CERN-TH.4177/85
LPTENS 85/17

FLAT AND POSITIVE POTENTIALS IN $N = 1$ SUPERGRAVITY

R. Barbieri^{*)} and E. Cremmer

Laboratoire de Physique Théorique
Ecole Normale Supérieure, Paris⁺⁾

and

S. Ferrara

Laboratoire de Physique Théorique
Ecole Normale Supérieure, Paris⁺⁾
and
CERN - Geneva

A B S T R A C T

We find the necessary and sufficient conditions in $N=1$ supergravity for theories having flat or semi-positive definite potentials. These conditions involve quantities naturally occurring in the minimal formulation of $N=1$ supergravity and may be relevant in recent attempts at realistic compactifications of super- string theories.

^{*)} Permanent address: Istituto di Fisica, Università di Pisa, 56100 Pisa, Italy.

⁺⁾ Laboratoire Propre du C.N.R.S., associé à l'Ecole Normale Supérieure et à l'Université de Paris-Sud.

Recently it has been realized [1-3] that certain supergravity theories may have semipositive definite potentials, then implying, at the classical level, the vanishing of the cosmological constant, even in presence of supersymmetry breaking. The construction of these theories originated in the observation that [1,4,5] in N=1 supergravity one may have a "flat" potential with an undetermined gravitino mass. This fact has led to the construction of a certain class of phenomenologically interesting models [1,2,3], the so-called no-scale models, in which one could hope to get all low energy scale parameters from the Planck mass through the mechanism of dimensional transmutation. Interestingly enough, in recent attempts to connect superstring theories [6] to low energy N=1 supergravity [7] in 4 dimensions, the low energy effective lagrangian seems to fall in a certain class of no-scale models in which the supergravity potential, including the hidden and observable sector, is semipositive definite. In an interesting attempt to simulate what superstring compactification on a Calabi-Yau manifold could be [8], Witten has recently derived a consistent truncation of 10-dimensional supergravity coupled to N=1 Yang-Mills in which a N=1 supersymmetry in 4 dimensions survives. The resulting lagrangian, with the possible inclusion of an effective superpotential term, has a semipositive definite potential.

In the present note, we elucidate the necessary and sufficient conditions for semipositive definite potentials or vanishing potentials in N=1 supergravity. These conditions are of course satisfied by all previously exhibited cases and are useful to construct all possible theories having these properties.

Let us consider a general N=1 supergravity theory coupled to chiral multiplets and gauge fields. The overall scalar potential is given by [9]

$$V(z_i) = e^{\mathcal{G}} [g^i (g^{-1})_i{}^j g_j - 3] + \text{Re } f_{\alpha\beta}^{-1} D^\alpha D^\beta \quad (1)$$

in terms of the Kahler potential $J(z, z^*)$, the superpotential $f(z)$, the "kinetic" Yang-Mills metric $f_{\alpha\beta}(z)$. The function

$$\mathcal{G} = J + \text{Log } |f(z)|^2 \quad (2)$$

is the basic object needed to construct the potential. The Yang-Mills part contains D^α gauge coefficients

$$D^\alpha = g_\alpha g^i (T^a)_i{}^j z_j \quad (3)$$

and is already semipositive definite in the positivity domain of the kinetic-energy of the Yang-Mills fields. The only part which is not positive definite is the chiral part

$$V_{ch} = e^g [g^i (g^a)_i{}^j g_j - 3] \quad (4)$$

which is in fact the difference between two positive definite terms due to the positivity properties of the $g^i{}_j$ matrix which defines the kinetic energy of the spin 0 and spin 1/2 particles of the chiral multiplets.

In the absence of supersymmetry breaking the quantities $e^g g_i$ vanish and the vacuum energy is given by [9]

$$V_{vac} = -3 e^g \quad (5)$$

corresponding to anti-De Sitter ($g \neq -\infty$) or Poincaré ($g = -\infty$) supergravity. On the other hand if supersymmetry is broken, it may happen that $V_{ch} \geq 0$ then implying a positive or vanishing vacuum energy.

It is easy to find such a situation. The Polony model [10] is a particular example of a potential that, by fine tuning of the parameters, can become semipositive definite. In this case the function g is

$$g = z z^* + \text{Log} |z + \beta|^2 \quad (6)$$

By a fine tuning of the β parameter to $\beta = 2 - \sqrt{3}$, one can get

$$V_{ch} = 0 \quad \text{at} \quad V_{,z} = 0 \quad (7)$$

otherwise $V_{ch} > 0$.

It is obvious that the g function is not appropriate to discuss positivity properties of the potential. By a simple manipulation of our formula, we will show that the appropriate function to discuss the positivity of the potential is the ϕ function defined through

$$g = -3 \text{Log} \phi \quad (8)$$

and naturally occurring in the superfield formulation of N=1 supergravity. The necessary and sufficient condition for having a semipositive definite potential turns out to be

$$\det (-\varphi) \leq 0 \quad (9)$$

in which φ is the hermitian matrix $\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}^j}$

In particular the necessary and sufficient condition for vanishing potential is

$$\det \varphi = 0 \quad (10)$$

To show this property we need to rewrite the potential V_{ch} in terms of determinants of the φ and $g = g_{i\bar{j}}$ matrices. Since

$$g_{i\bar{j}} = -3 \left(\frac{\phi_{i\bar{j}}}{\phi} - \frac{\phi_i \phi_{\bar{j}}}{\phi^2} \right) \quad (11)$$

if one defines

$$\rho = \phi^i (\varphi^{-1})_{i\bar{j}} \phi_{\bar{j}} \quad (12)$$

$$\rho \det \varphi = \phi^i A_{i\bar{j}} \phi_{\bar{j}} \quad (13)$$

one finds

$$(g^{-1})_{i\bar{j}} = -\frac{1}{3} \left[\phi (\varphi^{-1})_{i\bar{j}} + \frac{(\varphi^{-1})_{i\bar{k}} \phi_{\bar{k}} \phi^{\ell} (\varphi^{-1})_{\ell \bar{j}}}{1 - \rho/\phi} \right] \quad (14)$$

In deriving (14) we used the two properties of determinants of matrices in which the elements are themselves matrices

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det a \det (d - c a^{-1} b) \quad (15)$$

$$\det (I - |\alpha\rangle\langle\beta|) = 1 - \langle\alpha|\beta\rangle \quad (16)$$

By inserting (14) in the previous expression for V one gets

$$V = -\frac{3}{\phi^2} \frac{1}{\phi - \rho} \quad (17)$$

Positivity of the kinetic energy implies

$$g^i (g^{-1})_i{}^j g_j = - \frac{3\varphi}{\phi - \varphi} > 0 \text{ i.e. } \varphi(\phi - \varphi) < 0 \quad (18)$$

This can be satisfied by $\varphi < 0$ or $\varphi > \phi$ ($\phi > 0$). The two situations correspond to $V < 0$ and $V > 0$ respectively, the vanishing situation only occurring for $\varphi \rightarrow \infty$ i.e. with φ not invertible. The previous formula can be simplified by noting that (14) implies that

$$\det g = \det \left(-\frac{3\varphi}{\phi} \right) \left(1 - \frac{\varphi}{\phi} \right) \quad (19)$$

so that Eq. (17) can be rewritten as

$$V = - \frac{3}{\phi^3} \det \left(-\frac{3\varphi}{\phi g} \right) \quad (20)$$

This expression for V is true also for φ not invertible. Since $\phi > 0$ and $\det g > 0$ in the positive energy domain of the theory, we get that

$$V > 0 \text{ is equivalent to } \det(-\varphi) \leq 0 \quad (21)$$

For positive potentials Eq.(21) obviously implies that the φ matrix has an odd number of positive eigenvalues. The relation (11) actually implies that φ has one positive and $(n-1)$ negative eigenvalues as can be seen from the expression $\alpha^i \varphi_i{}^j \bar{\alpha}_j$ for an arbitrary α^i which can be written as

$$\alpha^i \varphi_i{}^j \bar{\alpha}_j = - \frac{\phi}{3} \beta^i g_i{}^j \bar{\beta}_j + \frac{|\alpha^k \phi_k|^2}{\varphi} \quad (22)$$

with β^i defined by

$$\beta^i = \alpha^i - \frac{(\alpha^p \phi_p) (g^{-1})^i{}_k \phi^k}{\phi_m (g^{-1})^m{}_n \phi^n}$$

satisfying obviously $\beta^k \phi_k = 0$ so that there are only $(n-1)$ independent β 's.

For $V > 0$ φ is > 0 and we obtain immediately the result on the properties of the eigenvalues of φ . On the other hand for negative potential $V < 0$, since $\varphi < 0$, the matrix φ is a negative definite matrix.

For flat potentials

$$V \equiv 0 \quad \text{is equivalent to} \quad \det \varphi = 0 \quad (23)$$

Therefore φ is a matrix of rank at most $n-1$. In fact if the rank of φ is lower than $n-1$, one also gets $\rho \det \varphi = 0$ which would imply $\det g = 0$ from Eq.(19). From Eq.(22), since $\rho = \infty$, we get that φ is a semidefinite negative matrix, more precisely it has one vanishing and $(n-1)$ negative eigenvalues.

We now want to show how all known examples can be easily understood from our condition on the determinant of the φ matrix.

Let us first consider flat potentials in which case $V = 0$. The first examples were [1]

$$\phi = f(z) + \bar{f}(z^*) + \rho(y, y^*) \quad (24)$$

$$\phi = \prod_{a=1}^n [f_a(z_a) + \bar{f}_a(z_a^*)]^{\frac{1}{n}} \quad (25)$$

In (24) $\det \varphi = 0$ trivially since $\varphi_z z^* = \varphi_{z^*} z = 0$. The solution (25) is equivalent to

$$\phi = \prod_{a=1}^n (z_a + z_a^*)^{\frac{1}{n}} \quad (26)$$

which implies that the ϕ function is homogeneous of degree 1 in the

$x_a = z_a + z_a^*$ variables, and satisfies

$$\phi_a x^a = \phi \quad (27)$$

$$\phi_{ab} x^b = 0 \quad (28)$$

and therefore $\det \varphi = 0$.

Actually Eq.(26) is a particular case of the solution

$$\phi^M = C_{\alpha_1 \dots \alpha_M} (z + z^*)_{\alpha_1} \dots (z + z^*)_{\alpha_M} \quad (29)$$

in which $C_{\alpha_1 \dots \alpha_M}$ are constant coefficients. For $M=3$ we get a class of flat potentials of $N=2$ supergravity coupled to vector multiplets [11].

It should be noted that the positivity condition on the kinetic energy gives restriction on the $c_1, \dots, c_m$ coefficients as it in fact happens in N=2 supergravity.

Another situation is the case in which ϕ^M is harmonic so that

$$\phi_i^j = (1-M) \frac{\phi_i \phi^j}{\phi} \quad (30)$$

Since ϕ_i^j must be a matrix of rank $n-1$ where n is the number of fields, the models exist only for $M=1, n=1$ and for $M > 1, n=1, 2$.

We now turn to the more interesting case of semipositive definite potentials. Let us first consider superpotential modifications which preserve the semipositivity of the scalar potential. For this purpose, let us write the function ϕ as

$$\phi(z, z^*) = \psi(z, z^*) |h(z)|^2 \quad (31)$$

where the matrix $\psi_i^j = \frac{\partial^2 \psi}{\partial z_i \partial z^{*j}}$ satisfies by itself $\det(-\psi) < 0$.

By means of Eq.(15,16) one gets

$$\det(-\psi) = \det(-\psi) |h(z)|^{2n} X \quad (32)$$

where

$$X = 1 + a^2 + b(\psi - e(\psi))$$

$$a = \psi^k (g^{-1})_k^e \frac{\bar{h}_e}{h}$$

$$b = \frac{h^k}{h} (g^{-1})_k^e \frac{\bar{h}_e}{h}$$

and

$$e(\psi) = \psi^k (g^{-1})_k^e \bar{\psi}_e \quad (33)$$

The potential is semipositive definite if and only if $X \geq 0$. From Eq. (19)

$\psi - e(\psi) < 0$ so that a sufficient condition for a positive potential is $b \leq 0$.

By use of the relation

$$\frac{h^k}{h} (g^{-1})_k^e \frac{\bar{h}_e}{h} = - \frac{\psi}{3(\psi - e)} (1 + a^2 + b(\psi - e)) \quad (34)$$

we can rewrite

$$X = 1 + a + \bar{a} - 3 \frac{\psi - \bar{\psi}}{\psi} \frac{h^k}{h} (g^{-1})_k^l \frac{\bar{h}_l}{h} \quad (35)$$

Therefore from the positivity of the metric the condition $a + \bar{a} + 1 \geq 0$ also implies $\det(-\psi) \leq 0$.

We consider now under which circumstances the potential may vanish at some point. If $\det(-\psi) \neq 0$ everywhere and $b \leq 0$, then $V = 0$ requires $b = 0$ and $a + 1 = 0$, which can occur only for $h_i \neq 0$. In the second situation for which $a + \bar{a} \geq -1$ V can never vanish.

Of special interest is the case where $\det \psi \equiv 0$ in eq.(31). In this limiting case, introducing the cofactor matrix A

$$\psi_i^e A_e^k = A_i^e \psi_e^k = 0 \quad (36)$$

one has, from Eqs.(32,24),

$$\det \psi = |h|^{2n} \left[\psi^k A_k^e \frac{\bar{h}_e}{h} + \frac{h^k}{h} A_k^e \bar{\psi}_e + \frac{3}{\psi} \psi^e A_e^k \bar{\psi}_k \frac{h^l}{h} (g^{-1})_l^m \frac{\bar{h}_m}{h} \right] \quad (37)$$

$$\det g = \left(-\frac{3}{\psi}\right)^{n+1} \psi^k A_k^e \bar{\psi}_e \quad (38)$$

A sufficient condition for $\det(-\psi) \leq 0$ therefore is

$$(-1)^n \left(\psi^k A_k^e \frac{\bar{h}_e}{h} + \frac{h^k}{h} A_k^e \bar{\psi}_e \right) \leq 0 \quad (39)$$

The special case

$$\psi^k A_k^e \frac{\bar{h}_e}{h} = 0 \quad (40)$$

leads to

$$\det \psi = -\frac{3}{\psi} (\psi^e A_e^k \bar{\psi}_k) \frac{h^k}{h} (g^{-1})_k^l \frac{\bar{h}_l}{h} \quad (41)$$

which vanishes if and only if $h^k = 0$. This case includes, in a form invariant under analytic reparametrizations, the examples considered in the

context of no-scale models [2]

$$\phi(z, z^*, t, t^*) = (z + z^* - \rho(t, t^*)) |h(t)|^2 \quad (42)$$

Condition (40) is trivially satisfied since A_{zz^*} is the only non-vanishing entry of the cofactor matrix A and $h_z = 0$. Notice on the other hand that all models satisfying (40) are no-scale models: if at one point z_k $h^k = 0$, i.e. $V(z_k) = 0$, then Eq.(40) gives a full line where the potential vanishes since h^k identically vanishes in the direction of the vector $\varphi^i A_{ik}$.

The definite sign of the eigenvalues of the matrix φ_i^j previously discussed suggests an alternative example. Namely

$$\phi = |t_0|^2 - |t_1|^2 - C \quad (43)$$

It is easy to check that if $C > 0$ then the kinetic energy is positive in the domain $\phi > 0$ (which is actually a hyperbola). This is not incompatible with the fact that the theory has a linear $SU(1,1)$ invariance. The potential is easily computed to be

$$V = \frac{3}{C \phi^2} \quad (44)$$

The above potential is positive definite and never vanishes. However we can easily extend this to obtain a semipositive potential by modifying the function ϕ accordingly

$$\phi = f(t_0, t_0^*) - |t_1|^2 \quad (45)$$

with $f_{t_0, t_0^*} \geq 0$. Then the positivity domain of the kinetic energy is defined by

$$\phi > 0, \quad |f_{t_0}|^2 > f f_{t_0, t_0^*} \quad (46)$$

The potential can vanish if f_{t_0, t_0^*} can vanish as can be checked by the explicit expression of the potential

$$V = \frac{3 f_{t_0, t_0^*}}{\phi^2 (|f_{t_0}|^2 - f f_{t_0, t_0^*})} \quad (47)$$

It is in fact a no-scale model since t_1 is undetermined. This can be easily extended to several variables t_i with a negative signature. Furthermore, the addition of an arbitrary superpotential $h(t_1)$ to (45) preserves the semipositivity of the scalar potential since the quantity b defined by eq.(33) is automatically negative due to the signature of the matrix φ . In this case however, the vanishing of the potential requires fixing both t_0 and t_1 variables.

Finally we remark that all examples with positive potentials considered in this paper, associated with a function $\phi_1(x, x^*)$ with $\det(-\varphi_1) < 0$, can be trivially generalized to

$$\Phi(z, z^*) = \phi_1(x, x^*) \phi_2(y, y^*) \quad z = (x, y) \quad (48)$$

since the x and y sectors are uncoupled in the metric matrix g . From the point of view of the overall determinant this is seen as follows

$$\det(-\varphi) = \det(-\Phi_2 \varphi_1) \det(-\Phi_1 \varphi_2) \left(1 - \frac{\rho_1 \rho_2}{\phi_1 \phi_2}\right) \quad (49)$$

which is manifestly negative, since

$$\begin{aligned} \det(-\varphi) &= (\phi_1 \phi_2)^{n+m+1} \det\left(-\frac{\varphi_1}{\phi_1}\right) \det\left(-\frac{\varphi_2}{\phi_2}\right) \times \\ &\quad \times \left[1 - \frac{\rho_2}{\phi_2} + \left(1 - \frac{\rho_1}{\phi_1}\right) \frac{\rho_2}{\phi_2}\right] \end{aligned} \quad (50)$$

and

$$\det\left(-\frac{\varphi_2}{\phi_2}\right) \left(1 - \frac{\rho_2}{\phi_2}\right) > 0, \quad \det\left(-\frac{\varphi_2}{\phi_2}\right) \frac{\rho_2}{\phi_2} < 0 \quad (51)$$

The ansatz (48) trivially generalizes to the case in which the x, y fields are coupled through a superpotential term

$$\Phi(z, z^*) = \phi_1(x, x^*) |h(x, y)|^2 \phi_2(y, y^*) \quad (52)$$

in which $\phi_1 |h|^2$ is semipositive definite. This is most easily seen from the formulation in terms of the overall g function and the form given by eq.(4) for the overall chiral potential.

To summarize, we have shown that the positivity properties of supergravity potentials are dramatically simplified in terms of the properties of the matrix Φ_{ij} and of its eigenvalues. This allows the construction of general models with vanishing cosmological constant, broken supersymmetry and no-fine tuning of parameters.

REFERENCES

- 1 E.Cremmer, S.Ferrara, C.Kounnas, D.V.Nanopoulos, Phys.Lett. 133B (1983) 61.
- 2 J.Ellis, A.B.Lahanas, D.V.Nanopoulos, K.A.Tamvakis, Phys.Lett. 134B (1984) 61; J.Ellis, C.Kounnas, D.V.Nanopoulos, Nucl.Phys. B241 (1984) 406, Nucl.Phys. B247 (1984) 373, Phys.Lett. 143B (1984) 61.
- 3 N.Dragon, M.G.Schmidt, U.Ellwanger, Phys.Lett. 145 (1984) 192. CERN preprint TH-4129 (1985).
- 4 R.Barbieri, S.Ferrara, D.V.Nanopoulos, K.Stelle, Phys.Lett. 113B (1982) 219.
- 5 N.P.Chang, S.Ouvry, X.Wu, Phys.Rev.Lett. 51 (1983) 327.
- 6 M.B.Green and J.H.Schwarz, Phys.Lett. 149B (1984) 117.
- 7 P.Candelas, G.Horowitz, A.Strominger, E.Witten, Santa Barbara preprint (1984).
- 8 E.Witten, "Dimensional reduction of superstring models", Princeton preprint Feb. (1985).
- 9 E.Cremmer, S.Ferrara, L.Girardello, A.Van Proeyen, Phys.Lett. 116B (1982) 231; Nucl.Phys. B212 (1983) 413.
- 10 J.Polony, Budapest preprint K-FKI-1977-83 (1977).
- 11 E.Cremmer et al., Nucl.Phys. B250 (1985) 385.