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STOCHASTIC IDENTITIES IN THE LIGHT CONE GAUGE^{*)}

V. de Alfaro
Istituto di Fisica Teorica, Università di Torino
INFN, Sezione di Torino

S. Fubini
CERN - Geneva

and

G. Furlan
Dipartimento di Fisica Teorica, Università di Trieste
INFN, Sezione di Trieste
International Centre for Theoretical Physics, Trieste

A B S T R A C T

The correspondence between stochastic quantization and supersymmetry is extended to $N = 1$ gauge supersymmetry coupled with matter. This applies to $N = 2$ extended gauge supersymmetry and also to $N = 1$ SUSY in six dimensions. The special rôle of the light plane gauge is emphasized.

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1. Supersymmetric field theories exhibit remarkable and unique features. In particular the cancellations occurring between boson and fermion contributions in the perturbation expansion are at the origin of their milder ultraviolet behaviour and lead in some special cases (the $N=4$ theory in 4 dimensions) to a completely finite theory⁽¹⁾.

A potentially important property of some supersymmetric theories closely related to those cancellations was first pointed out by Nicolai⁽²⁾, who suggested the possibility that, after integration over the fermionic fields, the resulting functional integral is cast in gaussian form in terms of a new set of bosonic variables (Nicolai map).

Conversely, Parisi and Sourlas⁽³⁾ realized that a supersymmetric theory emerges in some cases starting from classical stochastic equations in space-time.

This approach has been applied with success to supersymmetric quantum mechanics and to the $N=2$ Wess-Zumino scalar theory in 2 space-time dimensions; in both cases the new bosonic variables are local and polynomial functions of the old ones.

In two previous papers⁽⁴⁾ it was shown by G. Veneziano and the present authors that the Nicolai-Parisi Sourlas procedure could be extended to include a four dimensional field theory, the $N=1$ pure gauge supersymmetry. A novel feature which has emerged is the fundamental role of the light cone gauge. The unique role of this gauge has been stressed in an investigation by Amati and Veneziano⁽⁵⁾ and has been further examined by R. Floreanini, J.P. Leroy, J. Micheli and G.C. Rossi⁽⁶⁾ in a perturbation analysis.

In the present paper the previous results are extended to the $N = 1$ gauge theory coupled with matter, establishing the existence of a Nicolai map in which the stochastic variables are exhibited in explicit, closed form. This applies in particular to the $N = 2$ extended theory and to the 6-dimensional $N = 1$ gauge theory that goes by dimensional reduction into $N = 2$.

The element which allows this new step is the introduction of a new auxiliary field which allows to write the whole set of new bosonic variables (the stochastic variables) in a local form.

We are thus able to provide in closed form the complete set of stochastic variables required by quadratization. As in refs. I and II we shall proceed both by direct use of quadratization and by constructing the appropriate supersymmetric Ward identities.

2A. We shall start discussing the N=1 supersymmetric gauge theory coupled to a supersymmetric set of matter fields; for details we refer to section 5 of II. The ingredients are a gauge supermultiplet A_μ^a, λ^a of fields belonging to the regular representation of the colour group, and a matter supermultiplet of a complex scalar field ϕ^i and a Weyl spinor $\psi^i, "i"$ being the index of a generic representation .

The action contains both a bosonic and a fermionic piece

$$I = I_B + I_F = \int d^4x (\mathcal{L}_B + \mathcal{L}_F) \quad (1)$$

where

$$\mathcal{L}_B = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - D_\mu \phi^* D^\mu \phi - \frac{1}{2} g^2 (\phi^* \times \phi)^2 \quad (2)$$

and

$$\begin{aligned} \mathcal{L}_F = & -i\lambda(\sigma D)\bar{\lambda} - i\psi(\sigma D)\bar{\psi} + \\ & + i\sqrt{2}g(\bar{\lambda}\bar{\psi}\phi - \phi^*\lambda\psi) \end{aligned} \quad (3)$$

Summation over colour indices is understood . The definitions of some symbols are

$$D_\mu = \partial_\mu - ig(t^a A_\mu^a)$$

$$\phi^* \times \phi = \phi^*_i t^a_{ij} \phi_j$$

$$\phi^* \lambda \psi = \phi^*_i t^a_{ij} \lambda^a \psi_j$$

Let us deal first with the bosonic action. The discussion will be formulated in the basis

$$\begin{aligned} A_{R,L} &= A_0 \pm A_3, \\ A_{\pm} &= A_1 \pm i A_2. \end{aligned} \quad (4)$$

Using several partial integrations (*) $\mathcal{L}_B$ can be transformed into

$$\mathcal{L}_B = \mathcal{L}_0 - D_+ \phi^* D_- \phi \quad (5)$$

where

$$\mathcal{L}_0 = \frac{1}{2} N_3^2 + \frac{1}{2} F_L + F_R + D_L \phi^* D_R \phi \quad (6)$$

and

$$\begin{aligned} N_3 &= F_3 + i g \phi^* \times \phi, \\ F_3 &= \frac{1}{2} (F_{RL} + F_{+-}). \end{aligned} \quad (7)$$

In order to proceed further it is necessary to choose a particular gauge, the light cone gauge

$$A_L = 0$$

(*) among others we have used the identity

$$F_{RL} F_{+-} + F_{R-} F_{L+} + F_{R+} F_{-L} = 0$$

which is valid under integration since the l.h.s. is $\propto \vec{E} \cdot \vec{H}$, i.e. a four-divergence.

The expression of the functional integral Z is then (with $\mathcal{L}_B$ evaluated at $A_L=0$)

$$Z = \int dA_+ dA_- dA_R d\phi^* d\phi (\text{Det } \partial_L) \cdot \exp \left\{ i \int d^4x \mathcal{L}_B \right\} Z_F \quad (8)$$

Let us now introduce the two auxiliary fields M_+ and M_- which are defined through the new effective Lagrangian

$$\hat{\mathcal{L}}_B = \mathcal{L}_B + (\partial_L M_+ - D_+ \phi^*)(\partial_L M_- - D_- \phi). \quad (9)$$

This can be easily made to appear in the starting generating functional Z if two extra functional integrations are considered namely

$$Z = \int dA_+ dA_- dA_R d\phi^* d\phi dM_+ dM_- \cdot (\text{Det } \partial_L)^3 \exp \left\{ i \int d^4x \hat{\mathcal{L}}_B \right\} Z_F \quad (10)$$

We are now ready to set Z in a quadratic form. Taking eq.(5) into account one finds

$$\hat{\mathcal{L}}_B = \mathcal{L}_0 + \partial_L M_+ (\partial_L M_- - D_- \phi) - D_+ \phi^* \partial_L M_- \quad (11)$$

Putting further

$$\begin{aligned} m_+ &= \sqrt{2} \partial_L M_+ \\ m_- &= \sqrt{2} (\partial_L M_- - D_- \phi) \end{aligned} \quad (12)$$

and integrating once more by parts we are finally led to the following formula for the effective Lagrangian $\hat{\mathcal{L}}_B$

$$\hat{\mathcal{L}}_B = \frac{1}{2} N_3^2 + \frac{1}{2} N_+ N_- + \frac{1}{2} m_+ m_- + \frac{1}{2} M_+ M_-, \quad (13)$$

where

$$\begin{aligned} N_+ &= \partial_- A_+ \quad , \quad m_+ = \sqrt{2} \partial_- \phi^*, \\ N_- &= F_{R-} + 2 i g \phi^* \times M_-, \\ m_- &= \sqrt{2} (D_R \phi - D_+ M_-). \end{aligned} \quad (14)$$

Eq.(13) expresses the reduction of the bosonic lagrangian to a quadratic form in the light cone gauge, apart from surface terms. To the seven fields $A_+, A_-, A_R, \phi^*, \phi, M_+, M_-$ correspond the seven Nicolai variables N_3, N_+, n_+, m_- . If these quantities are taken as integration variables in the functional integral of eq.(10), one needs to construct the Jacobian of the transformation, that can be easily obtained writing equations (12) and (14) in differential form, A simplifying feature is represented by the trivial connection of N_+, n_+ and m_+ to A_+, ϕ^* and M_+ , so that the factor attached to the relevant transformation just corresponds to the term $(\text{Det } \partial_-)^3$ appearing in the expression for Z of eq.(10). Thus, keeping A_+, ϕ^* and M_+ fixed we have

$$\begin{aligned} dN_3 &= \frac{1}{2} (D_+ dA_- - \partial_- dA_R + 2 i g \phi^* \times d\phi), \\ dN_- &= D_R dA_- - D_- dA_R + 2 i g \phi^* \times dM_- \\ dm_- &= \sqrt{2} (-i g \phi dA_R + D_R d\phi - D_+ dM_-) \\ dm_- &= \sqrt{2} (i g \phi dA_- - D_- d\phi + \partial_- dM_-) \end{aligned} \quad (15)$$

from which the matrix $J = \frac{d(N, n, m)}{d(A, \phi, M)}$ can be read.

The final step consists in computing the Matthews-Salam determinant. This comes from integration over the fermionic fields $\lambda^a, \bar{\lambda}^a$ and $\psi^i, \bar{\psi}^i$ that appear in Z_F present in eq.(10)

$$Z_F = \int d\lambda d\bar{\lambda} d\psi d\bar{\psi} \exp(i I_F) \quad (16)$$

Conversely one can easily verify that the Jacobian determinant when parametrized by fermionic functional integration just gives the fermionic action.

We are thus allowed to use for the functional integral the purely bosonic form

$$Z = \int dN_3 dN_+ dN_- dm_+ dm_- dm_- \quad (17)$$

$$\exp\left\{\frac{i}{2} \int d^4x (N_3^2 + N_+ N_- + m_+ m_- + m_- m_+)\right\}$$

This achieves our goal.

If we now adopt, as in most applications of the functional method, a Euclidean metric $s = it$, $A_4 = iA_0$ etc ⁽⁺⁾ Z takes on a gaussian form, $\exp(i\hat{I}_B) \rightarrow \exp(-\hat{I}_B)$. As a final consequence we are then led to a complete set of stochastic identities of the kind

$$\begin{aligned} \langle N_{\alpha_1}(x_1) N_{\alpha_2}(x_2) \dots \rangle &= \\ &= \langle N_{\alpha_1}(x_1) N_{\alpha_2}(x_2) \dots \rangle_{\text{FRA}} \end{aligned} \quad (18)$$

(+) We shall not dwell upon the delicate point connected to the need of complex paths in the gauge $A_L=0$: this has been discussed in II.

2B. It has been shown in II that one can arrive directly to eq.(18) by an appropriate use of the supersymmetry transformations. In particular we recall that in the light-cone gauge $A_L=0$ it was possible to select a supersymmetric generator Δ_2 with the following properties

$$\begin{aligned} \text{a) } \Delta_2^2 &= 0, \\ \text{b) } \Delta_2 A_L &= 0. \end{aligned} \tag{19}$$

We then proved that if a bosonic variable N can be written in form $N = \Delta_2 \Psi$, then it is a stochastic variable i.e. it can be inserted in a stochastic identity.

From all that the further result followed that "if a field variable φ is annihilated by Δ_2 then $\partial_L \varphi$ is a stochastic variable". Using such technique we were able to show (see II eq. 5.14) that N_3, N_+, n_+ are indeed stochastic variables.

The new fact here is represented by the presence of the effective Lagrangian $\widehat{\mathcal{L}}_B$ defined in eq.(13) which contains the auxiliary fields M_+ . In order to adapt the "stochastic" method recalled above to our case, we have to discuss the invariance of $\widehat{\mathcal{L}}_B + \mathcal{L}_F$ under the transformation Δ_2 .

It is easy to show that $\widehat{\mathcal{L}}_B + \mathcal{L}_F$ is invariant under the following set of transformations:

$$\Delta_2 A_L = \Delta_2 A_+ = 0, \tag{20}$$

$$\Delta_2 \phi^* = \Delta_2 M_+ = 0.$$

$$\Delta_2 A_2 = -2i\bar{\lambda}^2, \quad \Delta_2 A_- = 2i\bar{\lambda}^1, \tag{21}$$

$$\Delta_2 \phi = \sqrt{2} \psi_2, \quad \Delta_2 M_- = \sqrt{2} \psi_1.$$

$$\begin{aligned}\Delta_2 \lambda_1 &= N_3, & \Delta_2 \lambda_2 &= N_+, \\ \Delta_2 \bar{\psi}_2 &= -im_+, & \Delta_2 \bar{\psi}_1 &= -i\mu_+.\end{aligned}\tag{22}$$

One has to remark that equations (20), (21), (22) differ from the usual transformations listed in II, eq. (5.8), in the following respects
-the new fields M_+ and M_- take part in the transformations
- $\bar{\psi}_1$ goes into $-im_+$ and not into $-i\sqrt{2}D_+\phi^*$.

Equations (22) show that N_3 , N_+ , n_+ and m_+ are indeed stochastic variables

Using the above transformation laws it is also easy to compute the effect of Δ_2 on these quantities. We find

$$\begin{aligned}\Delta_2 N_3 &= -\frac{\delta I}{\delta \lambda^2}, & \Delta_2 N_- &= -2\frac{\delta I}{\delta \lambda^1}, \\ \Delta_2 m_- &= 2i\frac{\delta I}{\delta \bar{\psi}_2}, & \Delta_2 \mu_- &= 2i\frac{\delta I}{\delta \bar{\psi}_1}.\end{aligned}\tag{23}$$

This result confirms the stochastic nature of the variables N_3 , N_- since the r.h.s. of eq. (23) vanish on the mass shell.

As in II we are only able to derive in this way stochastic relations which involve the L -derivatives of the previous variables.

On the other hand we are convinced on the basis of the results obtained in the previous section that this limitation is a fictitious one and can be overcome with an appropriate use of boundary conditions.

3. The previous arguments have led to the identification in the light cone gauge of five effective Nicolai variables built with the 3 independent gauge fields and with the spinless objects ϕ, ϕ^* . The framework was the N=1 gauge theory coupled supersymmetrically with matter.

Among the large class of such N=1 theories we shall now fix our attention on the N=2 extended supersymmetry. In this model just one matter superfield (ϕ, ψ) belonging to the regular colour representation is added to the pure N=1 theory. Furthermore it is well known that the N=2 extended supersymmetry (in four dimensions) can be obtained by dimensional reduction of the N=1 gauge theory in six dimensions.

Some features of the six dimensional theory will now be sketched: 5 effective Nicolai stochastic variables will be identified and it will be indicated that the functional integral can be given a purely gaussian form.

These six-dimensional stochastic variables after dimensional reduction are recognized to reproduce the Nicolai quantities already determined in the previous section.

Notice that the result is not trivial in the sense that there is no strict correspondence between the stochastic properties of a given model and those of its dimensional reduction. For instance for the N=1 Wess-Zumino model in 4 dimensions there is no Nicolai map which on the contrary exists for its 2-dimensional reduction.

The spectrum of the N=2 model consists of a Yang-Mills field $A_\mu^a, \mu=0,1,\dots,5$ and a four component spinor $\psi_\alpha^a, \alpha=1,2,3,4^{(*)}$. "a" is the color index and the fields belong to the regular representation of the gauge group.

A Minkowski metric $g_{00}=1 = -g_{55} = -g_{kk} \quad k=1,2,3,4$, will be adopted.

The supersymmetric 6-dimensional Lagrangian is

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma \cdot D \psi. \quad (24)$$

(*) In 6-dimensions a spin 1/2 field λ has 8 components but in order to match the number of vector and spinor degrees of freedom a Weyl condition has to be imposed. This reduces λ to a 4-dimensional Dirac spinor ψ and allows the use of the familiar 4x4 γ matrices. For a complete discussion one can consult Ref.(7).

As usual the color summation is understood.

Our treatment will be conveniently formulated in the basis

$$A_{R,L} = A_0 \pm A_5, \quad (25)$$

$$A_{\pm}^{(\Delta)} = A_{\Delta} \pm i A_{\Delta+1}, \quad \Delta = 1, 3.$$

We shall also define

$$F_{R-L+}^{(\Delta)} = [D_R, D_+^{(\Delta)}]$$

$$F_{+-}^{(\Delta)} = [D_+^{(\Delta)}, D_-^{(\Delta)}] \quad (26)$$

$$F_{++} = [D_+^{(1)}, D_+^{(3)}]$$

(D_μ is the covariant derivative $D_\mu = \partial_\mu - i t^a A_\mu^a \equiv \partial_\mu + A_\mu$).

After a repeated and clever use of partial integration one can cast the bosonic part $\mathcal{L}_B$ in the form

$$\mathcal{L}_B = \mathcal{L}_0 - \frac{1}{2} F_{++} F_{--} \quad (27)$$

where^(*)

$$\mathcal{L}_0 = \frac{1}{8} (F_{RL} + F_{+-}^{(1)} + F_{+-}^{(3)})^2 + \frac{1}{2} \sum_{\Delta} F_{R-}^{(\Delta)} F_{L+}^{(\Delta)}. \quad (28)$$

(*) This result can be generalized to a Yang-Mills Lagrangian in any even number of dimensions, (G. Veneziano, private communication).

We now parallel the 4-dimensional treatment given in Sect. 2A. Let us choose the light cone gauge $A_L=0$ and let us introduce the auxiliary fields M_+ and M_- , by modifying both the Lagrangian and the functional measure in such a way that Z remains unchanged:

$$Z = \int \pi_n dA_+^{(n)} dA_-^{(n)} dA_R dM_+ dM_- . \quad (29)$$

$$(\text{Det } \partial_L)^3 \exp\left\{ i \int d^6x \hat{\mathcal{L}}_B \right\} Z_F$$

with

$$\begin{aligned} \hat{\mathcal{L}}_B &= \mathcal{L}_B + \frac{1}{2} (\partial_L M_+ - F_{++}) (\partial_L M_- - F_{--}) = \\ &= \mathcal{L}_0 + \frac{1}{2} \partial_L M_+ (\partial_L M_- - F_{--}) - \frac{1}{2} \partial_L M_- F_{++} . \end{aligned} \quad (30)$$

This then leads, apart from surface terms, to the final result

$$\hat{\mathcal{L}}_B = \frac{1}{2} N_3^2 + \frac{1}{2} \sum_a N_+^{(a)} N_-^{(a)} + \frac{1}{2} m_+ m_- \quad (31)$$

where the Nicolai variables are

$$\begin{aligned} N_3 &= \frac{1}{2} (F_{RL} + F_{+-}^{(1)} + F_{+-}^{(3)}) \\ N_+^{(a)} &= \partial_L A_+^{(a)} , \\ N_-^{(a)} &= F_{R-}^{(a)} + \epsilon_{ab} D_+^{(b)} M_- , \quad (\epsilon_{13}=1) \\ m_+ &= \partial_L M_+ , \quad m_- = \partial_L M_- - F_{--} . \end{aligned} \quad (32)$$

Again it is easy to check that the Jacobian of the transformation between the Nicolai variables $N_+^{(s)}$, N_3 , m_+ and the fields $A_+^{(s)}$, A_R , M_+ is matched by the fermionic determinant.

We are thus allowed to use for the functional integral the purely bosonic expression

$$Z = \int \pi_+ dN_+^{(s)} dN_-^{(s)} dN_3 dm_+ dm_- \cdot \exp\{i \int d^6x \hat{\mathcal{L}}_B\}. \quad (33)$$

4. The results of this letter represent a further step in the programme of extending the Nicolai procedure to field theories of physical interest. In our analysis the unique role of the light cone gauge emerges in an even cleaner way.

There are several, well known, reasons make which such configuration so attractive.

The cancellations which lead to a completely finite theory for the $N=4$ gauge supersymmetry are actually achieved in this gauge, with a suitable prescription to handle the singularities of propagators⁽¹⁾.

The light cone i.e. the infinite momentum limit has also been recognized since a long time to constitute an optimal frame of reference to obtain a simple picture of physical phenomena⁽⁸⁾.

This special status of the light like gauge is indeed confirmed by the existence of stochastic identities which can lead to a very detailed information about the structure of the vacuum.

It is natural, to inquire whether similar results, namely the validity of stochastic identities, can be established for other supersymmetric theories.

A case which seems to defy all attempt is the Wess-Zumino model in 4 dimensions which consists of a scalar-spinor supermultiplet. As a consequence of the fact that the spin 1/2 fields are Majorana spinors, the functional integration of the fermionic sector does not lead to a determinant but to a Pfaffian.

A similar difficulty is expected to arise in the particularly interesting case of the $N=4$ gauge supersymmetric theory. We hope to be able to come back to those problems.

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