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SPINOR TECHNIQUES FOR CALCULATING $p\bar{p} \rightarrow W^\pm/Z^0 + \text{JETS}$

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A B S T R A C T

We present techniques which enable one to calculate quickly the amplitudes for many scattering processes in the high energy limit. As an illustration of the method, these are applied to the diagrams for $p\bar{p} \rightarrow V + 0, 1$ or 2 jets, where $V = W^\pm$ or Z^0 . The form of the results lends itself to immediate numerical evaluation.

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1. - INTRODUCTION

This paper deals with complicated tree-level amplitudes for high energy scattering processes in the framework of the standard theories of electroweak and strong interactions; in particular we discuss how to evaluate such amplitudes. The motivation for this is as follows. In the first place, as our understanding of the gauge theories describing the interactions develops, we will have to calculate ever higher orders in the perturbation series in order to refine the theoretical predictions. Apart from multi-loop contributions, high-order tree graphs then come into play. The evaluation of such diagrams by "standard algebraic techniques", although straightforward, becomes impractical when both the number of external lines and the number of diagrams involved become large. There will be an increasing need for methods which make it easier to obtain from the diagrams a value for the cross-section. (Throughout this paper we shall use the term "cross-section" for the absolute value squared of the amplitude, summed and averaged over spins and colours in some appropriate manner, but without any integral over the phase space momenta.)

In the second place a number of processes for which the cross-section is best evaluated by such "novel" techniques has already become observable. Examples are the production of multilepton states (such as $e^+e^+e^-e^-$, $e^+e^-\mu^+\mu^-$) in e^+e^- scattering¹⁾, multijet production in both e^+e^- and $p\bar{p}$ interactions, and the production of W and Z bosons in concurrence with jets at the CERN $p\bar{p}$ collider²⁾. It is this last process which we will discuss in some detail.

The outline of the paper is as follows. In Section 2 we argue that the "standard" way of calculating cross-sections has to be modified in order to make complicated sets of Feynman diagrams amenable to evaluation, and we enumerate the various ingredients that go into our alternative approach. These comprise: a particular representation of the spinors describing massless fermions (from which follow some identities for spinorial quantities by which we can rewrite and simplify diagrams); a representation for the polarization vectors of spin-1 particles, both massive ($W^\pm$, Z^0) and massless (photon, gluon); and, finally, a representation of the spinors for massive fermions such as top quarks. These various points will be discussed in Sections 3 through 6.

In Sections 7 and 8 we apply the techniques developed to an actual non-trivial calculation: that of the parton-level diagrams which contribute to $p\bar{p} \rightarrow V + 0, 1, \text{ or } 2 \text{ jets}$ (in this paper, V will denote a $W^\pm$ or Z^0 boson).

Apart from providing an illustration of our method, these processes are of sufficient physical interest to make knowledge of the exact cross-section important. The formulae presented here form the basis of numerical programs the results of which have already been discussed extensively. In this paper therefore we shall limit ourselves to describing the Feynman diagrams and to deriving the numerical expressions corresponding to them. We refer to a later publication for the complete enumeration of the 396 processes contributing to a $V + (0,1,2)$ jet final state. Finally, the Appendix contains the proofs of the statements that underly our technique. Included for completeness, these can also serve as illustrations for those readers who wish to make themselves familiar with our method.

2. - THE NEED FOR AMPLITUDE-LEVEL CALCULATIONS

In this section we discuss the evaluation of tree amplitudes, in particular those involving electroweak interactions. These usually consist of one or more spinor sandwiches contracted with each other by boson propagators. External on-shell vector particles are represented by their polarization vectors.

After writing down the Feynman diagrams corresponding to a given amplitude M one usually proceeds to derive an analytic expression for the cross-section $\Sigma |M|^2$, with an appropriate spin and/or colour sum or average. The result, which is usually a function of Minkowski products of the particle four-momenta, is then evaluated numerically at phase-space points in the region of interest.

The derivation of the cross-section formula is facilitated by two important considerations. First, the polarization vectors ϵ^μ of external spin-1 particles can be summed covariantly. For a heavy vector boson with mass m and momentum q^μ this results in the spin sum

$$\Sigma \epsilon^\mu \epsilon^{*\nu} = -g^{\mu\nu} + q^\mu q^\nu / m^2, \quad (2.1)$$

while for a photon with momentum k^μ we have in the axial gauge:

$$\Sigma \epsilon^\mu \epsilon^{*\nu} = -g^{\mu\nu} + [p^\mu k^\nu + k^\mu p^\nu] / (p \cdot k), \quad (2.2)$$

where p^μ is some four-vector different from k^μ . Second, a product of spinor

sandwiches can be rewritten as a trace over a string of Dirac matrices, using spinor projection operators. For a massive spin- $\frac{1}{2}$ particle with mass m and momentum p^μ we have the spin sum

$$\sum u(p) \bar{u}(p) = \not{p} \pm m, \quad (2.3)$$

where the sign is $+$ ($-$) for a particle (antiparticle). For a massless fermion^{*)} with momentum p^μ and helicity $\lambda = \pm 1$, we have

$$u_\lambda(p) \bar{u}_\lambda(p) = \omega_\lambda \not{p}, \quad \omega_\lambda = \frac{1}{2}(1 + \lambda \gamma^5). \quad (2.4)$$

We have given these well-known expressions explicitly because we shall refer to them in what follows.

The above approach has many attractive features. The unobserved spins and polarizations do not occur in the final result; also the arbitrary overall complex phases of the spinors and polarization vectors cancel. The algebraic evaluation of the trace expressions is straightforward and can be performed for any amplitude, given enough computing capacity³⁾.

It is this last condition which limits the complexity of the problems that can be handled. Since we have to square the amplitude before we can use Eqs (2.1) to (2.4) both the number of traces and their length increases very rapidly with the order of perturbation theory. It is usually very hard to keep the calculation systematic and reasonably compact at the same time, even if the final result turns out to be simple^{**)} . Consequently, it is easy to make errors that are hard to trace. Moreover, in many cases the expressions are so large as to defy a good analytical understanding of them apart from the crudest features^{***)}.

It is clear that any method which evaluates M instead of $\sum |M|^2$ will eventually become superior to the standard approach. Indeed, several authors have stressed this point and proposed alternatives⁷⁾⁻¹²⁾. These approaches, and their relationship to the one under discussion, will be reviewed elsewhere.

*) We shall use the term "fermion" and the symbol u for both particles and antiparticles, unless stated otherwise.

**) The results presented in Ref. 4) are very simple but were arrived at in an extremely tortuous way; see Ref. 5).

***) See, for instance, the as yet unexplained results of Ref. 6).

In the evaluation procedure sketched above, the cross-section is considered "known" when the Feynman diagrams have been reduced to an expression in terms of four-vector products. This is based on the fact that these products are familiar objects with well-known properties, and are moreover linear in the momenta. Of course the diagrams themselves, and the unevaluated trace expressions, are also well-defined objects, but they have much less physical appeal. Moreover, the spinors at the edges of the sandwiches are only proportional to the square root of the momentum (for zero mass), as follows from Eq. (2.4). It is therefore generally impossible to rewrite even a simple spinor sandwich in terms of the familiar vector products.

We want to argue that it is useful to employ in the evaluation of the amplitude not vector products like $p_1 \cdot p_2$ but rather spinor products like $\bar{u}(p_1)u(p_2)$. These objects are sufficiently simple to become "familiar", at least to those who follow the present discussion. Because of their more suitable dependence on the momentum arguments it is very easy to write a Feynman graph in terms of these products. They are more fundamental to calculations involving spinors than are vector products in the sense that vector products are easily obtained from spinor products [cf. Eq. (3.12)], but not vice versa.

Of course, just the introduction of the concept of spinor products does not in itself guarantee that a calculational approach is feasible. We have to find a form for the spinor products in terms of the momentum components which must not, from a computational point of view, be much more complicated than the simple form of the Minkowski product of two four-vectors. Moreover, in the standard approach unobserved spins are neatly summed over, but on the amplitude level this is not the case, and we have to take into account explicitly polarized states. For massless fermions the most expedient of these are helicity states. For massive fermions we cannot define the helicity in a covariant way, and so another choice of the spin vector will be described. For massless gauge bosons, i.e., photons and gluons, we shall use polarization vectors related to those introduced by the CALKUL collaboration^{4),5),13)}, which express the polarization covariantly in terms of the particle momenta, thus avoiding the introduction of yet another four-vector into the problem. For massive gauge bosons (i.e., $W^\pm$ and Z^0) the most straightforward generalization of the massless case is not very useful. Instead, we shall use a form which apart from giving simple expressions in terms of spinor products also allows a direct physical interpretation in terms of the decay of these massive particles. Finally, one other problem has to

be overcome. In general the spinor sandwiches in a given amplitude are not scalars, but contain "free" Lorentz indices contracted not with a momentum but with a free Lorentz index of another sandwich. We must have methods for manipulating such expressions so that the free indices disappear and scalar sandwiches remain. Such methods as have been developed to this end^{13),14)} are seen to arise naturally from our treatment of the spinors, which we present in the next section.

3. - SPINOR PHASES AND SPINOR PRODUCTS

In this section we derive expressions for the spinor products. These are essentially the same as those used in Ref. 1), the only difference being in the connection between the positive- and negative-helicity states. In this section we shall only consider massless fermions; massive fermions are discussed in Section 6.

We begin the discussion by setting up a convention for the overall complex phase of the spinors. As remarked before these are arbitrary, and we propose to choose them in the following way. Let us choose two four-vectors k_0^μ and k_1^μ with the following properties:

$$k_0 \cdot k_0 = 0 \quad , \quad k_1 \cdot k_1 = -1 \quad , \quad k_0 \cdot k_1 = 0 \quad . \quad (3.1)$$

Next we define one basic spinor $u_-(k_0)$ up to its complex phase by specifying the corresponding dyadic projection operator:

$$u_-(k_0) \bar{u}_-(k_0) = \omega_- \mathcal{K}_0 \quad , \quad (3.2)$$

with ω_- as in Eq. (2.4). The object $u_-(k_0)$ is therefore the negative-helicity state of a massless fermion with momentum k_0 . The corresponding positive-helicity state we now choose to be

$$u_+(k_0) = \mathcal{K}_1 u_-(k_0) \quad , \quad (3.3)$$

which also satisfies Eq. (2.4) because of (3.1). From these two spinors we can construct spinors for any other lightlike momentum p as follows:

$$u_{\lambda}(p) = \not{p} u_{\lambda}(k_0) / \sqrt{2 p \cdot k_0} \quad . \quad (3.4)$$

Again, spinors defined this way satisfy Eq. (2.4). The only restriction on the applicability of (3.4) is that $p \cdot k_0$ must be non-zero, and (for purposes of computer evaluation) not extremely small. In actual calculations we shall choose k_0 to be appreciably different from all particle momenta in the problem: this is easy if there is not an overwhelming number of momenta involved.

Using Eqs. (3.1) to (3.4) we can derive some useful identities for spinor sandwiches, which we shall use in the following. Let Γ be an arbitrary string of γ matrices, and let Γ^R be the same string in reversed order. We then have, for arbitrary (lightlike) momenta and helicities:

$$\bar{u}_{\lambda_1}(p_1) \Gamma u_{\lambda_2}(p_2) = \lambda_1 \lambda_2 \bar{u}_{-\lambda_2}(p_2) \Gamma^R u_{-\lambda_1}(p_1) \quad . \quad (3.5)$$

This is the equivalent of the "line-reversal" trick of Ref. 12). Since Eqs. (3.3) and (3.4) in fact only establish a particular relation between the complex phases of positive- and negative-helicity spinors, an identity like (3.5) always holds, with some complex phase instead of $\lambda_1 \lambda_2$. It is the fact that in our case this phase factor is simply ± 1 (depending only on the helicities) which makes Eq. (3.5) so useful in practice. With it, we can relate different helicity amplitudes to each other. The second useful identity reads

$$\left\{ \bar{u}_{\lambda}(p_1) \gamma^{\mu} u_{\lambda}(p_2) \right\} \gamma_{\mu} = 2 u_{\lambda}(p_2) \bar{u}_{\lambda}(p_1) + 2 u_{-\lambda}(p_1) \bar{u}_{-\lambda}(p_2) \quad , \quad (3.6)$$

which is in fact the so-called Chisholm identity. With the help of Eqs. (3.5) and (3.6); which are proved in the Appendix, we can reduce many amplitudes to expressions involving only spinor products. We give one example. Let $p_1 \dots p_7$ be lightlike momenta. Then,

$$\begin{aligned} & \bar{u}_{+}(p_1) \gamma^{\mu} u_{+}(p_2) \cdot \bar{u}_{+}(p_3) \gamma_{\mu} \not{p}_4 \gamma_{\nu} u_{+}(p_5) \cdot \bar{u}_{+}(p_6) \gamma^{\nu} u_{+}(p_7) \\ &= 4 \bar{u}_{+}(p_3) \left[u_{+}(p_2) \bar{u}_{+}(p_1) + u_{-}(p_1) \bar{u}_{-}(p_2) \right] \not{p}_4 \left[u_{+}(p_7) \bar{u}_{+}(p_6) + u_{-}(p_6) \bar{u}_{-}(p_7) \right] u_{+}(p_5) \\ &= 4 \left[\bar{u}_{+}(p_3) u_{-}(p_1) \right] \left[\bar{u}_{-}(p_2) u_{+}(p_4) \right] \left[\bar{u}_{+}(p_4) u_{-}(p_6) \right] \left[\bar{u}_{-}(p_7) u_{+}(p_5) \right] \quad . \quad (3.7) \end{aligned}$$

Here we have used the fact that we can write $\not{p}_4$ as a sum of the two projections $u_{+}(p_4) \bar{u}_{+}(p_4)$ and $u_{-}(p_4) \bar{u}_{-}(p_4)$, one of which gives zero because of its helicity structure.

We now proceed to discuss the spinor products themselves. For massless fermions with momenta p_1 and p_2 there are two non-zero products:

$$\begin{aligned} s(p_1, p_2) &\equiv \bar{u}_+(p_1) u_-(p_2) = -s(p_2, p_1), \\ t(p_1, p_2) &\equiv \bar{u}_-(p_1) u_+(p_2) = [s(p_2, p_1)]^*. \end{aligned} \quad (3.8)$$

It is enough to derive an expression for $s(p_1, p_2)$. Due to the definitions (3.2) to (3.4), we can immediately evaluate $s(p_1, p_2)$ as a trace:

$$\begin{aligned} s(p_1, p_2) &= \bar{u}_-(k_0) \not{p}_1 \not{p}_2 u_+(k_0) / \sqrt{4(p_1 \cdot k_0)(p_2 \cdot k_0)} \\ &= \text{Tr}(\omega_- \not{k}_0 \not{p}_1 \not{p}_2 \not{k}_1) / \sqrt{4(p_1 \cdot k_0)(p_2 \cdot k_0)} \\ &= [(p_1 \cdot k_0)(p_2 \cdot k_1) - (p_1 \cdot k_1)(p_2 \cdot k_0) - i \epsilon_{\mu\nu\rho\sigma} k_0^\mu k_1^\nu p_1^\rho p_2^\sigma] / \sqrt{4(p_1 \cdot k_0)(p_2 \cdot k_0)}. \end{aligned} \quad (3.9)$$

The last line in this equation is in itself not particularly simple. However, in an actual computation we can specify k_0^μ and k_1^μ so that the form of $s(p_1, p_2)$ becomes quite compact. Let us take, for instance:

$$\begin{aligned} p_i^\mu &= (p_i^0, p_i^x, p_i^y, p_i^z), \\ k_0^\mu &= (1, 1, 0, 0), \\ k_1^\mu &= (0, 0, 1, 0). \end{aligned} \quad (3.10)$$

This results in

$$s(p_1, p_2) = (p_1^y + i p_1^z) \left[\frac{p_2^0 - p_2^x}{p_1^0 - p_1^x} \right]^{1/2} - (p_2^y + i p_2^z) \left[\frac{p_1^0 - p_1^x}{p_2^0 - p_2^x} \right]^{1/2}, \quad (3.11)$$

and $t(p_1, p_2)$ is obtained by complex conjugation^{*)}. An expression like this is simple to program and not much more complicated to evaluate than a vector product like $p_1 \cdot p_2$. In fact when we have calculated the spinor product we immediately have the vector product as well:

$$\alpha(p_1, p_2) \equiv 2 p_1 \cdot p_2 = |s(p_1, p_2)|^2. \quad (3.12)$$

As remarked before, in this sense the spinor products are more fundamental than the vector products.

*) We use the metric $(+, -, -, -)$ and the Levi-Civita tensor $\epsilon_{0123} = +1$. Eqs. (3.10) and (3.11) also hold under cyclic permutation of the coordinates x , y and z .

We finish this section with some remarks on the computational aspects of our approach. The products $s(p_i, p_j)$ can be regarded not only as functions of the momenta p_i, p_j but (as soon as they have been calculated for a set of momenta $p_1, p_2, \dots, p_n$) also as a $n \times n$ matrix with entries i, j . The calculation of an amplitude will thus start with the evaluation of the $n \times n$ matrices s, t and d . From then on the four-vectors p_i can be regarded as just so many integer labels. The rest of the calculation of the diagrams will involve only simple multiplication and addition of the elements of s, t and d .

Another remark concerns the numerical stability of our approach. In electroweak and QCD processes cross-sections can exhibit quite drastic cancellations when two or more massless momenta become collinear. In such cases it is usually better to compute M instead of $\sum |M|^2$ because the cancellations will then take place over less orders of magnitude. Moreover the evaluation of $s(p_1, p_2)$ is itself numerically quite stable when p_1 and p_2 become collinear, because of the explicit antisymmetrization in Eq. (3.11). Indeed it is usually safer to employ Eq. (3.12) to obtain $p_1 \cdot p_2$ than the standard formula $p_1^\mu p_2^\mu - \vec{p}_1 \cdot \vec{p}_2$, for almost collinear lightlike momenta.

This completes our discussion on spinor manipulations. At this stage we are able to simplify and evaluate diagrams with only massless external fermions^{*)}. In the next two sections we shall describe how to treat the case where incoming or outgoing vector particles also occur.

4. - POLARIZATIONS FOR MASSLESS GAUGE BOSONS

We shall now explain how to make the polarization vectors of external photons^{**)} amenable to evaluation by the spinor techniques described in Section 3. Since we shall construct physical, transverse polarizations, the calculation of $\sum |M|^2$ will not require the introduction of ghost terms to cancel the spurious degrees of freedom.

^{*)} An example of this is the treatment of $e^+e^- \rightarrow e^+e^+e^-e^-$ in Ref. 1).

^{**)} In this section "photon" stands for any massless spin-1 particle.

We shall consider polarization vectors corresponding to states of definite helicity, denoted by $\epsilon_{\pm}^{\mu}$ and satisfying

$$\begin{aligned} \epsilon_{\lambda} \cdot k &= 0, & \epsilon_{\lambda} \cdot \epsilon_{\lambda} &= 0, \\ \epsilon_{-\lambda}^{\mu} &= (\epsilon_{\lambda}^{\mu})^*, & \epsilon_{\lambda} \cdot \epsilon_{-\lambda} &= -1, \end{aligned} \quad (4.1)$$

where k^{μ} is the photon momentum, and again $\lambda = \pm 1$. Any objects ϵ_{λ}^{μ} obeying the relations (4.1) make an acceptable choice for the polarization. We use the following:

$$\begin{aligned} \epsilon_{\lambda}^{\mu} &= N \bar{u}_{\lambda}(k) \gamma^{\mu} u_{\lambda}(p), \\ N &= [4(p \cdot k)]^{-1/2}, \end{aligned} \quad (4.2)$$

where p^{μ} is any lightlike vector not collinear to k^{μ} or k_0^{μ} . In the Appendix we prove that the choice (4.2) indeed satisfies the above conditions; moreover, the polarization sum is shown to be precisely of the form (2.2). It is usually simplest to choose for p^{μ} one of the other momenta occurring in the problem. As was first proved in Ref. 13), two different choices for p^{μ} result in polarization vectors differing by at most a gauge transformation and a complex phase. The precise interrelation is given in the Appendix. In practice we can use this freedom of choice to optimize the calculation: we shall come back to this point in our discussion of the amplitudes for $p\bar{p} \rightarrow V + \text{jets}$.

It is evident that Eq. (4.2) can be applied to calculate amplitudes with external photons, using the spinor products of the previous section. Before concluding this section we wish to comment on the relation between Eq. (4.2) and the polarization conventions used by the CALKUL collaboration in Refs. 4), 5) and 13). There, two lightlike four-vectors p_1^{μ}, p_2^{μ} are used to define the polarization. The object ϵ_{+} , for instance, is

$$\epsilon_{+} \propto \omega_{+} p_1 p_2 k - \omega_{-} k p_1 p_2. \quad (4.3)$$

Actually the right-hand side of (4.3) contains not only a part proportional to γ^{μ} but also terms proportional to $\gamma^5 \gamma^{\mu}$. It is obtained from a purely vectorial expression by assuming that not only the vector current but also the axial-vector current is conserved:

$$\omega_+ p_1 p_2 k + \omega_- k p_2 p_1 = \omega_+ p_1 p_2 k - \omega_- k p_1 p_2 + 2(p_1 p_2) \omega_- k, \quad (4.4)$$

where the last term is dropped under the above assumption. This is only really justified for Abelian interaction such as QED¹⁵⁾. The left-hand side of Eq. (4.4) can be rewritten as follows:

$$\omega_+ p_1 p_2 k + \omega_- k p_2 p_1 = \left[u_+(p_1) \bar{u}_+(k) + u_-(k) \bar{u}_-(p_1) \right] s(p_1, p_2) t(p_2, k). \quad (4.5)$$

The relation between the CALKUL convention and our representation is now clear: the term in brackets on the right-hand side of Eq. (4.5) is just what we expect for $\not{e}_+$ on the basis of Eq. (4.2). The vector p_2^μ is seen to just introduce yet another complex phase into the polarization. The spectacularly simple results for the single bremsstrahlung cross-sections reported in Refs. 4), 5) and 13) are in fact realized by the replacement in Eq. (4.4), leading to a considerable simplification of diagrams where $p_{1,2}$ are the external momenta in the fermion line which emits the bremsstrahlung.

5. - POLARIZATIONS FOR MASSIVE GAUGE BOSONS

We now discuss how to obtain expressions like (4.2) for the case of massive external bosons^{*)}. These have an additional longitudinally polarized state, so that the equivalent of Eq. (4.1) becomes, for a boson with momentum q^μ and mass m ,

$$\begin{aligned} \epsilon_\lambda \cdot q &= \epsilon_0 \cdot q = 0, & \epsilon_\lambda \cdot \epsilon_\lambda &= \epsilon_0 \cdot \epsilon_\lambda = 0, \\ \epsilon_{-\lambda}^\mu &= (\epsilon_\lambda^\mu)^* & \epsilon_\lambda \cdot \epsilon_{-\lambda} &= \epsilon_0 \cdot \epsilon_0 = -1. \end{aligned} \quad (5.1)$$

Again, ϵ_λ^μ denotes the helicity combination of the transversely polarized states, and ϵ_0^μ is the longitudinally polarized one.

As a first try, one might take some form like (4.2) to serve as the polarization vector. However, this turns out to be not so expedient. The problem is that for a timelike momentum q^μ we need both spinors $u(q)$ and antispinors

*) In this section, "boson" stands for any massive spin-1 particle.

$v(q)$ neither of which can be assigned a definite helicity in a covariant way. The techniques of Section 3 can be generalized to handle this (see the following section) but the simplicity of the approach of Section 4 is then lost.

We prefer instead to base our polarization convention on the following reasoning. Since we are at this moment not concerned with the boson polarization itself but only with the cross-section for unpolarized bosons, we really have as the only requirement on the $\varepsilon_{0,\lambda}^\mu$ that their spin sum should be Eq. (2.1). Any way by which we arrive at that expression gives us an acceptable polarization representation. We therefore proceed as follows. Let r_1^μ, r_2^μ be two lightlike four-vectors that together make up q^μ :

$$r_1^2 = r_2^2 = 0 \quad , \quad r_1^\mu + r_2^\mu = q^\mu . \quad (5.2)$$

We now introduce the object

$$a^\mu = \bar{u}_-(r_1) \gamma^\mu u_-(r_2) \quad , \quad (5.3)$$

which we shall use, after proper normalization, as the polarization vector belonging to q^μ . Of course there are as many possible choices for $r_{1,2}$ as there are different spatial directions in the rest frame of q^μ . Hence we replace the spin sum over $\varepsilon^\mu \varepsilon^{*\nu}$ in Eq. (2.1) by an integral over this solid angle. The result is of the desired form:

$$\begin{aligned} \int d\Omega \, a^\mu a^{*\nu} &= \int d\Omega \, \text{Tr}(\omega_- \gamma^\mu \gamma^\nu) \\ &= \frac{8}{3} \pi m^2 (-g^{\mu\nu} + q^\mu q^\nu / m^2) . \end{aligned} \quad (5.4)$$

This implies that we will arrive at the correct result for the cross-section if we make the following replacements for outgoing bosons:

$$\begin{aligned} \epsilon^\mu &\rightarrow a^\mu \\ \sum \epsilon^\mu \epsilon^{*\nu} &\rightarrow \frac{3}{8\pi m^2} \int d\Omega \, a^\mu a^{*\nu} . \end{aligned} \quad (5.5)$$

This representation fits perfectly in our method of evaluating the amplitude since it contains only the lightlike four-vectors r_1^μ and r_2^μ . In order to obtain the cross-section we have to perform an additional two-dimensional integral, but since the accuracy of Monte Carlo integration does not depend on the

dimensionality of the integrand but only on its variance¹⁶⁾, no accuracy will be lost as long as the dependence on Q is not extremely strong, implying huge polarization effects.

Up to this point we have treated r_1^μ , r_2^μ and a^μ as purely mathematical artefacts. It turns out that we can give them an extremely interesting and useful interpretation. To this end, let us consider the case of W^- emission. In the actual production process the W^- is not a free particle, but decays immediately into, say, an electron and an antineutrino. Therefore the physical outgoing state that should enter into a realistic calculation is not the polarization vector of the W^- , but its resonant propagator^{*}) coupled to the current describing the decay $W^- \rightarrow e^- \bar{\nu}_e$:

$$j_w^\mu = i g_w \bar{u}(p_e)(1 + \gamma^5) \gamma^\mu v_\nu(p_\nu) \quad (5.6)$$

where $p_e(p_\nu)$ is the electron (antineutrino) momentum, and g_w the coupling constant. Comparing Eqs. (5.3) and (5.6) we can identify r_1^μ with p_e^μ and r_2^μ with p_ν^μ . The angular integral in (5.4) relates to the fact that the W^- can be considered unpolarized as long as we do not study its decay in any detail. Thus our treatment of the boson polarization gives us, as a somewhat surprising bonus, all the correlations of the decay products without any extra work at all! In the "standard" approach to calculating the cross-section, such additional information can only be had with a considerable increase in computational effort.

Some final remarks should be made here. In the case of W^+ production we have to identify r_1^μ with the neutrino momentum, and r_2^μ with the positron momentum. For Z^0 bosons the identification is less trivial. For $Z^0(q) \rightarrow e^+(p_+)e^-(p_-)$ the decay current is

$$j_z^\mu(\lambda) = i \bar{u}_\lambda(p_-)(g_v + g_a \gamma^5) \gamma^\mu v_\lambda(p_+) , \quad (5.7)$$

in which the $e^\pm$ can occur with both + and - helicity because the current is not purely $V \pm A$. For negative helicity we have

$$j_z^\mu(-) = i(g_v + g_a) \bar{u}(p_-) \gamma^\mu v(p_+) , \quad (5.8a)$$

^{*}) The propagator can either be taken in the peaking approximation or included exactly, if one assumes that the decay under consideration saturates the decay width (see Appendix).

while for positive helicity we use Eq. (3.5) to obtain

$$j_2^\mu(+)=i(g_v-g_a)\bar{v}_-(p_+)\gamma^\mu u_-(p_-) \quad (5.8b)$$

In the cross-section these two helicity configurations have to be summed over incoherently. In a sample of Monte Carlo events we can therefore make the following identifications:

$$\begin{aligned} p_-^\mu &= r_1^\mu, \quad p_+^\mu = r_2^\mu \text{ in a fraction } F \text{ of the events,} \\ p_-^\mu &= r_2^\mu, \quad p_+^\mu = r_1^\mu \text{ in a fraction } 1-F \text{ of the events,} \\ F &= (g_v+g_a)^2/[(g_v+g_a)^2+(g_v-g_a)^2]. \end{aligned} \quad (5.9)$$

An algorithm like this is extremely well suited to implementation in Monte Carlo integrations and event generators. Of course when we do not discriminate between e^+ and e^- we may make either of the two identifications.

6. - SPINORS FOR MASSIVE FERMIONS

All fermions that we have considered up to now have been massless. For many purposes this is quite sufficient: at CERN collider energies the mass of even the b quark can be considered very small. The zero-mass approximation breaks down in the case of top quarks with a putative mass of around 40 GeV¹⁷⁾. Clearly, in neglecting masses of this kind we would be making serious errors not only in the calculation of $\Sigma|M|^2$ but also in the scattering kinematics. We therefore describe in this section how to treat massive fermions.

Our procedure will be similar to that of the previous section: we construct a spinorial object which leads to the correct spin sum, namely Eq. (2.3). Our starting point is the following. For any lightlike four-vector k^μ , the object

$$u(q,s)=(\not{q}+m)u_-(k)/\sqrt{2(q.k)} \quad (6.1)$$

is a good spinor for a massive fermion (not antifermion) with momentum q^μ and mass m , i.e., we can find a unique spin vector s^μ such that the Dirac equations

hold^{*}). The proof of this statement, and the form of s^μ are given in the Appendix. Note that in the limit $m \rightarrow 0$ we recover Eq. (3.4). We now make a choice for k^μ . As before we choose two lightlike momenta that build up q^μ :

$$p_1^2 = p_2^2 = 0 \quad , \quad p_1^\mu + p_2^\mu = q^\mu \quad . \quad (6.2)$$

Choosing p_2^μ for k^μ and inserting in Eq. (6.1), gives

$$\begin{aligned} u(q, s) &= \frac{1}{m} (\not{p}_1 + \not{p}_2 + m) u_-(p_2) \\ &= \frac{s(p_1, p_2)}{m} u_+(p_1) + u_-(p_2) \end{aligned} \quad (6.3)$$

Since $|s(p_1, p_2)| = m$, the coefficient of $u_+(p_1)$ is just a complex phase. The result for s^μ in the Appendix gives for this choice:

$$s^\mu = (p_1^\mu - p_2^\mu)/m \quad (6.4)$$

In this way we have an expression for the massive-fermion spinor as a linear combination of two massless-fermion spinors with opposite helicities. By making the appropriate changes in Eqs. (6.1) through (6.3), we find the complete set of definitions for massive (anti-) fermions:

$$\begin{aligned} u(q, +s) &= [s(p_1, p_2)/m] u_+(p_1) + u_-(p_2) \quad , \\ u(q, -s) &= [t(p_1, p_2)/m] u_-(p_1) + u_+(p_2) \quad , \\ v(q, +s) &= [s(p_1, p_2)/m] u_+(p_1) - u_-(p_2) \quad , \\ v(q, -s) &= [t(p_1, p_2)/m] u_-(p_1) - u_+(p_2) \quad . \end{aligned} \quad (6.5)$$

It goes without saying that we can easily implement these definitions into our approach. As an illustration, we calculate the weak current involved in the decay $W^+ \rightarrow t(q_t) \bar{b}(q_b)$. It reads, with $\lambda = \pm 1$:

$$j_w^\mu(\lambda) = i g_w \bar{u}(q_t, \lambda s) (1 + \gamma^5) \gamma^\mu u(q_b) \quad , \quad (6.6)$$

^{*}) In Refs. 18) and 19), this formula was used to calculate $e^+e^- \rightarrow e^+e^+e^-e^-$ keeping the complete mass dependence.

where we have neglected the b quark mass as stated above. Now both polarization states of the t quark give a non-zero result. We have, with $p_{1,2}^\mu$ as before,

$$\begin{aligned} j_w^\mu(+)&= 2i g_w \bar{u}_-(p_2) \gamma^\mu u_-(q_b) , \\ j_w^\mu(-)&= 2i g_w \bar{u}_-(p_1) \gamma^\mu u_-(q_b) \cdot [t(p_1, p_2)/m]^* . \end{aligned} \quad (6.7)$$

These currents can be treated in the standard way. Moreover, we may drop the complex phase factor t/m in $j_w^\mu(-)$ since it does not occur in the cross-section.

The simplicity of the result (6.7) is due to the fact that the V-A structure of the charged current will always pick out only one of the two terms in eq. (6.5). If the current is not purely V±A the situation is somewhat more complicated. Consider the decay $Z^0 \rightarrow t(q)\bar{t}(q')$ for which the current is given by

$$j_z^\mu(\lambda, \lambda') = i \bar{u}(q, \lambda s) (g_v + g_a \gamma^5) \gamma^\mu v(q', \lambda' s') , \quad (6.8)$$

where the primed quantities refer to the antiquark, the unprimed ones to the quark. In the four possible polarization cases, Eq. (6.8) goes over into

$$\begin{aligned} j_z^\mu(+, +) &= i(g_v - g_a) [t(p_2, p_1) s(p_1', p_2')/m^2] \bar{u}_+(p_1) \gamma^\mu u_+(p_1') - i(g_v + g_a) \bar{u}_-(p_2) \gamma^\mu u_-(p_2'), \\ j_z^\mu(+, -) &= -i(g_v - g_a) [t(p_2, p_1)/m] \bar{u}_+(p_1) \gamma^\mu u_+(p_2') + i(g_v + g_a) [t(p_1', p_2')/m] \bar{u}_-(p_2) \gamma^\mu u_-(p_1'), \\ j_z^\mu(-, +) &= -i(g_v + g_a) [s(p_2, p_1)/m] \bar{u}_-(p_1) \gamma^\mu u_-(p_2') + i(g_v - g_a) [s(p_1', p_2')/m] \bar{u}_+(p_2) \gamma^\mu u_+(p_1'), \\ j_z^\mu(-, -) &= i(g_v + g_a) [s(p_2, p_1) t(p_1', p_2')/m^2] \bar{u}_-(p_1) \gamma^\mu u_-(p_1') - i(g_v - g_a) \bar{u}_+(p_2) \gamma^\mu u_+(p_2'). \end{aligned} \quad (6.9)$$

To obtain the cross-section we have to calculate $|M|^2$ for the four different polarization cases, each amplitude consisting of two terms. This may again be simplified somewhat by a reasoning similar to that in Section 5. Upon interchange of p_1^μ and p_2^μ , spinors for the two opposite polarization states transform into each other modulo a complex phase:

$$\left[u(q, +s) \right]_{p_1 \leftrightarrow p_2} = \frac{t(p_2, p_1)}{m} u(q, -s) , \quad (6.10)$$

as was already suggested by Eq. (6.4). This means that we can restrict ourselves to one polarization only, provided that we integrate over the p_1^μ, p_2^μ system as before:

$$\sum_\lambda u(q, \lambda s) \bar{u}(q, \lambda s) = \frac{i}{2\pi} \int d\Omega u(q, +s) \bar{u}(q, +s) , \quad (6.11)$$

with a similar identity for antiparticles. Using this trick we can obtain the complete polarization sum using only the current $j_z^\mu(+, +)$ of Eqs. (6.9). Again the two additional integrals are not expected to reduce the accuracy of the Monte Carlo integration significantly.

Before we finish this section a remark is in order concerning the physical interpretation of the above spinor representation. For heavy bosons one can picture the polarization representation as the current of an actual two-body decay; one may wonder whether such a picture is possible in this case as well. This is not a trivial question. The physically most appealing decay channel is a weak three-body decay, like $t \rightarrow b \mu^+ \nu_\mu$. Another alternative would be the two-body decay into a massless fermion and a massless scalar or vector particle, which does not lead to a gauge-invariant amplitude if we assume minimal coupling. In the Appendix it is shown that we can actually write down a gauge-invariant amplitude for the decay $t \rightarrow (\text{massless fermion}) + \text{photon}$, in such a way that we recover the desired spin sum of Eq. (2.3).

7. - W's, Z's AND PARTONS

In the previous sections we have developed a calculational apparatus. Such an exercise⁷⁾⁻¹²⁾ is bound to be sterile unless one can actually perform non-trivial calculations with it. Therefore we shall now describe a specific example; as stated in the introduction, this calculation concerns the cross-section for the process

$$p \bar{p} \rightarrow V + (0, 1, 2) \text{ jets} . \quad (7.1)$$

It is the fact that this calculation has already been applied successfully to study the phenomenology of the above process which is the justification of the

previous discussion. Moreover, the calculation that we are now going to present may serve as an illustration of our method.

In this paper we restrict ourselves to deriving compact expressions for the diagrams, which lend themselves immediately to numerical evaluation by computer. Of course, to obtain the cross-section for (7.1) one has to square the values resulting from these expressions and sum/average over spins and colours. Many amplitudes, related to each other by crossing (and sometimes described by a different number of diagrams) have to be included to obtain the complete prediction for all the 396 parton-level subprocesses^{*)} which contribute. Finally, the parton distribution functions (and, possibly, jet fragmentation) have to be taken into account in order to arrive at a definite phenomenological prediction.

Here we shall disregard all these points. The only phenomenological aspects in our treatment are the following. First, the parton masses are neglected in view of the fact that we do not consider top quark production. Second, the phase space cuts are assumed to be (and in fact, are^{2),20),21)} such that no two partons can become collinear. Hence we do not have to worry about the numerical stability of our approach, although we shall comment on it later.

We now proceed to prepare the calculation. It is easily seen that we have to consider only four distinct subprocesses contributing to (7.1). These are

$$\begin{aligned} u \bar{u} &\rightarrow Z^0, \\ u \bar{u} &\rightarrow Z^0 g, \\ u \bar{u} &\rightarrow Z^0 u \bar{u}, \\ u \bar{u} &\rightarrow Z^0 g g. \end{aligned} \quad (7.2)$$

All other subprocesses can be obtained from these by changing the weak coupling constants, leaving out certain Feynman graphs, or introducing Kobayashi-Maskawa mixing matrix elements; or by moving partons from the initial to the final state or vice versa. In order to facilitate this last manipulation we introduce the following expression for momentum conservation:

$$b(p_1) p_1^\mu + b(p_2) p_2^\mu + \dots + b(p_n) p_n^\mu = 0, \quad (7.3)$$

^{*)} Neglecting processes with b or t quarks in the initial state, or with t quarks in the final state²⁰⁾.

where the p_i^μ are the particle momenta, and

$$b(p_i) = b_i = \begin{cases} -1 & \text{if } p_i \text{ is incoming;} \\ +1 & \text{if } p_i \text{ is outgoing} \end{cases} . \quad (7.4)$$

If we include the coefficients b_i in our expressions all possible crossings are automatically accounted for. As stated in our discussion of the spinor products we shall regard the momentum names p_i as integer labels, so that the b_i become a simple array which can be initialized at the start of the calculation. From this point on, then, we can forget about which of the particles in a given amplitude are incoming. In this way we arrive at diagrams for the processes in (7.2) in which all the particles have been moved to the final state. These we call the generic diagrams. There are 10 of these, all of which have to be evaluated for arbitrary helicities and crossings. This evaluation will be the subject of the next section.

8. - TEN DIAGRAMS

The computation of the generic Feynman graphs in this section is structured as follows. We first write down the expressions for the diagrams. These consist of three parts. The first part contains the overall constants, combinations of coupling constants, and the denominators arising from the various propagators: these quantities are easy to evaluate, without need for any spinor techniques. The second part contains the spinors and Dirac matrices, for which we shall derive expressions in terms of the s and t product matrices. The third part is the colour factor. This is the only part of the cross-section which we do not include in the amplitude M but only in the colour sum and average in $\Sigma |M|^2$. For a more detailed discussion of this point we refer to the end of this section.

Our notation will be as follows. The labels i, j, k and l will denote the partons. Parton i is defined to have four-momentum p_i^μ and helicity λ_i (as defined in the previous sections); its b-factor, defined in Eq. (7.4), is denoted by b_i . The following notation will also be useful:

$$f_{ij} = b_i b_j d_{ij} , \quad (8.1)$$

where $d_{ij} = d(p_i, p_j)$ as defined in Eq. (3.12). Throughout this section we shall use the symbols s_{ij} and t_{ij} for the products $s(p_i, p_j)$ and $t(p_i, p_j)$ respectively. We shall denote the four momentum of V by q^μ , and its mass by m ; the "decay momenta" introduced in Eq. (5.2) are r_1^μ and r_2^μ [the symbol s_{1i} standing for $s(r_1, p_i)$, and so on]. We can consistently use $b(r_{1,2}) = +1$. The vector and axial-vector coupling constants of V to the parton with label i are written as g_v^i and g_a^i , respectively. The QCD coupling constant is denoted by g . In principle we also have to multiply amplitudes for $W^\pm$ production with an appropriate element of the Kobayashi-Maskawa matrix. This rather trivial overall factor will be neglected in the following.

We are now ready to discuss the diagrams. This we shall do separately for the four processes in Eq. (7.2), in ascending order of complexity.

1: Vqq

The one generic diagram, given in Fig. 1, reads

$$M_1(i, j) = D_1 \cdot T_1(\lambda_i) \cdot C_1 \quad . \quad (8.2)$$

With helicity conservation along the fermion line taken into account D_1 is given by

$$D_1 = i(g_v^i - \lambda_i g_a^i) \delta_{\lambda_i \lambda_j} \quad , \quad (8.3)$$

where δ is the Kronecker symbol. The colour factor is

$$C_1 = \delta_{c_i c_j} \quad , \quad (8.4)$$

where c_i and c_j are the colours of the quarks i and j respectively. The c 's can take on three different values in QCD. The spinorial part T_1 depends on the helicity along the fermion line:

$$T_1(\lambda) = \bar{u}_\lambda(p_i) \gamma^\mu u_\lambda(p_j) \cdot \bar{u}_-(r_1) \gamma_\mu u_-(r_2) \quad . \quad (8.5)$$

With the results of Section 3, we can easily rewrite T_1 :

$$\begin{aligned} T_1(+) &= 2 S_{i2} t_{1j} , \\ T_1(-) &= 2 t_{i1} s_{2j} . \end{aligned} \quad (8.6)$$

This completes our evaluation of the matrix element. As an illustration we note that for W production only $T_1(-)$ contributes. Consequently, we have in the cross-section:

$$|T_1(-)|^2 \propto |t_{i1} s_{2j}|^2 \propto (p_i \cdot r_1)(p_j \cdot r_2) \propto (1 + \cos \theta^*)^2, \quad (8.7)$$

where θ^* is the angle between $\vec{r}_1$ and $\vec{p}_j$ in the W rest frame. The ease with which this well-known result is derived indicates that our spinor product formalism can be useful even if the diagrams are not particularly complicated.

Before proceeding with the next process we remark that the above cross-section is of course already well known and can be derived in many ways. We have included it here for didactic reasons: recall our argument that the spinor products must become familiar objects if we want to use them.

2: $Vq\bar{q}g$

We can easily write down the generic diagrams of Fig. 2. We first, however, have to specify the auxiliary vector p^μ in the gluon polarization [cf. Eq. (4.2)]. It is convenient to choose p^μ to be one of the other parton momenta in the process: here, we choose $p^\mu = p_1^\mu$. Now the symmetry between the diagrams M_2 and M_3 is lost; on the other hand, M_3 will be zero for some helicities, as we shall see. Of course the value of the cross-section should not depend on the gauge choice. Such properties actually provide powerful numerical checks on calculations like these. The generic diagrams are

$$M_n(i, j, k) = D_n \cdot T_n(\lambda_i, \lambda_k) \cdot C_n, \quad n = 2, 3. \quad (8.8)$$

The factors D_n now also include the normalization N in Eq. (4.2), and read

$$\begin{aligned} D_2 &= i g (g_v^i - \lambda_i g_a^i) \delta_{\lambda_i \lambda_j} f_{jk}^{-1} (2d_{ik})^{-1/2}, \\ D_3 &= -i g (g_v^i - \lambda_i g_a^i) \delta_{\lambda_i \lambda_j} f_{ik}^{-1} (2d_{ik})^{-1/2}. \end{aligned} \quad (8.9)$$

The colour factor now depends also on the colour label a_k of the gluon, which can take on eight values in QCD:

$$C_2 = C_3 = (T^{a_k})_{c_i c_j} , \quad (8.10)$$

where the T's are the colour matrices [defined as $\frac{1}{2}$ times the Gell-Mann matrices - see for instance Ref. 22)]. The spinorial parts read:

$$\begin{aligned} T_2(\lambda, \lambda') &= \bar{u}_-(r_1) \gamma_\mu u_-(r_2) \cdot \bar{u}_\lambda(p_i) \gamma^\mu (b_j \not{p}_j + b_k \not{p}_k) \gamma^\nu u_\lambda(p_j) \cdot \bar{u}_\lambda'(p_k) \gamma_\nu u_\lambda'(p_i) , \\ T_3(\lambda, \lambda') &= \bar{u}_\lambda'(p_k) \gamma_\mu u_\lambda'(p_i) \cdot \bar{u}_\lambda(p_i) \gamma^\mu (b_j \not{p}_j + b_k \not{p}_k) \gamma^\nu u_\lambda(p_j) \cdot \bar{u}_-(r_1) \gamma_\nu u_-(r_2) . \end{aligned} \quad (8.11)$$

The evaluation is now very similar to that in Eq. (3.7):

$$\begin{aligned} T_2(+, +) &= 4 b_j S_{i2} t_{1j} s_{jk} t_{ij} , \\ T_2(+, -) &= 4 S_{i2} [b_j t_{1j} s_{ji} + b_k t_{1k} s_{ki}] t_{kj} , \\ T_2(-, +) &= 4 t_{i1} [b_j s_{2j} t_{ji} + b_k s_{2k} t_{ki}] s_{kj} , \\ T_2(-, -) &= 4 b_j t_{i1} s_{2j} t_{jk} s_{ij} , \end{aligned} \quad (8.12)$$

and

$$\begin{aligned} T_3(+, +) &= 4 b_k s_{ik} t_{ik} s_{k2} t_{1j} , \\ T_3(+, -) &= T_3(-, +) = 0 , \\ T_3(-, -) &= 4 b_k t_{ik} s_{ik} t_{k1} s_{2j} . \end{aligned} \quad (8.13)$$

Before we finish our discussion of this process a few remarks are in order. First, if we had chosen the gauge $p^\mu = p_j^\mu$, two of the T_2 would have vanished instead of the T_3 . Second, the T_3 's contain a factor $p_i \cdot p_k$, while the T_2 's always contain a factor $\sqrt{p_i \cdot p_k}$, as follows from Eq. (3.12). Consequently the effective denominators are $[(p_i \cdot p_k)(p_j \cdot p_k)]^{-\frac{1}{2}}$ for M_2 , and $[p_i \cdot p_k]^{-\frac{1}{2}}$ for M_3 . The cross-section is therefore guaranteed to behave as $(p_i \cdot p_k)^{-1} (p_j \cdot p_k)^{-1}$ without double poles. It is the above two properties that were more fully exploited by the CALKUL collaboration^{4), 5), 13)} to obtain simple results for the

cross-sections. For the present discussion, the most noteworthy fact is that we obtain the correct peaking of the cross-section without needing any additional cancellations between M_2 and M_3 . This ensures that our approach is numerically stable over a much larger region of phase space than we need to integrate over.

3: Vqqqq

The two generic diagrams, given in Fig. 3, are

$$M_n(i, j, k, \ell) = D_n \cdot T_n(\lambda_i, \lambda_k) \cdot C_n, \quad n = 4, 5; \quad (8.14)$$

$$\begin{aligned} D_4 &= -i g^2 (g_v^i - \lambda_i g_a^i) \delta_{\lambda_i \lambda_j} \delta_{\lambda_k \lambda_\ell} f_{ke}^{-1} (f_{1i} + f_{2i} + f_{12})^{-1}, \\ D_5 &= +i g^2 (g_v^i - \lambda_i g_a^i) \delta_{\lambda_i \lambda_j} \delta_{\lambda_k \lambda_\ell} f_{ke}^{-1} (f_{1j} + f_{2j} + f_{12})^{-1}; \end{aligned} \quad (8.15)$$

$$C_4 = C_5 = \sum_a (T^a)_{c_i c_j} (T^a)_{c_k c_\ell}; \quad (8.16)$$

$$\begin{aligned} T_4(\lambda, \lambda') &= \bar{u}_-(r_1) \gamma_\mu u_-(r_2) \cdot \bar{u}_j(p_i) \gamma^\mu (\not{q} + b_i \not{p}_i) \gamma^\nu u_j(p_j) \cdot \bar{u}_j(p_k) \gamma_\nu u_j(p_\ell), \\ T_5(\lambda, \lambda') &= \bar{u}_j(p_k) \gamma_\mu u_j(p_\ell) \cdot \bar{u}_j(p_i) \gamma^\mu (\not{q} + b_j \not{p}_j) \gamma^\nu u_j(p_j) \cdot \bar{u}_-(r_1) \gamma_\nu u_-(r_2). \end{aligned} \quad (8.17)$$

The strings $T_{4,5}$ have the same structure as $T_{2,3}$. Using $\not{q} = \not{p}_1 + \not{p}_2$ (one of these always giving zero) we can write them out immediately. A slightly more systematic approach is to use the reversal identity (3.5) to realize that there are really only two distinct expressions:

$$\begin{aligned} A_+(i, j, k, \ell) &\equiv T_4(+, +) = 4 s_{i2} (t_{12} s_{2k} + b_i t_{1i} s_{ik}) t_{\ell j}, \\ A_-(i, j, k, \ell) &\equiv T_4(-, -) = 4 t_{i1} (s_{21} t_{1k} + b_i s_{2i} t_{ik}) s_{\ell j}. \end{aligned} \quad (8.18)$$

The six other possibilities correspond to permutations of the arguments of A_+ and A_- . We have:

$$\begin{aligned}
T_4(+,+) &= A_+(i,j,k,\ell) & T_5(+,+) &= A_-(j,i,\ell,k) \\
T_4(+,-) &= A_+(i,j,\ell,k) & T_5(+,-) &= A_-(j,i,k,\ell) \\
T_4(-,+) &= A_-(i,j,\ell,k) & T_5(-,+) &= A_+(j,i,k,\ell) \\
T_4(-,-) &= A_-(i,j,k,\ell) & T_5(-,-) &= A_+(j,i,\ell,k)
\end{aligned} \tag{8.19}$$

We want to remark that there are processes (such as $u\bar{u} \rightarrow Z^0 u\bar{u}$) in which the fermion lines are connected in two different ways, so that in the amplitude we will encounter not only $M_{4,5}(i,j,k,\ell)$ but also $M_{4,5}(i,\ell,k,j)$. Consequently all possible allowed permutations in the arguments of $A_{\pm}$ are actually used during the calculation. The use of the functions $A_{\pm}$ ensures, therefore, a compact and efficient program structure.

4: Vqqgg

We now come to the most complicated process so far. Of the five generic diagrams (Fig. 4), two contain the non-Abelian three-gluon coupling. We shall first discuss the "Abelian" diagrams $M_{6,7,8}$, and then study the non-Abelian ones, M_9 and M_{10} .

First of all we have to decide on a choice for the polarization vectors $(\epsilon^k)^\mu$ and $(\epsilon^\ell)^\mu$ of the gluons k and ℓ respectively. As before, we could choose $p^\mu = p_1^\mu$. However, in this case a more attractive option is available. We choose

$$(\epsilon^k)^\mu = (\epsilon^\ell)^\mu = \bar{u}_\lambda(p_k) \gamma^\mu u_\lambda(p_\ell) / \sqrt{4p_k \cdot p_\ell} \quad , \tag{8.20}$$

so that $p^\mu = p_k^\mu$ for ϵ^k , and $p^\mu = p_\ell^\mu$ for ϵ^ℓ . Apart from being symmetric, this choice means that the gauges of both polarizations are identical: both ϵ^k and ϵ^ℓ are exactly transverse in a frame in which $\vec{p}_k + \vec{p}_\ell = 0$. We now write down the Abelian graphs:

$$\begin{aligned}
M_n(i,j,k,\ell) &= D_n \cdot T_n(\lambda_i, \lambda_k, \lambda_\ell) \cdot C_n \quad , \quad n = 6, 7, 8 \quad ; \\
D_6 &= i g^2 (g_\nu^i - \lambda_i g_a^i) \delta_{\lambda_i \lambda_j} f_{j\ell}^{-1} (f_{j\ell} + f_{jk} + f_{k\ell})^{-1} (2\alpha_{k\ell})^{-1/2} ,
\end{aligned} \tag{8.21}$$

$$\begin{aligned} D_7 &= -ig^2(g_v^i - \lambda_i g_a^i) \delta_{\lambda_i \lambda_j} f_{ik}^{-1} f_{jl}^{-1} (2\alpha_{kl})^{-1/2}, \\ D_8 &= ig^2(g_v^i - \lambda_i g_a^i) \delta_{\lambda_i \lambda_j} f_{ik}^{-1} (f_{ik} + f_{il} + f_{kl})^{-1} (2\alpha_{kl})^{-1/2}; \end{aligned} \quad (8.22)$$

$$C_6 = C_7 = C_8 = (T^a_k T^a_l)_{c_i c_j}; \quad (8.23)$$

$$T_n(\lambda, \lambda_1, \lambda_2) = T_n^{\mu\nu\rho}(\lambda) \bar{u}_\mu(r_1) \gamma_\mu u_\nu(r_2) \cdot \bar{u}_{\lambda_1}(\rho_k) \gamma_\nu u_{\lambda_1}(\rho_l) \cdot \bar{u}_{\lambda_2}(\rho_l) \gamma_\rho u_{\lambda_2}(\rho_k),$$

(8.24)

$n = 6, 7, 8$;

$$\begin{aligned} T_6^{\mu\nu\rho}(\lambda) &= \bar{u}_\lambda(\rho_i) \gamma^\mu (b_j \not{p}_j + b_k \not{p}_k + b_l \not{p}_l) \gamma^\nu (b_j \not{p}_j + b_l \not{p}_l) \gamma^\rho u_\lambda(\rho_j), \\ T_7^{\mu\nu\rho}(\lambda) &= \bar{u}_\lambda(\rho_i) \gamma^\nu (b_i \not{p}_i + b_k \not{p}_k) \gamma^\mu (b_j \not{p}_j + b_l \not{p}_l) \gamma^\rho u_\lambda(\rho_j), \\ T_8^{\mu\nu\rho}(\lambda) &= \bar{u}_\lambda(\rho_i) \gamma^\nu (b_i \not{p}_i + b_k \not{p}_k) \gamma^\rho (b_i \not{p}_i + b_k \not{p}_k + b_l \not{p}_l) \gamma^\mu u_\lambda(\rho_j). \end{aligned} \quad (8.25)$$

Although the expressions for $T_{6,7,8}$ are again more complicated, we can still write down the result immediately. In order to reduce the number of results that have to be given explicitly, we note that

$$T_n(-\lambda, -\lambda_1, -\lambda_2) = [T_n(\lambda, \lambda_1, \lambda_2)]_{r_1 \leftrightarrow r_2}^*, \quad n = 6, 7, 8. \quad (8.26)$$

We therefore have to list only 12 of the 24 expressions. For T_6 we have

$$\begin{aligned} T_6(+, +, +) &= 8b_j s_{i2} (b_j t_{ij} s_{jk} + b_l t_{il} s_{lk}) d_{lj} t_{kj}, \\ T_6(+, +, -) &= 8b_j s_{i2} (b_j t_{ij} s_{jk} + b_l t_{il} s_{lk}) t_{lj} s_{jk} t_{lj}, \\ T_6(+, -, +) &= 8b_j s_{i2} (b_j t_{ij} s_{jl} + b_k t_{ik} s_{kl}) t_{kj} s_{jl} t_{kj}, \\ T_6(+, -, -) &= 8s_{i2} (b_j t_{ij} s_{jl} + b_k t_{ik} s_{kl}) (b_j d_{kj} + b_l d_{kl}) t_{lj}. \end{aligned} \quad (8.27)$$

As stated above, the other four possibilities follow from Eq. (8.26). For T_7 the result is

$$\begin{aligned}
T_7(+,+,+) &= 8 b_j s_{ik} (b_i t_{li} s_{i2} + b_k t_{lk} s_{k2}) t_{aj} s_{jl} t_{kj} , \\
T_7(+,+,-) &= 8 s_{ik} (b_i t_{li} s_{i2} + b_k t_{lk} s_{k2}) (b_j t_{aj} s_{jk} + b_l t_{al} s_{lk}) t_{lj} , \\
T_7(+,-,+) &= 8 b_i b_j s_{il} t_{ki} s_{i2} t_{aj} s_{jl} t_{kj} , \\
T_7(+,-,-) &= 8 b_i s_{il} t_{ki} s_{i2} (b_j t_{aj} s_{jk} + b_l t_{al} s_{lk}) t_{lj} . \quad (8.28)
\end{aligned}$$

Finally, the expressions for T_8 read:

$$\begin{aligned}
T_8(+,+,+) &= 8 s_{ik} (b_i t_{li} + b_k t_{lk}) (b_i t_{ki} s_{i2} + b_l t_{kl} s_{l2}) t_{aj} , \\
T_8(+,+,-) &= 8 b_i s_{ik} t_{li} s_{ik} (b_i t_{li} s_{i2} + b_k t_{lk} s_{k2}) t_{aj} , \\
T_8(+,-,+) &= 8 b_i s_{il} t_{ki} s_{il} (b_i t_{ki} s_{i2} + b_l t_{kl} s_{l2}) t_{aj} , \\
T_8(+,-,-) &= 8 b_i s_{ik} t_{li} s_{ik} (b_i t_{li} s_{i2} + b_k t_{lk} s_{k2}) t_{aj} . \quad (8.29)
\end{aligned}$$

At this point a comment is in order on the peaking structure. As for $Vqqg$, the poles in the cross-section when a quark and a gluon become collinear turn out to be less divergent than one might expect from considering the denominators only. We showed how the numerators in each diagram separately go to zero in the collinear limit, thus reducing the degree of divergence. In our approach, this behaviour is numerically stable. In the present case, the number of possible collinear peaking situations is larger: we can have $j//l$, $i//k$, $i//k//l$, $j//k//l$, or $i//k + j//l$ (with obvious notation). For this amplitude as well, in each of these cases the $T_{6,7,8}$ behave precisely so as to reduce the divergence of the diagrams $M_{6,7,8}$ to the correct degree, so that no further cancellations are needed between the diagrams. This stability is quite surprising since in the polarization choice (8.20) we introduced yet another denominator that vanishes when $k//l$. This divergence, too, which is an artefact of our calculation, is neatly cancelled in each separate diagram. So again the numerical behaviour of our approach is even much better than was either required or expected.

We now come to the two "non-Abelian" diagrams. It will turn out to be useful to study first the three-gluon vertex part of these diagrams in some detail. It is depicted in Fig. 5 and reads

$$-g f^{a_k a_\ell c} T_c \left[(b_k p_k - b_\ell p_\ell)^\mu (\epsilon_k \cdot \epsilon_\ell) + (b_k p_k + 2b_\ell p_\ell \cdot \epsilon_k) \epsilon_\ell^\mu - (b_\ell p_\ell + 2b_k p_k \cdot \epsilon_\ell) \epsilon_k^\mu \right], \quad (8.30)$$

where we have used momentum conservation and included the colour matrix T_c with which the gluon is coupled to the fermion line. f is the structure constant of the colour group. We obtain a dramatic simplification in Eq. (8.30) by noting that, in the present gauge, $\epsilon_k^+ = \epsilon_\ell^-$. Inserting this, and using the definition of f as a structure constant, we reduce Eq. (8.30) to

$$-ig (T^{a_k} T^{a_\ell} - T^{a_\ell} T^{a_k}) \delta_{\lambda_k \lambda_\ell} (b_k p_k - b_\ell p_\ell)^\mu. \quad (8.31)$$

Hence the expressions for $M_{9,10}$ are written as follows:

$$M_n(i, j, k, \ell) = D_n \cdot T_n(\lambda_i) \cdot C_n, \quad n=9, 10; \quad (8.32)$$

$$\begin{aligned} D_9 &= -ig^2 (g_\nu^i - \lambda_i \cdot g_a^i) \delta_{\lambda_i \lambda_j} \delta_{\lambda_k \lambda_\ell} f_{k\ell}^{-1} (f_{jk} + f_{j\ell} + f_{k\ell})^{-1}, \\ D_{10} &= +ig^2 (g_\nu^i - \lambda_i \cdot g_a^i) \delta_{\lambda_i \lambda_j} \delta_{\lambda_k \lambda_\ell} f_{k\ell}^{-1} (f_{ik} + f_{i\ell} + f_{k\ell})^{-1}; \end{aligned} \quad (8.33)$$

$$C_9 = C_{10} = (T^{a_k} T^{a_\ell} - T^{a_\ell} T^{a_k})_{c_i c_j}; \quad (8.34)$$

$$\begin{aligned} T_9(\lambda) &= \bar{u}(\tau_1) \gamma_\mu u(\tau_2) \cdot \bar{u}_\lambda(p_i) \gamma^\mu (b_j p_j + b_k p_k + b_\ell p_\ell) (b_k p_k - b_\ell p_\ell) u_\lambda(p_j), \\ T_{10}(\lambda) &= \bar{u}_\lambda(p_i) (b_k p_k - b_\ell p_\ell) (b_i p_i + b_k p_k + b_\ell p_\ell) \gamma^\mu u_\lambda(p_j) \cdot \bar{u}(\tau_1) \gamma_\mu u(\tau_2). \end{aligned} \quad (8.35)$$

It is easy to write down the explicit form for $T_{9,10}$:

$$\begin{aligned} T_9(+) &= 2S_{i2} \{ [b_j t_{ij} s_{jk} + b_{\ell i \ell} s_{\ell k}] b_k t_{kj} - [b_j t_{ij} s_{j\ell} + b_{\ell i k} s_{k\ell}] b_{\ell} t_{\ell j} \}, \\ T_{10}(+) &= 2 \{ b_k s_{ik} [b_i t_{ki} s_{i2} + b_{\ell k \ell} s_{\ell 2}] - b_{\ell} s_{i\ell} [b_i t_{\ell i} s_{i2} + b_k t_{\ell k} s_{k2}] \} t_{ij}, \quad (8.36) \end{aligned}$$

and again $T_{9,10}(-)$ are obtained by taking the complex conjugate and interchanging r_1 and r_2 , according to Eq. (8.26). In this case as well, the $T_{9,10}$ behave in collinear peaking situations just so as to temper the degree of divergence.

This finishes the main part of our discussion of the generic diagrams. We have shown that we can write them as expressions in terms of the products s and t , with very little Dirac algebra actually being required. The formulae can immediately be entered into a computer program. The only part which remains is the colour factors.

Given an explicit representation of the colour matrices T^a , the colour factor for each diagram is completely determined for given colour labels of the relevant partons. In principle we could therefore perform an explicit sum over the colour degrees of freedom in order to obtain $\Sigma |M|^2$. However, on the one hand this is inefficient since the colour labels take on many different values, and on the other hand it is unnecessary because the colour factors in the cross-section are constants, not depending on the phase space momenta, and can therefore be fixed once and for all. We shall now discuss the colour factors that have to be included in the definition of $\Sigma |M|^2$ for our four processes.

(i): V_{qq}

The overall colour factor is very simple^{*)}

$$\sum_{c_1, c_2} (\delta_{c_1 c_2})^2 = 3. \quad (8.37)$$

This has to be included as a factor in the cross-section.

^{*)} Three colours for the quarks are assumed.

(ii): Vqgg

Again we have one overall factor for the cross-section

$$\sum_{a, c_1, c_2} (T_{c_1 c_2}^a)^2 = \sum_a T_R T^a T^a = 4 \quad (8.38)$$

(iii): Vqggg

This case is more complicated. Since in the most general case the quarks can be connected in two different ways, the interference between these two sets of diagrams will be colour-suppressed. Consider the process

$$\bar{u}(p_i) u(p_j) \rightarrow u(p_k) \bar{u}(p_l) Z_0 \quad (8.39)$$

The eight diagrams contributing to the amplitude are grouped as follows:

$$\begin{aligned} B_1 &= M_4(i, j, k, l) + M_5(i, j, k, l) + M_4(k, l, i, j) + M_5(k, l, i, j) , \\ B_2 &= M_4(i, l, k, j) + M_5(i, l, k, j) + M_4(k, j, i, l) + M_5(k, j, i, l) . \end{aligned} \quad (8.40)$$

The cross-section is therefore constructed as

$$\sum |M|^2 = C_{11} \sum_{spins} |B_1|^2 + C_{22} \sum_{spins} |B_2|^2 - C_{12} \sum_{spins} 2 \text{Re}(B_1 B_2^*) \quad (8.41)$$

with

$$\begin{aligned} C_{11} &= C_{22} = \sum_{ab} T_R (T^a T^b) T_R (T^a T^b) = 2 , \\ C_{12} &= \sum_{ab} T_R (T^a T^b T^a T^b) = -\frac{2}{3} . \end{aligned} \quad (8.42)$$

In Eq. (8.41) the minus sign takes the Fermi statistics into account.

(iv): Vqggg

In this case we always have eight Feynman graphs which are most conveniently grouped according to the symmetry in the gluon colour labels. For the process

$$\bar{u}(p_i) u(p_j) \rightarrow g(p_k) g(p_l) Z_0 \quad (8.43)$$

this gives us the following two groups:

$$\begin{aligned}
 B'_1 &= \frac{1}{2} [M_6(i, j, k, l) + M_7(i, j, k, l) + M_8(i, j, k, l) \\
 &\quad + M_6(i, j, l, k) + M_7(i, j, l, k) + M_8(i, j, l, k)] , \\
 B'_2 &= \frac{1}{2} [M_6(i, j, k, l) + M_7(i, j, k, l) + M_8(i, j, k, l) \\
 &\quad - M_6(i, j, l, k) - M_7(i, j, l, k) - M_8(i, j, l, k)] \\
 &\quad + M_9(i, j, k, l) + M_{10}(i, j, k, l)
 \end{aligned} \tag{8.44}$$

The colour factor of B'_1 is symmetric in the gluon colour labels, while B'_2 is antisymmetric. Consequently they do not interfere in the colour sum, and we can write

$$\sum |M|^2 = C'_{11} \sum_{spins} |B'_1|^2 + C'_{22} \sum_{spins} |B'_2|^2 , \tag{8.45}$$

with

$$\begin{aligned}
 C'_{11} &= \sum_{ab} T_R (T^a T^b + T^b T^a) (T^a T^b + T^b T^a) = \frac{20}{3} , \\
 C'_{22} &= \sum_{ab} T_R (T^a T^b - T^b T^a) (T^b T^a - T^a T^b) = 12 .
 \end{aligned} \tag{8.46}$$

Finally, to obtain the correct spin and colour average, we have to divide the cross-sections by 36 for a qq , $q\bar{q}$ or $\bar{q}q$ initial state, by 96 for a qg or $\bar{q}g$ initial state, and by 256 for a gg initial state.

This finishes our discussion of the amplitudes for the processes in Eq. (7.2). The formulae presented in this section can be directly programmed into a computer. As we have stated before, the rest of the calculation, which includes an enumeration of all 396 processes, their Kobayashi-Maskawa flavour mixing factors, the parton distribution functions, the choice of the QCD scale, and the Monte Carlo integration, are discussed elsewhere, together with the phenomenological consequences.

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APPENDIX

Here we present proofs of the various equations presented in this paper. The reader may want to follow them as an exercise to become familiar with spinor techniques.

Section 3

Eq. (3.3):

$$u_+(k_0) \bar{u}_+(k_0) = \not{k}_0 \omega_- \not{k}_0 \not{k}_1 = -\omega_+ \not{k}_0 k_1^2 = \omega_+ \not{k}_0 . \quad (\text{A.1})$$

Eq. (3.4):

$$u_\lambda(p) \bar{u}_\lambda(p) = \not{p} \omega_\lambda \not{k}_0 \not{p} / (2 p \cdot k_0) = \omega_\lambda \not{p} . \quad (\text{A.2})$$

Eq. (3.5): Let S be any string of γ matrices, and S^R its reverse. Then,

$$\bar{u}_-(k_0) S u_-(k_0) = \text{Tr}(\omega_- \not{k}_0 S) = \text{Tr}(S^R \not{k}_0 \omega_-) = \bar{u}_+(k_0) S^R u_+(k_0) , \quad (\text{A.3})$$

and

$$\begin{aligned} \bar{u}_-(k_0) S u_+(k_0) &= \text{Tr}(\omega_- \not{k}_0 S \not{k}_1) = \text{Tr}(S^R \not{k}_0 \omega_- \not{k}_1) = \\ &= -\text{Tr}(S^R \not{k}_1 \omega_- \not{k}_0) = -\bar{u}_-(k_0) S^R u_+(k_0) . \end{aligned} \quad (\text{A.4})$$

If we choose

$$S = \not{p}_1 \not{\Gamma} \not{p}_2 / [4 (p_1 \cdot k_0)(p_2 \cdot k_0)]^{1/2} ,$$

Eq. (3.5) follows.

Eq. (3.6): Let S be any string of an odd number of γ matrices. Then there are two four-vectors V^μ and A^μ such that

$$S = V_\mu \gamma^\mu + A_\mu \gamma^5 \gamma^\mu . \quad (\text{A.5})$$

From the derivation

$$2(S + S^R) = 2[V_\mu(2\gamma^\mu) + A_\mu(\gamma^5\gamma^\mu + \gamma^\mu\gamma^5)] = 4V_\mu\gamma^\mu = \text{Tr}(S\gamma^\mu), \quad (\text{A.6})$$

Eq. (3.6) follows if we choose

$$S = \not{p}_2 \omega_- \not{k}_0 \not{p}_1 / [4(p_1 \cdot k_0)(p_2 \cdot k_0)]^{1/2}.$$

Eq. (3.12):

$$S(p_1, p_2) \bar{t}(p_2, p_1) = \bar{u}_+(p_1) u_-(p_2) \bar{u}_-(p_2) u_+(p_1) = \text{Tr}(\omega_+ \not{p}_1 \not{p}_2) = 2p_1 \cdot p_2. \quad (\text{A.7})$$

Section 4

Eq. (4.1): We prove the two non-trivial properties of Eq. (4.1) for the polarization choice (4.2):

$$\epsilon_+ \cdot \epsilon_+ \propto \bar{u}_+(k) \gamma^\mu [\bar{u}_+(k) \gamma_\mu u_+(p)] u_+(p) = 2\bar{u}_+(k) [u_+(p) \bar{u}_+(k) + u_-(k) \bar{u}_-(p)] u_+(p) = 0, \quad (\text{A.8})$$

where the first term vanishes because of its helicity structure, and the second one because k^μ and p^μ are lightlike. The other property follows most easily from Eq. (3.5):

$$\begin{aligned} \epsilon_+ \cdot \epsilon_- &= N^2 \bar{u}_+(k) \gamma^\mu u_+(p) \bar{u}_-(k) \gamma_\mu u_-(p) = N^2 \bar{u}_+(k) \gamma^\mu u_+(p) \bar{u}_+(p) \gamma_\mu u_+(k) = \\ &= -2N^2 \bar{u}_+(k) \not{p} u_+(k) = -4N^2(p \cdot k) = -1. \end{aligned} \quad (\text{A.9})$$

The spin sum (2.2) follows from explicit evaluation:

$$\sum_\lambda \epsilon_\lambda^\mu (\epsilon_\lambda^\nu)^* = \sum_\lambda N^2 \text{Tr}(\omega_\lambda \not{k} \gamma^\mu \not{p} \gamma^\nu) = -g^{\mu\nu} + \frac{p^\mu k^\nu + p^\nu k^\mu}{(p \cdot k)}. \quad (\text{A.10})$$

It might appear that the polarization choice (4.2) could be modified in order to avoid choosing a p^μ :

$$\epsilon_\lambda^\mu = \lambda \bar{u}_\lambda(k) \gamma^\mu u_\lambda(k_0) / [4(k \cdot k_0)]^{1/2}. \quad (\text{A.11})$$

However, this introduces $u_{\pm}(k_0)$ explicitly into the calculation. Since these behave so differently from the other spinors we want to avoid this here.

The interrelation between two different choices of p^μ is the following. Define two choices as

$$\epsilon_i^\mu = N_i \bar{u}_+(k) \gamma^\mu u_+(p_i) , \quad N_i = [4(p_i \cdot k)]^{-1/2} , \quad i=1,2 . \quad (\text{A.12})$$

We can choose a frame in which $\vec{k}$ points along the z axis. We can make $\vec{\epsilon}_{1,2}$ lie in the xy plane by a suitable gauge transformation (i.e., adding a term proportional to k^μ). They can then differ by at most a complex phase. Consequently there are numbers ϕ and β such that

$$\epsilon_2^\mu = e^{i\phi(p_2, p_1)} \epsilon_1^\mu + \beta(p_2, p_1) k^\mu . \quad (\text{A.13})$$

We find ϕ from the normalization:

$$\begin{aligned} e^{i\phi(p_2, p_1)} &= -\epsilon_2 \cdot \epsilon_1^* = -N_1 N_2 \bar{u}_+(k) \gamma^\mu u_+(p_2) \cdot \bar{u}_+(p_1) \gamma_\mu u_+(k) \\ &= 2 N_1 N_2 s(k, p_1) t(p_2, k) \\ &= \left[\frac{s(k, p_1)}{t(k, p_1)} \cdot \frac{t(k, p_2)}{s(k, p_2)} \right]^{1/2} . \end{aligned} \quad (\text{A.14})$$

As expected, ϕ changes sign under interchange of p_1 and p_2 . Contracting both sides of Eq. (A.13) with p_1^μ we obtain

$$\epsilon_2 \cdot p_1 = N_2 \bar{u}_+(k) \not{p}_1 u_+(p_2) = N_2 s(k, p_1) t(p_1, p_2) = \beta(p_2, p_1) (p_1 \cdot k) , \quad (\text{A.15})$$

from which we have

$$\beta(p_2, p_1) = 2 N_2 t(p_1, p_2) / t(p_1, k) \quad (\text{A.16})$$

The consistency condition

$$\beta(p_1, p_2) = - e^{i\phi(p_1, p_2)} \beta(p_2, p_1) \quad (\text{A.17})$$

is left to the reader as an exercise.

Section 5

We illustrate the relation between the use in the cross-section of free on-shell bosons, and the use of decaying resonances. Let a boson with momentum q^μ and mass m decay into two fermions with momenta p_1^μ and p_2^μ , by way of the current

$$j^\mu = i g \bar{u}(p_1)(1+\gamma^5)\gamma^\mu v(p_2) \quad . \quad (\text{A.18})$$

The corresponding decay width is

$$\Gamma = m g^2 / 6\pi \quad . \quad (\text{A.19})$$

In the cross-section an on-shell boson is represented by

$$\sum \epsilon^\mu \epsilon^{*\nu} \frac{1}{(2\pi)^3} d^4 q \delta(q^2 - m^2) = (-g^{\mu\nu} + \frac{q^\mu q^\nu}{m^2}) \frac{d^3 \vec{q}}{(2\pi)^3 2q^0} \quad . \quad (\text{A.20})$$

If we treat the boson as a resonance, we find, instead:

$$\frac{1}{(q^2 - m^2)^2 + m^2 \Gamma_t^2} j^\mu j^{*\nu} \frac{1}{(2\pi)^6} d^4 p_1 \delta(p_1^2) d^4 p_2 \delta(p_2^2) \\ \frac{g^2 \text{Tr}(\omega_+ \gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1)}{(q^2 - m^2)^2 + m^2 \Gamma_t^2} \cdot \frac{d^4 q d\Omega}{2(2\pi)^6} \quad , \quad (\text{A.21})$$

where Γ_t is the total decay width of the resonance, and Ω the decay angle in its rest frame. Performing the angular integral as in Eq. (5.4) we reduce this to

$$\frac{g^2 m^2}{6\pi^2} \cdot \frac{(-g^{\mu\nu} + q^\mu q^\nu / m^2)}{(q^2 - m^2)^2 + m^2 \Gamma_t^2} \cdot \frac{d^4 q}{(2\pi)^3} \sim \frac{\Gamma}{\Gamma_t} (-g^{\mu\nu} + \frac{q^\mu q^\nu}{m^2}) \frac{d^4 q \delta(q^2 - m^2)}{(2\pi)^3} \quad . \quad (\text{A.22})$$

In this last step the so-called peaking approximation has been made, in which the Breit-Wigner shape is replaced by a delta function. Comparing Eqs. (A.20) and (A.22) we see that the two approaches are equivalent if the above decay saturates the total width, i.e., $\Gamma = \Gamma_t$.

Section 6

We prove Eq. (6.1) by writing out the projection operator:

$$\begin{aligned}
 u(q,s)\bar{u}(q,s) &= (\not{q}+m)\omega_{-}\not{k}(\not{q}+m)/(2q\cdot k) \\
 &= [(\not{q}+m)\not{k} + \gamma^5(\not{q}-m)\not{k}](\not{q}+m)/(4q\cdot k) \\
 &= [2(q\cdot k) + 2(q\cdot k)\gamma^5 - 2m\gamma^5\not{k}](\not{q}+m)/(4q\cdot k) \\
 &= \frac{1}{2}(1+\gamma^5\not{s})(\not{q}+m) , \tag{A.23}
 \end{aligned}$$

where s^μ is defined as

$$s^\mu = \frac{1}{m} q^\mu - \frac{m}{(q\cdot k)} k^\mu . \tag{A.24}$$

It is easy to see that $s\cdot s = -1$ and $q\cdot s = 0$ so that Eq. (A.23) indeed describes a pure spin state. Substituting p_2^μ for k^μ as in Eq. (6.2) we arrive at the form (6.4) for s^μ . By changing the sign of the γ^5 we obtain the other spin state. The spin sum (2.3) then follows immediately. Equation (6.11) is proved by writing out the integral and some Dirac algebra:

$$\begin{aligned}
 \int d\Omega u(q,s)\bar{u}(q,s) &= \int d\Omega (\not{q}/m+1)\omega_{-}\not{p}_2(\not{q}/m+1) \\
 &= (\not{q}/m+1) \left[\int d\Omega \omega_{-}\not{p}_2 \right] (\not{q}/m+1) \\
 &= 2\pi (\not{q}/m+1) \omega_{-}\not{q} (\not{q}/m+1) = 2\pi (\not{q}+m) . \tag{A.25}
 \end{aligned}$$

By replacing m by $-m$ we obtain the same derivation for antiparticles. Finally, which decay of a heavy fermion can lead to the correct spin sum? Let the massive fermion with momentum q^μ and mass m decay into a massless fermion with momentum p^μ and a photon with momentum k^μ and polarization vector ϵ^μ . The simplest gauge-invariant amplitude for this decay is $u(q)$ coupled to a spinorial object ξ :

$$\xi \propto \left\{ \alpha[(p\cdot k)\not{\epsilon} - (p\cdot \epsilon)\not{k}] + \beta[\not{k}\not{\epsilon} - \not{\epsilon}\not{k}] \right\} u_\lambda(p) , \tag{A.26}$$

with arbitrary complex numbers α and β . We choose ϵ^μ as in Eq. (4.2), with helicity λ' . We then obtain

$$\begin{aligned}
 \xi &\propto (\alpha + 2\beta \not{k}) (u_{\lambda}(p) \bar{u}_{\lambda}(k) + \underline{u}_{\lambda}(k) \underline{u}_{\lambda}(p)) u_{\lambda}(p) \\
 &= -\not{\epsilon}_{\lambda, \lambda} \bar{u}_{\lambda}(k) u_{\lambda}(p) (\alpha + 2\beta \not{k}) \underline{u}_{\lambda}(p) \\
 &\propto \left[\frac{2\beta}{m} \bar{u}_{\lambda}(k) \underline{u}_{\lambda}(p) \right] u_{\lambda}(k) + \underline{u}_{\lambda}(p) \quad . \quad (A.27)
 \end{aligned}$$

which reduces to the form (6.3) if we take $p_1^\mu = k^\mu$, $p_2^\mu = p^\mu$, and $\alpha = 2m\beta$. the choice $\alpha = -2m\beta$ results in the choice for the antiparticle.

REFERENCES

- 1) R. Kleiss - Nucl.Phys. B241 (1984) 61.
- 2) S.D. Ellis, R. Kleiss and W.J. Stirling - CERN Preprints TH. 4096 (1985) and TH. 4144 (1985).
- 3) A.C. Hearn - REDUCE 2 User's manual, University of Utah (1973);
M. Veltman - SCHOONSCHIP (unpublished).
- 4) CALKUL Collaboration - Phys.Lett. 103B (1981) 124.
- 5) P. de Causmaecker - Ph.D. Thesis, Katholieke Universiteit Leuven (1982).
- 6) B.L. Combridge and C.J. Maxwell - Phys.Lett. 151B (1985) 299.
- 7) J.D. Bjorken and M.C. Chew - Phys.Rev. 154 (1967) 1335;
O. Reading Henry - Phys.Rev. 154 (1967) 1534.
- 8) H.W. Fearing and R.R. Silbar - Phys.Rev. D6 (1972) 471.
- 9) K.J.F. Gaemers and D. Gounaris - Z.Phys. C1 (1979) 259.
- 10) J.O. Eeg - J.Math.Phys. 21 (1980) 170.
- 11) G. Passarino - Nucl.Phys. B237 (1984) 249;
M. Caffo and E. Remmidì - Helv.Phys.Acta 55 (1982) 339.
- 12) G. Farrar and F. Neri - Phys.Lett. 130B (1983) 109.
- 13) CALKUL Collaboration - Phys.Lett. 105B (1981) 215; 114B (1982) 203; Nucl.
Phys. B206 (1982) 53; B206 (1982) 61; B239 (1984) 382; B239 (1984) 395.
- 14) A. Sirlin - Nucl.Phys. B192 (1981) 93.
- 15) Z. Xu, D.-H. Zhang and L. Chang - Tsinghua University preprints TUTP-84/3,
TUTP-84/4, TUTP-84/5, TUTP-84/6.
- 16) F. James - Rep.Progr.Phys. 43 (1980) 1145, and references quoted therein.
- 17) UA1 Collaboration, CERN-EP/84-135 (1984).
- 18) F.A. Berends, P.H. Daverveldt and R. Kleiss - Nucl.Phys. B253 (1985) 441.
- 19) P.H. Daverveldt - Ph.D. Thesis, Leiden University (1985).
- 20) R.K. Ellis, S.D. Ellis, R. Gonsalves, R. Kleiss and W.J. Stirling - in
preparation.
- 21) S.D. Ellis, R. Kleiss and W.J. Stirling - CERN Preprint TH. 4185 (1985).
- 22) F.J. Yndurain - Quantum Chromodynamics (Springer, 1983).
- 23) Z. Kunszt and J.F. Gunion - Oregon University Preprint OITS-296 (1985).

FIGURE CAPTIONS

Fig. 1 Generic Feynman diagram associated with $u\bar{u} \rightarrow Z^0$.

Fig. 2 Generic Feynman diagrams associated with $u\bar{u} \rightarrow Z^0 g$.

Fig. 3 Generic Feynman diagrams associated with $u\bar{u} \rightarrow Z^0 u\bar{u}$.

Fig. 4 Generic Feynman diagrams associated with $u\bar{u} \rightarrow Z^0 gg$.

Fig. 5 The three-gluon vertex occurring in M_9 and M_{10} .

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5