

The next step is to write a representation for $\delta\eta$. This one does using a local lemma proved in Ref. 6) which states

$$\delta\eta = \delta \left\{ \sum_{i=1}^{N-1} A_i \alpha_i \right\} \quad (6)$$

where A_i are smooth functions and $\{\alpha_i\}$ a set of $N-1$ symplectic forms in R^N with the properties

- (i) $d\alpha_i = 0$
- (ii) $\alpha_i \wedge \dots \wedge \alpha_i = (N/2)! v$
- (iii) $*\alpha_i = \frac{1}{(N/2-1)!} \alpha_i \wedge \dots \wedge \alpha_i$
- (iv) $*\alpha_i \wedge \alpha_j = 0 \quad i \neq j$

[v is the volume form in R^N and * denotes Hodge duality].

A constructive prescription to obtain a set of two-forms obeying (i)-(iv) in any even dimension is given in Ref. 6). For $N = 2, 4$ and 6 , for example, the non-vanishing components are:

$$N = 2 \quad (\alpha_1)_{12} = 1$$

$$N = 4 \quad (\alpha_1)_{12} = (\alpha_1)_{34} = (\alpha_2)_{13} = (\alpha_2)_{42} = (\alpha_3)_{14} = (\alpha_3)_{23} = 1$$

$$\begin{aligned} N = 6 \quad (\alpha_1)_{12} &= (\alpha_1)_{34} = (\alpha_1)_{56} = (\alpha_2)_{13} = (\alpha_2)_{25} = (\alpha_2)_{46} = \\ &= (\alpha_3)_{14} = (\alpha_3)_{26} = (\alpha_3)_{35} = (\alpha_4)_{15} = (\alpha_4)_{24} = \\ &= (\alpha_4)_{36} = (\alpha_5)_{16} = (\alpha_5)_{23} = (\alpha_5)_{45} = 1 \end{aligned}$$

The result one wants to prove [Eq. (3)] follows now from combining Eqs. (5) and (6). ■

The result gives a differential representation of a map in even-dimensional Euclidean space. For a map operating in an odd dimensional space, one may embed it in R^{N+1} , apply Eq. (3) and then discard the irrelevant variables.

I will now work out this representation for three known bidimensional maps. For two dimensions, the representation is simply

$$\gamma = \frac{\partial S(x, \bar{x})}{\partial x} + \frac{\partial A(x, \bar{x})}{\partial \bar{x}} \quad (7a)$$

$$\bar{\gamma} = - \frac{\partial S(x, \bar{x})}{\partial \bar{x}} + \frac{\partial A(x, \bar{x})}{\partial x} \quad (7b)$$

1) For the Hénon map, $\bar{x} = 1 + y - ax^2$; $\bar{y} = -bx$

$$S_1 = \frac{a}{3} x^3 - x + x\bar{x} \quad A_1 = \frac{1}{2} (1-b) x^2 \quad (8)$$

2) For the standard map (with dissipation), $\bar{x} = bx + y + \alpha \sin(2\pi x)$;
 $\bar{y} = y + \alpha \sin(2\pi x)$

$$S_2 = \frac{\alpha}{2\pi} \cos(2\pi x) - \frac{1}{4} (1+b) (x - \bar{x})^2 \quad (9a)$$

$$A_2 = \frac{1}{4} (1-b) (x + \bar{x})^2 \quad (9b)$$

3) For one of the maps studied in Ref. 7), $\bar{x} = \sigma x + \alpha \{ \cos(2\pi x) + by \}$;
 $\bar{y} = by + \cos(2\pi x)$

$$S_3 = -\frac{1}{2\pi b} \sin(2\pi x) - \frac{(1+\sigma)(1+b)}{8\alpha b} (x - \bar{x})^2 + \frac{(1-\sigma)(1-b)}{8\alpha b} (x + \bar{x})^2 \quad (10a)$$

$$A_3 = \frac{(1+\sigma)(1-b)}{8\alpha b} (x - \bar{x})^2 + \frac{(1-\sigma)(1+b)}{8\alpha b} (x + \bar{x})^2 \quad (10b)$$

Of course, these representations are not unique, but neither in general are the generating functions for canonical maps.

REFERENCES AND FOOTNOTES

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- 2) R.S. Mackay, J.D. Meiss and I.C. Percival - Physica 13D (1984) 55, and references therein.
- 3) J.-P. Amiet and P. Huguenin - Helv.Phys.Acta 53 (1980) 377.
- 4) F.W. Warner - "Foundations of Differentiable Manifolds and Lie Groups", Ch. 6, Scott Foresman, Glenview (1971).
- 5) Actually Hodge's theorem asserts the decomposition into an exact, a coexact and a harmonic form. The harmonic form in the open set $\mathbb{Q}$, can, however, by Poincaré's lemma, be absorbed in the exact or the coexact components.
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