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NON-ABELIAN COMPACTIFICATION OF THE INTERACTING BOSONIC STRING

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A B S T R A C T

Compactification of the weight lattice of any simply-laced group G leads to a theory of open and closed interacting bosonic strings with full local G symmetry.

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Usual compactifications of space-time dimensions in quantum field theories generally break the symmetry group of the non-compactified theory, leaving only as residual symmetry the group of isometries of the compactified dimensions. For example, in an ordinary D space-time dimensional theory, compactification of d space dimensions on an arbitrary flat torus breaks the full original Lorentz symmetry $SO(D-1,1)$ to $SO(D-1-d,1) \otimes (U(1))^d$. Less trivial compactifications, in which the residual symmetry is non-Abelian, instead of the Abelian $(U(1))^d$ of this trivial example, have been used in supergravity theories¹⁾. A new approach to this problem has been made possible by the work of Refs. 2) and 3) on the so-called vertex operator of the dual string model. This work suggests, for example, that compactification of the ordinary bosonic string model onto an appropriate spacelike d -dimensional flat torus could be used to build an interacting ghost free string model in $26-d$ dimensions with any rank d simply laced compact Lie group as internal symmetry group. These are the groups for which all roots have equal lengths, $SU(d+1)$, $SO(2d)$, E_6 , E_7 and E_8 , or direct products thereof.

Furthermore, one would expect the symmetry to be a local one: indeed, in the closed string sector, one should find, in addition to the local space-time symmetry implied by the graviton, a local Yang-Mills invariance generated by transverse vectors belonging to the adjoint representation of the group considered. Thus, in contradistinction to the usual Kaluza-Klein compactification, this procedure would lead, for the compactified theory, to a local symmetry group which would be larger than the symmetry of the classical action.

However, a classical treatment of the closed string in compactified dimensions apparently invalidates the above conjecture, except for the group $SU(2)$ ⁴⁾. We shall briefly review this treatment. We shall then proceed to actually construct the closed string spectrum of the compactified theory by computing the non-planar loop of open strings and studying the poles in the closed string channel. We show that the group is present in this channel, which displays the expected Yang-Mills gauge bosons for any G [and not for $SU(2)$ only]. We then discuss the interpretation and consequences of our result, in particular the significance of the heterotic string truncation.

Following Ref. 3), we briefly recall how the symmetry group acts on open string states. We then go to the naive closed string analysis, which would indicate that the closed string sector is not compatible with this symmetry unless $G = SU(2)$.

In the light cone formulation, the independent degrees of freedom of an open bosonic string are parametrized by the familiar 24 transverse co-ordinates [see Ref. 5) for a review and notations]

$$X^i(\sigma, \tau) = x^i + p^i \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^i \cos n\sigma e^{-in\tau} \quad (1)$$

where we have taken the slope α' to be $\frac{1}{2}$. The mass formula is

$$\frac{1}{2} m^2 = \sum_{n=1}^{\infty} \sum_{i=1}^{24} \alpha_{-n}^i \alpha_n^i - 1 \equiv N - 1 \quad (2)$$

When r dimensions are compactified, the corresponding components of the centre-of-mass momentum p^γ take discrete values only, and the mass formula becomes

$$\frac{1}{2} m^2 = \frac{1}{2} p_\gamma^2 + N - 1 \quad (3)$$

For straightforwardness, we take p^γ to lie on the root lattice Λ_R of the simple, simply-laced rank r Lie group G , and we normalize the roots to $p_\gamma^2 = 2$. One basic result of Ref. 3) is that formula (3) is group invariant, although not obviously so.

The lowest state of (3) is the familiar tachyon at $m^2 = -2$, which is a group scalar. At mass zero, there are r Lorentz scalars obtained by taking harmonic excitations in the compactified dimensions $\alpha_{-1}^\gamma |0\rangle$, where $\gamma = 1, 2, \dots, r$. At the same mass, there are $\dim G - r$ other states $|p_\gamma\rangle$ obtained by taking for p_γ one of the roots. These $\dim G$ states combine together to form an adjoint representation of G , and the corresponding generators are the vertex operators constructed in Refs. 2) and 3). These vertex operators commute with the Virasoro generators L_n , and ensure that at all mass levels, the G symmetry is maintained.

When going to closed strings, one must allow for non-trivial winding in the compactified dimensions, and one writes⁵⁾

$$X^i(\sigma, \tau) = x^i + p^i \tau + 2L^i \sigma + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \left[\alpha_{nR}^i e^{2in(\tau-\sigma)} + \alpha_{nL}^i e^{-2in(\tau-\sigma)} \right] \quad (4)$$

In this formula, the term $L^i \sigma$, which describes winding on the torus is non-zero only for the index i lying in a compactified direction and $\vec{L}$ belongs to $\Lambda_R^* = \Lambda_W$, where Λ_W is the weight lattice. Following the same light-cone treatment as for the open string, one obtains the mass formula⁵⁾

(5)

This is not the whole story: the parametrization (4) selects the point $\sigma = 0$ on the string. To express the fact that actually this point is not special, one imposes on all physical states the constraint

$$\frac{1}{4}(m_L^2 - m_R^2) \equiv i \frac{\partial}{\partial \sigma} = 0 \quad (6)$$

where

$$\frac{1}{2} m_L^2 = \frac{1}{8} (p_Y - 2L_Y)^2 + N_L - 1 \quad (7)$$

$$\frac{1}{2} m_R^2 = \frac{1}{8} (p_Y + 2L_Y)^2 + N_R - 1 \quad (8)$$

Equation (6) can be rewritten as

$$p_Y L_Y = N_R - N_L \quad (9)$$

One easily generates r massless vectors $\alpha_{-1R}^\mu \alpha_{-1L}^\gamma |0\rangle$ corresponding to gauging the maximal Abelian subgroup $U(1)^r$ of G . We consider here non-oriented open strings, and hence, one has to symmetrize these states between left- and right-moving modes. From the analysis of the open string and the general intuitive feeling that closed strings essentially behave like a direct product of two open strings (one for the R modes and one for the L modes), one expects that the remaining $\dim G - r$ gauge bosons are of the form $\alpha_{-1}^\mu |p_Y\rangle$, with $p_Y^2 = 2$ but one easily checks⁴⁾ that it is not possible to find an $\vec{L} \in \Lambda_W$ such that this state has zero mass. Indeed, one has $N_R - N_L = 1$, hence $p_Y L_Y = 1$, which means that $\vec{L} = \frac{1}{2} \vec{p}_Y + \vec{M}$, with $\vec{M}$ orthogonal to $\vec{p}_Y$. One immediately verifies that only $\vec{M} = 0$ is such that the mass is zero, but $\vec{L} = \frac{1}{2} \vec{p}_Y$ is a weight only for $SU(2)$. This was the conclusion of Ref. 4).

One has nevertheless the feeling that because a consistent theory of interacting open strings with G as a symmetry group exists at the tree level³⁾, closed strings should come in automatically via loops of open strings in a manner consistent with G . We shall now see that this is indeed the case for any

simply-laced G , and find how the spectrum (5) and the constraint (9) are correspondingly modified.

The non-planar loop of open strings moving in compactified dimensions was first computed in Ref. 6), where the winding numbers of the closed string were thus discovered. Hence, we just have to adapt the calculations of Ref. 6), which were done with a compactified momentum lying on a hypercubic lattice, to our case, where $p_Y \in \Lambda_r$.

We thus compute the loop diagram of Section (5.2) of Ref. 5), using the same conventions and notations. The external particles all satisfy Eq. (2) with $N = 0$. We first use the standard twist

$$\Omega = (-1)^N \quad (10)$$

The trace calculation over states circulating in the loop decomposes into three traces: the harmonic oscillator trace, the orbital momentum integration and the summation over the discrete compactified momenta.

The oscillator trace gives the usual answer, Eq. (5.37) of Ref. 5), since the oscillators ignore the compactification:

$$\text{trace/oscillators} \rightarrow \int (\omega)^{-24} \prod_{I < J} \prod_{m=1}^{\infty} \left[\frac{(1 \pm \omega^{m-1} c_{IJ})(1 \pm \omega^m / c_{IJ})}{(1 \pm \omega^m)^2} \right]^{k_I k_J} \quad (11)$$

with the + sign if I and J are on different boundaries, and the - sign if they are on the same boundary.

The orbital momentum integration is identical to the planar case, and reproduces formula (5.13) of Ref. 5):

$$M = \int d^d p \prod_{I=1}^M x_I^{\frac{1}{2} p_I^2} = \left(-\frac{2\pi}{L\omega} \right)^{\frac{1}{2}(26-d)} \prod_{1 \leq I < J \leq M} \left[c_{IJ}^{-\frac{1}{2}} \exp \frac{L\omega^2 c_{IJ}}{2L\omega} \right]^{k_I k_J} \quad (12)$$

For comparison with the forthcoming discrete sum over compactified momenta, we note that the integrand in (12) is a Gaussian of the form $\exp(Ap^2 + Bpr + C)$. The term C gives the $c_{JI}^{-\frac{1}{2}}$ piece of (12), the completion of the square

$$A p^2 + B p r = A \left(p + \frac{B}{2A} r \right)^2 - \frac{B^2}{4A} r^2 \quad (13)$$

gives the $\exp(\ln^2 C_{JI}/2 \ln w)$ term, and finally, after shifting p by $(B/2A)r$, the Gaussian integral gives $(-2\pi/\ln w)^{(26-d/2)}$.

The discrete sum over the momenta of the compactified directions also involves a term of the form $\exp(Ap^2 + Bpr + C)$. The C term is as before, but the discreteness of the sum over $p \in \Lambda_R$ forbids the shift by $(B/2A)r$ which in general does not belong to Λ_R . Hence, one is left with the following function

$$F_{\Lambda_R}(\vec{v}|\tau) = \sum_{p \in \Lambda_R} e^{i\pi \tau p^2} e^{2i\pi \vec{v} \cdot \vec{p}} \quad (14)$$

where

$$e^{i\pi \tau} = e^{\frac{1}{2} \ln w}, \quad 2i\pi \vec{v} = \sum_{I=1}^m p_I \ln p_I \quad (15)$$

This function is a multivariable generalization of the standard elliptic θ functions. An essential ingredient^{5),6)} in the analysis of the singularities of the closed string channel is the Jacobi transformation

$$\vec{v} \rightarrow -\frac{\vec{v}}{\tau}, \quad \tau \rightarrow -\frac{1}{\tau} \quad (16)$$

One can analyze the behaviour of $F_{\Lambda_R}(\vec{v}|\tau)$ under this transformation by the same method as in Ref. 5) for ordinary θ functions, and one obtains^{*)}

$$F_{\Lambda_R}(\vec{v}|\tau) = \frac{1}{(-i\tau)^{\frac{1}{2}V}} \exp\left(-i\pi \frac{V}{\tau}\right) F_{\Lambda_W}\left(-\frac{\vec{v}}{\tau} \middle| -\frac{1}{\tau}\right) \quad (17)$$

where F_{Λ_W} is the same function as F_{Λ_R} , defined by Eq. (3.5), except that the sum runs over the weight lattice $\Lambda_W = \Lambda_R^*$:

$$F_{\Lambda_W} = \sum_{\vec{w}} q^{\vec{w}^2 + 2\vec{v} \cdot \vec{w}} \quad (18)$$

*) $V \equiv \sqrt{|g^{ij}|}$ where g^{ij} is the Cartan matrix; hence, $V = \sqrt{Z}$, where Z is of the order of the centre of the covering group of G .

with

$$q \equiv e^{-\frac{i\pi}{\tau}} = e^{\frac{2\pi^2}{\ln \omega}} \quad (19)$$

Before proceeding, we must correct for the use of an improper twist operator; Eq. (10) is not the correct twist operator to be used in the case of compactification on the root lattice of G , because it is not group invariant. The massless states $\alpha_{-1}^Y |0\rangle$ and $|p_Y\rangle$, which form the adjoint representation of G have different eigenvalues of Ω . This can be corrected by replacing Ω by

$$\Omega' = \Omega(-1)^{\frac{1}{2} P_Y^2} \quad (20)$$

which trivially commutes with all the group generators of Goddard and Olive. Equation (20) adds a phase to every twisted propagator, and provides the required group invariant extension of the phase prescriptions of Goddard and Olive which were valid for the tree amplitudes.

The effect of Eq. (20) is to replace some variables ρ_I by $-\rho_I$ in Eq. (15). This turns the vector $\vec{v}$ of (3.6) into $\vec{v} + (\vec{p}_Y/2)$, where $\vec{p}_Y$ is the total compactified momentum flowing in the closed string channel. After performing the Jacobi transformation, one finds that the only modification is to transform the sum on $\vec{W}$ in Eq. (10) from the weight lattice to the weight lattice shifted by $\frac{1}{2}\vec{p}_Y$. The final answer for the non-planar loop is thus

$$A = \int_0^1 \frac{dq}{q^3} \prod_{I=1}^{N-1} d\nu_I \left(\frac{-2\pi^2}{\ln q} \right)^N [f(q^2)]^{-24} \prod_{I < J} (\psi_{IJ}^{\nu_I} + \psi_{IJ}^{\nu_J})^{k_I k_J} \\ \times \sum_{\vec{W}} q^{(W + \frac{1}{2} P_Y)^2} \exp \left[-2i\pi \sum_{I=1}^N \nu_I \vec{P}_I (\vec{W} + \frac{1}{2} \vec{P}_Y) \right] \quad (21)$$

The functions ψ_{IJ} and ψ_{IJ}^T can be found in Ref. 5).

The rest of the analysis is then straightforward, following Ref. 6): one finds poles at the following value of the mass

$$\frac{1}{4} m^2 = \frac{1}{4} P_Y^2 + W'^2 + N_L + N_R - 2 \quad (22)$$

where $\vec{W}' = \vec{W} + \frac{1}{2}\vec{p}_Y$, together with the constraint

$$\vec{p}_\gamma \vec{W}' = N_R - N_L \quad (23)$$

These are exactly the conditions of the naive classical analysis of Eqs. (5) and (9), except that $\vec{L} = \vec{W} + \frac{1}{2}\vec{p}_\gamma$ and is therefore no longer on the weight lattice [except for SU(2)]. The massless Yang-Mills bosons of the group G are obtained for $p_\gamma^2 = 2$, and $\vec{W} = 0$, as expected, which satisfies (22) and (23) for $m = 0$. Symmetrization between left and right movers is obtained by adding the solution $\vec{W} = -\vec{p}_\gamma$. Note that we would obtain the group $G \times G$ instead of G if the open strings were oriented by suitable Chan-Paton factors.

Thus, the non-planar loop tells us how closed strings accommodate the group G in the compactified dimensions. In particular, a transverse Lorentz vector in the full adjoint representation of G is produced at zero mass, turning invariance under G into a local gauge symmetry, and we are now in a position to discuss the general features of this construction.

At first sight, it may seem strange that $\vec{L}$ in Eq. (4), which is a topological quantity (winding around the torus) should contain $\vec{p}_\gamma$, which is dynamical, as we find. However, there is no inconsistency. The constraint (9), or (23), is only a constraint applied on states which is compatible with the equations of motion. The mixing of $\vec{W}$ and $\vec{p}$ in $\vec{L}$ reflects the fact that the topological stability of the winding number is restricted to the Cartan subgroup of G. This is analogous to the topological stability of monopoles in unbroken gauge theories, where all monopoles with magnetic charge differing by a root lattice vector are topologically equivalent in the full group⁷⁾.

We find that the closed string spectrum is labelled, in the compactified dimensions, by both the root and the weight lattice of G which are dual to each other. As in the simpler compactification of Chapter 7 of Ref. 5), one expects that when building loops of closed strings, one must allow non-zero values of $\vec{W}$ to circulate in the loop. This combined sum over roots and weights is in general necessary for invariance under the modular group and consistency of these loops of closed strings. The heterotic sector of the closed bosonic string can be defined by the truncation $\vec{W} = 0$ or $\vec{W} = -\vec{p}_\gamma$.

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