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LONGITUDINAL AND TRANSVERSE IMPEDANCES
OF BELLOWS IN THE LOW FREQUENCY RANGE

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1. INTRODUCTION

The stability of longitudinal and transverse oscillations of a bunch in high-current accelerators depends on the corresponding impedances of the bunch surroundings. For different technological reasons one is forced to design and build a non-smooth vacuum pipe which has cavities, bellows, transitions, flanges, etc. All such non-uniformities are sources of field disruptions, energy losses, and wake fields. The wake field in turn influences the bunch motion and may result in an instability, thus limiting the bunch current. Hence, it is very important to have a computational tool which may be used for reliable evaluation of the wake fields and/or the impedances (which are the Fourier transforms of the corresponding wake fields¹) for different geometries of the vacuum pipe.

The present work considers one important case of such a geometry -- the axially symmetric bellows, or, more generally, the axially symmetric and longitudinally periodic variations of a vacuum pipe.

Axially symmetric and longitudinally periodic structures were analytically treated before in two different ways. Firstly, there are some papers²⁻⁴ which treat the corrugations of a pipe as a perturbation of a smooth pipe of constant radius. The amplitude of the corrugations is assumed to be small compared to the pipe radius. The impedances are found to depend quadratically on the depth of the corrugations. The reason for this dependence is the absence of a synchronous field component moving with the bunch velocity in first order of perturbation theory. Secondly, there are other papers⁵⁻⁷ which solve the problem by the method of field matching. There the cross-section of the pipe wall is assumed to have a simple rectangular geometry. The cross-section is split into simple regions in which the solution of the Maxwell equations can be found in the form of infinite series with unknown coefficients. The coefficients of these series are found by matching the field components to each other on the imaginary lines dividing the regions. For values of the depth a little larger than the period, the transverse impedance is found to depend linearly on the depth of the non-uniformity of the wall.

The problem was also addressed by purely numerical methods. The most

suitable program at present for this aim is TBCI⁸. It has been extensively used to evaluate the wake fields and the loss factors⁹ for different geometries including bellows. In the latter case the bellows cross-section is sometimes assumed to have a rectangular shape with transverse depth d and longitudinal period g . To keep the computational time reasonable one has to take the mesh size not too small. In the region of parameters studied the transverse impedance was found to depend almost linearly on the depth d , but the straight line does not pass through zero for $d \rightarrow 0$.

Thus we see that different dependences for impedances are predicted by different approaches. The present work has been undertaken with the aim of resolving this discrepancy.

The method which is applied here is an extension of a method developed in a paper by Krinsky and Gluckstern¹⁰. In that work, only axially symmetric modes ($m = 0$) have been considered and correspondingly only the longitudinal ($m = 0$) impedance of the corrugated pipe is found. The direct extension of this method to the dipole mode ($m = 1$) fails, since a divergence appears in the expansion coefficients in the most interesting case of ultra-relativistic particles ($\gamma \rightarrow \infty$). This difficulty is overcome by employing hybrid modes for the synchronous field components. To obtain the impedances of bellows one needs to find the electromagnetic field excited by a bunch travelling inside a vacuum pipe. Mathematically, the field is described by the solution of the inhomogeneous Maxwell equation which has to satisfy certain boundary conditions. For a non-resistive metallic surface these conditions are

$$E_{\parallel} \Big|_S = 0 \quad (1.1)$$

$$H_{\perp} \Big|_S = 0 \quad (1.2)$$

where $E_{\parallel}$ and $H_{\perp}$ are the components of the electric and magnetic fields, which are locally parallel and perpendicular to the surface S , respectively.

Section 2 contains the general solution of the Maxwell equations in free space (without boundary, apart from the condition on the pipe axis). This solution has two infinite sets of arbitrary coefficients. These coefficients are used to fulfil the boundary conditions on the pipe wall (see Section 3). The results of the numerical solution of the equations in the "low-frequency range" are presented in Section 4, where one can also find a comparison with the purely numerical results obtained with the help of TBCI. It can be seen from the figures that a reasonable agreement is achieved both with the perturbation approach and with the field-matching method. Discussion of the complementary nature of both approaches, in Section 5, concludes the paper.

2. GENERAL SOLUTION OF THE MAXWELL EQUATIONS IN FREE SPACE

The general solution of the Maxwell equations, as of any linear differential equation, may be represented as the sum of a particular solution of the inhomogeneous equations (driven solution) and a general solution of the homogeneous equations which contains arbitrary coefficients (free solution).

The Maxwell equations will be solved first without taking the boundary conditions into account. In the subsequent sections the coefficients of the solution will be chosen in such a way as to satisfy the boundary conditions.

Cylindrical coordinates r , θ , z are used throughout this paper. The velocity of particles in the bunch v is assumed to be constant. Thus, we neglect the influence of the wake field on the particle motion. Figure 1 represents the geometry considered and the notation used.

2.1 Charge and current densities

Following Ref. 4 we represent the m -th Fourier component of the charge and current densities of an infinitesimally thin ring of radius a moving along the longitudinal axis z in the following form:

$$\varrho_m = \frac{qN}{\pi a v} \delta(t - \frac{z}{v}) \delta(r-a) \frac{\cos m\theta}{1 + \delta_{m0}}, \quad (2.1)$$

$$\vec{j}_m = \vec{e}_r 0 + \vec{e}_\theta 0 + \vec{e}_z \varrho_m v, \quad (2.2)$$

where N is the number of particles of charge q in the bunch, v is their velocity, and $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_z$ are the unit vectors of the cylindrical-coordinate system r, θ, z ; δ_{m0} is the Kronecker symbol, which is equal to one for $m = 0$ and 0 for $m \neq 0$; $\delta(t - z/v)$ and $\delta(r - a)$ are the usual Dirac δ -functions. The Fourier components of ϱ_m and $\vec{j}_m$ at frequency ω are

$$\tilde{\varrho}_m = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t} \varrho_m = \frac{qN}{\pi a v} \delta(r-a) e^{i\omega z/v} \frac{\cos m\theta}{1 + \delta_{m0}} \quad (2.3)$$

$$\tilde{j}_{mz} = \tilde{\varrho}_m v, \quad \tilde{j}_{mr} = \tilde{j}_{m\theta} = 0. \quad (2.4)$$

2.2 Driven solution

One simple way to obtain the fields produced by the charge (2.3) and current (2.4) densities is to use the integral formulae¹¹ for scalar $\tilde{\phi}_m$ and vector $\vec{A}_m$ potentials:

$$\tilde{\phi}_m(\vec{r}) = \frac{qN}{2\pi^2 a v (1 + \delta_{m,0})} \int d\vec{r}_0 \frac{\exp(ik|\vec{r} - \vec{r}_0| + ikz_0/\beta)}{|\vec{r} - \vec{r}_0|} \delta(r_0 - a) \cos m\theta_0 \quad (2.5)$$

$$\tilde{A}_{mz} = \tilde{\phi}_m \beta, \quad \tilde{A}_{mr} = \tilde{A}_{m\theta} = 0, \quad (2.6)$$

where $k = \omega/c$, $\beta = v/c$ and c is the light velocity. In cylindrical coordinates

$$|\vec{r} - \vec{r}_0| = \sqrt{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) + (z - z_0)^2}. \quad (2.7)$$

Performing the integrations in Eqs. (2.5) and (2.6) one finds

$$\tilde{\phi}_m = \frac{d_m}{\beta} e^{ikz/\beta} F_m(r) \cos m\theta \quad (2.8)$$

$$\tilde{A}_{mz} = \tilde{\phi}_m \beta, \quad (2.9)$$

where

$$d_m = 2qN/\pi c(1 + \delta_{m0}) \quad (2.10)$$

and

$$F_m(r) = \begin{cases} I_m(kr/\gamma\beta)K_m(ka/\gamma\beta), & r < a \\ I_m(ka/\gamma\beta)K_m(kr/\gamma\beta), & r > a \end{cases} \quad (2.11)$$

is Green's function for cylindrical geometry. In Eq. (2.11) and below γ is the Lorentz factor

$$\gamma = (1 - \beta^2)^{-1/2}, \quad (2.12)$$

and $I_m(x)$ and $K_m(x)$ are m -th order modified Bessel functions of the first and second kind, respectively.

The Fourier components of the fields can be obtained from the potentials $\tilde{\phi}_m$ and $\tilde{A}_m$ by differentiation:

$$\vec{\tilde{E}}_m = -\nabla \tilde{\phi}_m + ik\vec{\tilde{A}}_m \quad (2.13)$$

$$\vec{\tilde{H}}_m = \nabla \times \vec{\tilde{A}}_m. \quad (2.14)$$

One obtains

$$\tilde{E}_{mr} = -\frac{d_m}{\beta} e^{ikz/\beta} \frac{dF_m}{dr} \cos m\theta \quad (2.15)$$

$$\tilde{E}_{m\theta} = m \frac{d_m}{\beta} e^{ikz/\beta} \frac{F_m}{r} \sin m\theta \quad (2.16)$$

$$\tilde{E}_{mz} = -i \frac{k}{\gamma^2 \beta^2} d_m e^{ikz/\beta} F_m \cos m\theta \quad (2.17)$$

$$\tilde{H}_{mr} = -m d_m e^{ikz/\beta} \frac{F_m}{r} \sin m\theta \quad (2.18)$$

$$\tilde{H}_{m\theta} = -d_m e^{ikz/\beta} \frac{dF_m}{dr} \cos m\theta \quad (2.19)$$

$$\tilde{H}_{mz} = 0 \quad (2.20)$$

2.3 Solution of the homogeneous equations

To obtain a solution of the homogeneous Maxwell equations it is useful to introduce the Hertz vectors $\vec{\Pi}$ and $\vec{\Pi}^*$, which satisfy homogeneous wave equations. For the Fourier components $\vec{\Pi}$ and $\vec{\Pi}^*$ they look as follows:

$$\nabla^2 \vec{\Pi} + k^2 \vec{\Pi} = 0 \quad (2.21)$$

$$\nabla^2 \vec{\Pi}^* + k^2 \vec{\Pi}^* = 0 \quad (2.22)$$

Having in mind the periodicity of the boundary and the z -dependence of the driven solution, we will search for a solution of Eqs. (2.21) and (2.22) in the following form:

$$\tilde{\Pi} = e^{ikz/\beta} \sum_{p=-\infty}^{\infty} \tilde{\Pi}_{pm} e^{2\pi ipz/L} \quad (2.23)$$

and a similar expansion for $\tilde{\Pi}^*$. Substitution of Eq. (2.23) into Eq. (2.21) yields:

$$\Pi_{pmr} = 0 \quad (2.24)$$

$$\Pi_{pm\theta} = 0 \quad (2.25)$$

$$\frac{\partial^2 \Pi_{pmz}}{\partial r^2} + \frac{1}{r} \frac{\partial \Pi_{pmz}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Pi_{pmz}}{\partial \theta^2} - \Lambda_p^2 \Pi_{pmz} = 0. \quad (2.26)$$

where

$$\Lambda_p^2 = \frac{k^2}{\gamma^2 \beta^2} + (2\pi p/L)^2 + 2 \frac{k}{\beta} (2\pi p/L). \quad (2.27)$$

The solution of Eq. (2.26), which is finite on the axis $r = 0$, is well known¹¹:

$$\Pi_{pmz} = \Lambda_p I_m(\Lambda_p r) \cos m\theta. \quad (2.28)$$

Similarly, we get for $\tilde{\Pi}^*$:

$$\Pi_{pmr}^* = 0 \quad (2.29)$$

$$\Pi_{pm\theta}^* = 0 \quad (2.30)$$

$$\Pi_{pmz}^* = \Lambda_p I_m(\Lambda_p r) \sin m\theta. \quad (2.31)$$

The fields can be obtained from $\tilde{\vec{\Pi}}$ by means of the formulae:

$$\tilde{\vec{E}}^{(H)} = \nabla \times \nabla \times \tilde{\vec{\Pi}} \quad (2.32)$$

$$\tilde{\vec{H}}^{(H)} = -ik \nabla \times \tilde{\vec{\Pi}} \quad (2.33)$$

(TM modes); and similarly from $\tilde{\vec{\Pi}}^*$:

$$\tilde{\vec{E}}^{(E)} = -ik \nabla \times \tilde{\vec{\Pi}}^* \quad (2.34)$$

$$\tilde{\vec{H}}^{(E)} = \nabla \times \nabla \times \tilde{\vec{\Pi}}^* \quad (2.35)$$

(TE modes). Introducing the notations $E_p = \exp(2\pi ipz/L)$ and $I_m = I_m(\Lambda_p r)$ we get:

i) TM modes

$$\tilde{E}_{mr}^{(H)} = \sum_p b_p^m \left(\frac{2\pi p}{L} + \frac{k}{\beta} \right) \frac{dI_m}{dr} \cos m\theta E_p e^{ikz/\beta} \quad (2.36)$$

$$\tilde{E}_{m\theta}^{(H)} = - \sum_p b_p^m \left(\frac{2\pi p}{L} + \frac{k}{\beta} \right) \frac{I_m}{r} \sin m\theta E_p e^{ikz/\beta} \quad (2.37)$$

$$\tilde{E}_{mz}^{(H)} = i \sum_p b_p^m \Lambda_p^2 I_m \cos m\theta E_p e^{ikz/\beta} \quad (2.38)$$

$$\tilde{H}_{mr}^{(H)} = \sum_p b_p^m k \frac{I_m}{r} \sin m\theta E_p e^{ikz/\beta} \quad (2.39)$$

$$\tilde{H}_{m\theta}^{(H)} = \sum_p b_p^m k \frac{dI_m}{dr} \cos m\theta E_p e^{ikz/\beta} \quad (2.40)$$

$$\tilde{H}_{mz}^{(H)} = 0. \quad (2.41)$$

ii) TE modes

$$\tilde{E}_{mr}^{(E)} = - \sum_p g_p^m k m \frac{I_m}{r} \cos m\theta E_p e^{ikz/\beta} \quad (2.42)$$

$$\tilde{E}_{m\theta}^{(E)} = \sum_p g_p^m k \frac{dI_m}{dr} \sin m\theta E_p e^{ikz/\beta} \quad (2.43)$$

$$\tilde{E}_{mz}^{(E)} = 0. \quad (2.44)$$

$$\tilde{H}_{mr}^{(E)} = - \sum_p g_p^m \left(\frac{2\pi p}{L} + \frac{k}{\beta} \right) \frac{dI_m}{dr} \sin m\theta E_p e^{ikz/\beta} \quad (2.45)$$

$$\tilde{H}_{m\theta}^{(E)} = - \sum_p g_p^m \left(\frac{2\pi p}{L} + \frac{k}{\beta} \right) \frac{I_m}{r} \cos m\theta E_p e^{ikz/\beta} \quad (2.46)$$

$$\tilde{H}_{mz}^{(E)} = - i \sum_p g_p^m \Lambda_p^2 I_m \sin m\theta E_p e^{ikz/\beta}. \quad (2.47)$$

The coefficients b_p^m and g_p^m ($p = -\infty, \dots, \infty$) are the arbitrary constants to be found from the boundary conditions. Notation $\sum_p$ means a summation over all integer p from $-\infty$ to $+\infty$.

2.4 General solution

The general solution for each value of m can be obtained by summing up the driven solution [Eqs. (2.15) to (2.20)] and the free solutions for TM and TE modes [Eqs. (2.36) to (2.47)]. As we will see later, the m -th longitudinal impedance is proportional to the coefficient b_0^m in

the expression for $\tilde{E}_{mz}$. It is convenient to redefine the coefficients in such a way as to make all b_p^m and g_p^m vanish for the smooth pipe of a constant radius $R > a$. Then the impedance found will originate from corrugations only. It is achieved by redefining the function F in such a way that it vanishes at $r = R$:

$$\hat{F}_m(r) = \begin{cases} I_m\left(\frac{kr}{\gamma\beta}\right) \left[K_m\left(\frac{ka}{\gamma\beta}\right) - \frac{K_m(kR/\gamma\beta)}{I_m(kR/\gamma\beta)} I_m\left(\frac{ka}{\gamma\beta}\right) \right], & r < a \\ I_m\left(\frac{ka}{\gamma\beta}\right) \left[K_m\left(\frac{kr}{\gamma\beta}\right) - \frac{K_m(kR/\gamma\beta)}{I_m(kR/\gamma\beta)} I_m\left(\frac{kr}{\gamma\beta}\right) \right], & r > a \end{cases} \quad (2.48)$$

With the driven solution thus defined, the general solution looks as follows:

$$\tilde{E}_{mr} = d_m e^{ikz/\beta} \cos m\theta \left[-\frac{d\hat{F}_m}{\beta dr} + \sum_p E_p \frac{[B_p^m(\frac{2\pi p}{L} + \frac{k}{\beta}) \frac{dI_m(\Lambda_p r)}{dr} - G_p^m k \frac{I_m(\Lambda_p r)}{r}]}{\Lambda_p^2 I_m(\Lambda_p R)} \right] \quad (2.49)$$

$$\tilde{E}_{m\theta} = d_m e^{ikz/\beta} \sin m\theta \left[\frac{m\hat{F}_m}{\beta r} + \sum_p E_p \frac{[-B_p^m(\frac{2\pi p}{L} + \frac{k}{\beta}) \frac{I_m(\Lambda_p r)}{r} + G_p^m k \frac{dI_m(\Lambda_p r)}{dr}]}{\Lambda_p^2 I_m(\Lambda_p R)} \right] \quad (2.50)$$

$$\tilde{E}_{mz} = id_m e^{ikz/\beta} \cos m\theta \left[-\frac{k\hat{F}_m}{\gamma^2 \beta^2} + \sum_p E_p B_p^m \frac{I_m(\Lambda_p r)}{I_m(\Lambda_p R)} \right] \quad (2.51)$$

$$\tilde{H}_{mr} = d_m e^{ikz/\beta} \sin m\theta \left[-\frac{m\hat{F}_m}{r} + \sum_p E_p \frac{[B_p^m k \frac{I_m(\Lambda_p r)}{r} - G_p^m(\frac{2\pi p}{L} + \frac{k}{\beta}) \frac{dI_m(\Lambda_p r)}{dr}]}{\Lambda_p^2 I_m(\Lambda_p R)} \right] \quad (2.52)$$

$$\tilde{H}_{m\theta} = d_m e^{ikz/\beta} \cos m\theta \left[-\frac{d\hat{F}_m}{dr} + \sum_p E_p \frac{[B_p^m k \frac{dI_m(\Lambda_p r)}{dr} - G_p^m \left(\frac{2\pi p}{L} + \frac{k}{\beta} \right) \frac{mI_m(\Lambda_p r)}{r}]}{\Lambda_p^2 I_m(\Lambda_p R)} \right] \quad (2.53)$$

$$\tilde{H}_{mz} = -id_m e^{ikz/\beta} \sin m\theta \sum_p E_p G_p^m \frac{I_m(\Lambda_p r)}{I_m(\Lambda_p R)} \quad (2.54)$$

The unknown coefficients B_p^m and G_p^m ($p = -\infty, \dots, \infty$), defined in this way, are dimensionless.

2.5 Solution in the ultra-relativistic limit

We can simplify the general solution by taking the ultra-relativistic limit $\gamma \rightarrow \infty$. We will do this separately for the cases $m = 0$ and $m = 1$. Since we want them to match to the boundary we will use that branch of the solution which is valid for $r > a$.

It is convenient here to introduce the following dimensionless variables:

$$w = r/R \quad (2.55)$$

$$u = 2\pi z/L \quad (2.56)$$

$$\eta = 2\pi R/L \quad (2.57)$$

$$\kappa = k \cdot R \quad (2.58)$$

$$\alpha_p = \Lambda_p R = (\eta^2 p^2 + 2\kappa\eta p)^{1/2}. \quad (2.59)$$

i) $m = 0$. Only TM modes are excited in this case. There is no trouble in going to the limit $\gamma \rightarrow \infty$. Indeed, although $\alpha_p = 0$ for $p = 0$, the ratio $I_1(\alpha_p w)/\alpha_p I_0(\alpha_p)$ is finite and well defined. Expanding the functions $I_0(x)$ and $K_0(x)$ in the limit $x \rightarrow 0$, we find

$$\tilde{E}_{0r} = \frac{d_0}{R} e^{ikz} \left[\frac{1}{w} + B_0^0 \frac{\kappa}{2} w + \sum_p' B_p^0 e^{ipu} \frac{(\eta p + \kappa) I_1(\alpha_p w)}{\alpha_p I_0(\alpha_p)} \right] \quad (2.60)$$

$$\tilde{E}_{0\theta} = 0 \quad (2.61)$$

$$\tilde{E}_{0z} = i \frac{d_0}{R} e^{ikz} \left[B_0^0 + \sum_p' B_p^0 e^{ipu} \frac{I_0(\alpha_p w)}{I_0(\alpha_p)} \right] \quad (2.62)$$

$$\tilde{H}_{0r} = 0 \quad (2.63)$$

$$\tilde{H}_{0\theta} = \frac{d_0}{R} e^{ikz} \left[\frac{1}{w} + B_0^0 \frac{\kappa}{2} w + \sum_p' B_p^0 e^{ipu} \frac{\kappa I_1(\alpha_p w)}{\alpha_p I_0(\alpha_p)} \right] \quad (2.64)$$

$$\tilde{H}_{0z} = 0. \quad (2.65)$$

Here and below the symbol $\sum_p'$ means summation over all the negative and positive values of p excluding zero; B_p^0 ($p = -\infty, \dots, \infty$) is the infinite set of constants which will be determined from the boundary conditions.

ii) $m = 1$. Both TM and TE modes are excited in this case. There is a minor difficulty when one tries to go to the ultra-relativistic limit in this case. Indeed, the term $p = 0$ diverges in the limit $\gamma \rightarrow \infty$. To get around this difficulty we will use a hybrid mode for $p = 0$ by redefining the coefficient

$$B_0^1 \rightarrow B_0^1 - G_0^1 \cdot \alpha_0^2 \quad (2.66)$$

With the help of the identity

$$dI_1/dx = I_2 + I_1/x \quad (2.67)$$

it is easy to show that the divergent terms disappear. Expanding the functions $I_1(x)$ and $K_1(x)$ in the limit $x \rightarrow 0$ we obtain :

$$\begin{aligned} \tilde{E}_{1r} = \frac{d_0 a}{R^2} e^{ikz} \cos \theta & \left[1 + \frac{1}{w^2} + B_0^1 \left(\frac{1}{2\kappa} + \frac{\kappa}{4} w^2 \right) + G_0^1 \kappa \right. \\ & \left. + \sum_p' e^{ipu} \frac{[B_p^1 (\eta p + \kappa) dI_1(\alpha_p w)/dw - G_p^1 \kappa I_1(\alpha_p w)/w]}{\alpha_p^2 I_1(\alpha_p)} \right] \end{aligned} \quad (2.68)$$

$$\begin{aligned} \tilde{E}_{1\theta} = \frac{d_0 a}{R^2} e^{ikz} \sin \theta & \left[-1 + \frac{1}{w^2} - B_0^1 \left(\frac{1}{2\kappa} - \frac{\kappa}{4} w^2 \right) - G_0^1 \kappa \right. \\ & \left. - \sum_p' e^{ipu} \frac{[B_p^1 (\eta p + \kappa) I_1(\alpha_p w)/w - G_p^1 \kappa dI_1(\alpha_p w)/dw]}{\alpha_p^2 I_1(\alpha_p)} \right] \end{aligned} \quad (2.69)$$

$$\tilde{E}_{1z} = i \frac{d_0 a}{R^2} e^{ikz} \cos \theta \left[B_0^1 w + \sum_p' e^{ipu} B_p^1 \frac{I_1(\alpha_p w)}{I_1(\alpha_p)} \right] \quad (2.70)$$

$$\begin{aligned} \tilde{H}_{1r} = \frac{d_0 a}{R^2} e^{ikz} \sin \theta & \left[1 - \frac{1}{w^2} - B_0^1 \left(\frac{1}{2\kappa} + \frac{\kappa}{4} w^2 \right) + G_0^1 \kappa \right. \\ & \left. + \sum_p' e^{ipu} \frac{[B_p^1 \kappa I_1(\alpha_p w)/w - G_p^1 (\eta p + \kappa) dI_1(\alpha_p w)/dw]}{\alpha_p^2 I_1(\alpha_p)} \right] \end{aligned} \quad (2.71)$$

$$\tilde{H}_{1\theta} = \frac{d_0 a}{R^2} e^{ikz} \cos \theta \left[1 + \frac{1}{w^2} - B_0^1 \left(\frac{1}{2\kappa} - \frac{\kappa}{4} w^2 \right) + G_0^1 \kappa + \sum_p' e^{ipu} \frac{[B_p^1 \kappa dI_1(\alpha_p w)/dw - G_p^1 (\eta p + \kappa) I_1(\alpha_p w)/w]}{\alpha_p^2 I_1(\alpha_p)} \right] \quad (2.72)$$

$$\tilde{H}_{1z} = -i \frac{d_0 a}{R^2} e^{ikz} \sin \theta \left[B_0^1 w + \sum_p' e^{ipu} G_p^1 \frac{I_1(\alpha_p w)}{I_1(\alpha_p)} \right] \quad (2.73)$$

B_p^1 and G_p^1 ($p = -\infty, \dots, \infty$) are two infinite sets of constants which will be used in the next section to satisfy boundary conditions.

3. BOUNDARY CONDITIONS

Let us assume that the boundary in the plane $\theta = \text{const}$ is described by a curve

$$r_b = r_b(z) \quad (3.1)$$

or in the dimensionless units [Eqs. (2.55) to (2.56)]

$$w = w(u). \quad (3.2)$$

The function $w(u)$ is understood to be smooth and single-valued, and to have period 2π . Otherwise it may be quite an arbitrary function.

The boundary conditions (1.1) and (1.2) for our case give in general three equations

$$E_z(w) = - \frac{dw}{du} E_r(w) \quad (3.3)$$

$$H_r(w) = \frac{dw}{du} H_z(w) \quad (3.4)$$

$$E_{\theta}(w) = 0. \quad (3.5)$$

The first two equations are Eqs. (1.1) and (1.2) for the plane $\theta = \text{const.}$ The third one is Eq. (1.1) for the plane $z = \text{const.}$ Owing to the properties of the Maxwell equations, any one of these equations follows from the two others. These features may be used as a check for the correctness of the solution.

Consider, for example, the pair (3.3), (3.4). These equations should be satisfied identically for any value of u in the interval $0 < u < 2\pi$. Owing to the periodicity of the boundary function $w(u)$, Eqs. (3.3) to (3.5) are also periodic at least with the same period. Hence, we can find an equivalent set of algebraic equations by expanding both sides of each equation into the Fourier series.

In this way, we obtain two infinite sets of equations for two infinite sets of coefficients B_p and G_p . The solution of the Maxwell equations satisfying the boundary conditions are thus reduced to the solution of two sets of linear algebraic equations.

3.1 Longitudinal and transverse impedances

Suppose for a moment that the coefficients B_p and G_p are known. The longitudinal Z'_m and the transverse Z_m impedances per period of the structure can be defined then as integrals over z of the synchronous component of the forces

$$\tilde{E}_{mz} \text{ and } \tilde{E}_{mr} - \beta \tilde{H}_{m\theta},$$

respectively. In the limit $\gamma \rightarrow \infty$:

$$Z'_m = - \frac{2\pi}{qN} \int_0^L \tilde{E}_{mz} e^{-ikz} dz \quad (3.6)$$

$$Z_m = -i \frac{2\pi}{qN\delta} \int_0^L (\tilde{E}_{mr} - \tilde{H}_{m\theta}) e^{-ikz} dz, \quad (3.7)$$

where δ is the displacement of the bunch from the pipe axis. In the ultra-relativistic limit one should use Eq. (2.62) for the $m = 0$ case and Eqs. (2.68) and (2.73) for the $m = 1$ case. Equations (3.6) and (3.7) agree with the usual definitions for the impedances⁹.

i) $m = 0$. Only longitudinal impedance is not zero in this case. The integral over the infinite series vanishes and only the B_0^0 term remains:

$$Z'_0 = -i \frac{2L}{cR} B_0^0. \quad (3.8)$$

ii) $m = 1$. Using now Eqs. (2.68) and (2.73) we find:

$$Z'_1 = -i \frac{2L}{cR^3} B_0^1 r \delta \quad (3.9)$$

$$Z_1 = -i \frac{2L}{cR^3} \frac{B_0^1}{k}. \quad (3.10)$$

From Eqs. (3.9) and (3.10) follow the well-known relation⁹ between Z'_1 and Z_1 :

$$Z_1 = \frac{c}{R^2} \frac{Z'_1}{\omega} \quad (3.11)$$

Conditions (3.3) to (3.5) will now be imposed on the general solution obtained in the previous section to find equations for the unknown coefficients.

3.2 Equations for the coefficients B_p^0 ($m = 0$)

For $m = 0$ only TM modes exist and Eqs. (3.4) and (3.5) are fulfilled identically. The remaining equation (3.3) gives the following condition:

$$B_0^0 (1 - i\frac{\kappa}{2} \eta w' w) + \sum_p B_p^0 e^{ipu} \left[\frac{\alpha_p I_0(\alpha_p w) - i\eta(\eta p + \kappa) w' I_1(\alpha_p w)}{\alpha_p I_0(\alpha_p)} \right] = i\eta \frac{w'}{w} \quad (3.12)$$

where $w' = dw/du$. Multiplying both sides by $\exp(-inu)$ and averaging over u one gets:

$$\sum_{p=-\infty}^{\infty} M_{np}^0 B_p^0 = N_n^0, \quad n = -\infty, \dots, \infty \quad (3.13)$$

where

$$M_{np}^0 = \begin{cases} \langle e^{-inu} (1 - i\frac{\kappa\eta}{2} w' w) \rangle, & p = 0 \\ \frac{\eta^2 p n + \eta \kappa (p+n)}{\alpha_p^2} \langle e^{-i(n-p)u} I_0(\alpha_p w) / I_0(\alpha_p) \rangle, & p \neq 0 \end{cases} \quad (3.14)$$

$$N_n^0 = i\eta \langle e^{-inu} w' / w \rangle. \quad (3.15)$$

The brackets $\langle f \rangle$ here are used to define the averaging of f over u

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(u) du. \quad (3.16)$$

The infinite set of linear algebraic equations (3.12) for the coefficients B_p^0 was already obtained in Ref. 10. We only need the coefficient B_0^0 , which is proportional to the longitudinal impedance of the pipe corrugations.

It is instructive to see how Eqs. (3.12) produce the results of the perturbation theory²⁻⁴. Following these papers we represent the boundary curve by the following expansion:

$$w = 1 + \Delta = 1 + \sum_{q=-\infty}^{\infty} c_q e^{iqu} \quad (c_0=0). \quad (3.17)$$

The coefficients C_q are assumed to be small, or $|\Delta| \ll 1$. To the first order in the value Δ :

$$N_n^0 = -\eta n \langle e^{-inu} \ln w \rangle \simeq -\eta n C_n. \quad (3.18)$$

Assume now that not only is $|\Delta|$ small, but in addition

$$|\alpha_p \Delta| \ll 1. \quad (3.19)$$

Then to the first order in the parameter $|\alpha_p \Delta|$:

$$\frac{I_0(\alpha_p w)}{I_0(\alpha_p)} \simeq 1 + \alpha_p \Delta \frac{I_1(\alpha_p)}{I_0(\alpha_p)} \quad (3.20)$$

The matrix elements M_{np}^0 can now be evaluated as follows:

$$M_{np}^0 \simeq \left[\delta_{n,p} + \frac{\alpha_p I_1(\alpha_p)}{I_0(\alpha_p)} C_{n-p} \right] \frac{\eta^2 p n + \eta \kappa (p+n)}{\alpha_p^2}, \quad (p \neq 0) \quad (3.21)$$

With these matrix elements each n -th row of Eqs. (3.12) for $n \neq 0$ yields to the same order in $|\Delta|$:

$$B_n^0 \approx \frac{N_n^0}{M_{n,n}^0} \approx N_n^0, \quad n \neq 0. \quad (3.22)$$

The $n = 0$ row gives:

$$B_0^0 = - \sum_p M_{0p}^0 N_p^0 \quad (3.23)$$

or to the first non-vanishing order in $|\Delta|$

$$B_0^0 = \eta^2 \alpha \sum_p p^2 |C_p|^2 \frac{I_1(\alpha_p)}{\alpha_p I_0(\alpha_p)}. \quad (3.24)$$

Substituting this expression into Eq. (3.8), the answer agrees with the one obtained in Refs. 2-4. Hence, the smallness of both $|\Delta|$ and $|\alpha_p \Delta|$ are required for the validity of the perturbation method and its results. Since $|\alpha_p|$ grows with p , the inequality (3.19) may not be fulfilled for higher harmonic numbers in expansion (3.20), even if it holds for lower ones. Thus, the perturbation method is not suitable for a square wall configuration, even for very small depths of corrugations.

As one can see from Eq. (3.24), in the broad frequency range where α_p is real, B_0^0 is also real and antisymmetric in frequency. The same is true also for B_0^0 as a solution of Eqs. (3.12). Consequently, the longitudinal impedance Z_0' is purely imaginary and antisymmetric in frequency.

3.3 Equations for the coefficients B_p^1 and G_p^1 ($m = 1$)

In the case where $m = 1$ all three boundary conditions (3.3) to (3.5) should be satisfied. Substituting the fields (2.68) to (2.73) into them, multiplying both sides of all equations by $\exp(-inu)$ and then averaging over u one obtains:

$$\sum_p (M_{np}^1 B_p^1 + L_{np}^1 G_p^1) = N_n^1 \quad (3.25)$$

$$\sum_p (\bar{M}_{np}^1 B_p^1 + \bar{L}_{np}^1 G_p^1) = \bar{N}_n^1 \quad (3.26)$$

$$\sum_p (\bar{\bar{M}}_{np}^1 B_p^1 + \bar{\bar{L}}_{np}^1 G_p^1) = \bar{\bar{N}}_n^1, \quad (3.27)$$

where $n = -\infty, \dots, \infty$ and

$$M_{np}^1 = \begin{cases} \langle e^{-inu} (w - i \frac{\eta}{2\kappa} w' - i \frac{\eta\kappa}{4} w' w^2) \rangle, & p = 0 \\ \frac{\eta^2 p n + \eta\kappa(p+n)}{\alpha_p^2} \langle e^{-i(n-p)u} \frac{I_1(\alpha_p w)}{I_1(\alpha_p)} \rangle, & p \neq 0 \end{cases} \quad (3.28)$$

$$L_{np}^1 = \begin{cases} -i\eta\kappa \langle e^{-inu} w' \rangle, & p = 0 \\ i\eta \frac{\kappa}{\alpha_p^2} \langle e^{-i(n-p)u} \frac{w' I_1(\alpha_p w)}{w I_1(\alpha_p)} \rangle, & p \neq 0 \end{cases} \quad (3.29)$$

$$\bar{M}_{np}^1 = \begin{cases} \langle e^{-inu} \left(\frac{1}{2\kappa} + \frac{\kappa}{4} w^2 - i\eta w' w \right) \rangle, & p = 0 \\ -\frac{\kappa}{\alpha_p^2} \langle e^{-i(n-p)u} \frac{I_1(\alpha_p w)}{w I_1(\alpha_p)} \rangle, & p \neq 0 \end{cases} \quad (3.30)$$

$$\bar{L}_{np}^1 = \begin{cases} -\kappa \langle e^{-inu} \rangle, & p = 0 \\ \langle e^{-i(n-p)u} \left[\frac{(\eta p + \kappa) I_1'(\alpha_p w)}{\alpha_p I_1(\alpha_p)} - i\eta \frac{w' I_1(\alpha_p w)}{I_1(\alpha_p)} \right] \rangle, & p \neq 0 \end{cases} \quad (3.31)$$

$$\bar{M}_{np}^1 = \begin{cases} \langle e^{-inu} \left(\frac{\kappa w^2}{4} - \frac{1}{2\kappa} \right) \rangle, & p = 0 \\ -\frac{\eta p + \kappa}{\alpha_p^2} \langle e^{-i(n-p)u} \frac{I_1(\alpha_p w)}{w I_1(\alpha_p)} \rangle, & p \neq 0 \end{cases} \quad (3.32)$$

$$\bar{L}_{np}^1 = \begin{cases} -\kappa \langle e^{-inu} \rangle, & p = 0 \\ \frac{\kappa}{\alpha_p} \langle e^{-i(n-p)u} \frac{w I_1'(\alpha_p w)}{I_1(\alpha_p)} \rangle, & p \neq 0 \end{cases} \quad (3.33)$$

$$N_n^1 = i\eta \langle e^{-inu} w' (1 + 1/w^2) \rangle \quad (3.34)$$

$$\bar{N}_n^1 = \langle e^{-inu} (1 - 1/w^2) \rangle \quad (3.35)$$

$$\bar{\bar{N}}_n^1 = \langle e^{-inu} (1 - 1/w^2) \rangle \quad (3.36)$$

In the case where $m = 1$, only two infinite sets of linear algebraic equations are required while the third one is redundant. These define the two infinite sets of unknown coefficients B_p^1 and G_p^1 ($p = -\infty, \dots, \infty$). The coefficient B_0^1 is proportional to both longitudinal and transverse impedances for $m = 1$.

Let us again consider the region of parameters where the perturbation method is applicable. Expansion in $|\alpha_p \Delta|$ of the matrix elements of Eqs. (3.25) and (3.26) gives

$$M_{np}^1 = \left(\delta_{n,p} + \frac{\alpha_p I_1'(\alpha_p)}{I_1(\alpha_p)} \right) \frac{\eta^2 p n + \eta \kappa (p+n)}{\alpha_p^2}, \quad p \neq 0 \quad (3.37)$$

$$L_{np}^1 = -\frac{\eta \kappa (n-p)}{\alpha_p^2} C_{n-p}, \quad p \neq 0 \quad (3.38)$$

$$\bar{M}_{np}^1 = -\frac{\kappa}{\alpha_p^2} \left[\delta_{n,p} + \left(\alpha_p \frac{I_1'(\alpha_p)}{I_1(\alpha_p)} - 1 \right) C_{n-p} \right], \quad p \neq 0 \quad (3.39)$$

$$\bar{L}_{np}^1 = \frac{\eta p + \kappa}{\alpha_p} \left[\frac{I_1'(\alpha_p)}{I_1(\alpha_p)} \delta_{n,p} + \frac{\alpha_p I_1''(\alpha_p)}{I_1(\alpha_p)} C_{n-p} \right] + \eta(n-p) C_{n-p}, \quad p \neq 0 \quad (3.40)$$

$$N_n^1 = -2\eta n C_n \quad (3.41)$$

$$\tilde{N}_n^1 = 2C_n. \quad (3.42)$$

From the $n \neq 0$ rows of Eq. (3.25) we find to the first order in $|\Delta|$:

$$B_n^1 = N_n^1, \quad n \neq 0 \quad (3.43)$$

From the $n \neq 0$ rows of Eq. (3.26) with the same accuracy:

$$G_n^1 = \frac{\tilde{N}_n^1 - \bar{M}_{n,n}^1 B_n^1}{\bar{L}_{n,n}^1} = \frac{2C_n \alpha_n [1 - \eta \kappa / \alpha_n^2] I_1'(\alpha_n)}{(\eta n + \kappa) I_1'(\alpha_n)} \quad (3.44)$$

The $n = 0$ row of Eq. (3.25) gives

$$B_0^1 = - \sum_p' (M_{0p}^1 B_p^1 + L_{0p}^1 G_p^1)$$

or

$$B_0^1 = 2\eta^2 \kappa \sum_p' p^2 |C_p|^2 \left[\frac{I_1'(\alpha_p)}{\alpha_p I_1(\alpha_p)} - \frac{I_1(\alpha_p)}{\alpha_p^3 I_1'(\alpha_p)} \right] \quad (3.45)$$

Both longitudinal and transverse impedances (3.9) and (3.10) calculated with the help of expression (3.45) agree with those obtained in Refs. 3 and 4. The region of validity of expression (3.45) is the same as for the longitudinal case $m = 0$ discussed in Section 3.2, i.e. both $|\Delta|$ and $|\alpha_p \Delta|$ should be small compared to unity. From expression (3.45) it follows that in the low-frequency range where α_p is real, B_0^1 is also real and antisymmetric in frequency. The same is true also for B_0^1 as a solution of Eqs. (3.25) and (3.26). Consequently, for low frequencies the longitudinal impedance Z_1^1 is antisymmetric and the transverse impedance Z_1 is symmetric in frequency.

4. IMPEDANCES OF BELLOWS

We will approximate the boundary of bellows in the cross-section $\theta = \text{const}$ by a cosine function. For the reference boundary we will assume the cylindrical surface with constant radius R equal to the pipe radius:

$$w = 1 + \epsilon + b\epsilon \cos u. \quad (4.1)$$

Here

$$\epsilon = d/2R \quad (4.2)$$

and b is a numerical factor of the order of one.

Typical dimensions of bellows (for example LEP bellows) are as follows: the pipe radius $R = 40$ mm, the depth of the bellows $d = 8$ mm, and the period length $L = 8$ mm (the cut-off frequency is of the order of 3 GHz). For such bellows the dimensionless parameters defined earlier are $\eta = 2\pi R/L \approx 30$ and $\epsilon = d/2R \approx 0.1$. Their product is $\eta\epsilon \approx 3$. (The value of the dimensionless parameter κ corresponding to the cut-off frequency is of the order of three.) This situation

$$\eta \gg 1 \quad (4.3)$$

$$\epsilon \ll 1 \quad (4.4)$$

$$\eta\epsilon \gtrsim 1 \quad (4.5)$$

is typical for bellows. As it is discussed in Section 3, the perturbation approach breaks down for $\eta\epsilon \approx 1$.

4.1 Estimate of the transverse impedance ($m = 1$)

Here we present an estimate of B_0^1 by taking into account only the terms $p = 0, \pm 1$ of Eqs. (3.25) - (3.27). Although the result obtained in this way is not very exact, it should show the correct dependence of the impedance on the parameters.

When the inequality (4.4) holds, all values need only be evaluated to the lowest order in ϵ . Also, using the inequality (4.3) the asymptotic expression for the modified Bessel functions may be employed:

$$I_m(\alpha_p w)/I_m(\alpha_p) \simeq e^{\alpha_p \epsilon (1+b \cos u)} \quad (4.6)$$

In this approximation only the first equation (3.25) is sufficient. The evaluation of its matrix elements and its right-hand sides gives:

$$M_{np}^1 = \begin{cases} \delta_{n,0} & p = 0 \\ \frac{\eta^2 pn + \eta\kappa(p+n)}{\alpha_p^2} e^{\alpha_p \epsilon} I_{n-p}(\alpha_p b\epsilon), & p \neq 0 \end{cases} \quad (4.7)$$

$$L_{np}^1 = 0 \quad (4.8)$$

$$N_n^1 = -\eta b\epsilon(\delta_{n,1} - \delta_{n,-1}). \quad (4.9)$$

The $n = 0$ row gives

$$B_0^1 = -M_{0,-1}^1 B_{-1}^1 - M_{0,1}^1 B_1^1, \quad (4.10)$$

while the $n = 1$ and $n = -1$ rows constitute the system of two equations for B_1^1 and B_{-1}^1 :

$$\begin{cases} M_{1,1}^1 B_1^1 + M_{1,-1}^1 B_{-1}^1 = N_1^1 \\ M_{-1,1}^1 B_1^1 + M_{-1,-1}^1 B_{-1}^1 = N_{-1}^1 \end{cases} \quad (4.11)$$

From here one finds B_1^1 and B_{-1}^1 . Substituting them into Eq. (4.10) one finds B_0^1 . In the low-frequency limit

$$\kappa \ll \eta, \quad (4.12)$$

we get

$$B_0^1 = \frac{\epsilon \kappa}{\eta} f(y), \quad (4.13)$$

where $y = b\eta\epsilon$ and

$$f(y) = 2yI_1(y)/[I_0(y) + I_2(y)]. \quad (4.14)$$

Figure 2 shows this function. For the case $\eta\epsilon \gg 1$ this expression yields:

$$B_0^1 \simeq b\epsilon\kappa.$$

As one sees, B_0^1 is linear in ϵ and κ , which is reasonable for the considered range of parameters. This yields the following expression for the transverse impedance

$$Z_1 = -i \frac{bLd}{cR^3},$$

which agrees with the corresponding result obtained in Ref. 6.

4.2 Numerical results

The most straightforward way of finding the solution for the coefficients B_p and G_p is to solve the corresponding equations numerically. The program IMPASS (Impedances of Periodic Axialsymmetric Smooth Structures) has been written, which solves Eqs. (3.25) to (3.27) for $m = 0$ and $m = 1$ approximately by truncating the matrices to a certain size and then inverting them.

Figures 3-9 present the results of calculations by IMPASS. Dependences of B_0^0 [which is proportional to the longitudinal impedance for $m = 0$, see Eq. (3.8)] and B_0^1 [which is proportional to the longitudinal and the transverse impedances for $m = 1$, see Eqs. (3.9) and (3.10)] on κ , ϵ , and η are presented here.

As one can see the coefficients found are linear in κ only in the region of rather small $\kappa \lesssim 1$. For larger values the dependence becomes more and more non-linear. This feature should be kept in mind when evaluating the loss factor (see below).

One notices that the following approximate numerical relation holds

$$B_0^1 \simeq 2B_0^0$$

and a similar expression for Z_0^1 and Z_1^1 .

4.3 Comparison with TBCI

A similar geometry has also been studied with the help of a purely numerical program TBCI⁸, and it is interesting to compare the results of both approaches. Two factors complicate the comparison. Firstly, TBCI is suitable only for boundaries formed by pieces of straight lines, while IMPASS handles smoothly curved boundaries, in the simplest form a cosine-like function. An attempt to take into consideration higher

longitudinal harmonics to represent a more or less square-wall configuration causes a rapid increase in the required matrix size. One way to approximate the square-wall boundary by a smooth curve is to use a few of the first terms of its Fourier expansion. In what follows we use the first two terms of such a decomposition. If we consider the case $d = L$, then one finds $b = 4/\pi$ for the coefficient b introduced in Eq. (4.1). Secondly, TBCI calculates the loss factor

$$K_{\perp} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Im } Z_{\perp} e^{-\omega^2 \sigma^2 / c^2} d\omega \quad (4.15)$$

where σ is the r.m.s. of the longitudinal density distribution (assumed to be Gaussian) of particles in a bunch.

If $\text{Im } Z_{\perp}$ is constant in frequency

$$K_{\perp} = \frac{c}{2\sqrt{\pi}\sigma} \text{Im } Z_{\perp} \quad (4.16)$$

In terms of the coefficient B_0^1 the loss factor per period for one bellows convolution becomes

$$K_{\perp} = \frac{Z_0 c}{2\sqrt{\pi}\sigma} \frac{B_0^1}{\eta\kappa}, \quad (4.17)$$

where $Z_0 = 377 \Omega$ is the vacuum impedance. As was mentioned previously $\text{Im } Z_{\perp}$ does depend on frequency. Taking this into account changes the results only slightly (by 5-10% for the parameters' range of interest) and it is not included in the following figures.

Figures 10 and 11 show a comparison of the results obtained by Henke¹² with the help of TBCI (solid lines) and the present calculations done with IMPASS (broken lines).

Figure 12 shows a similar comparison with the results of calculations performed by one of the authors (BZ) for a different range of bellows parameters.

Given all the differences underlying the two approaches, the agreement between the results seems to be quite good.

5. CONCLUSION

Several concluding remarks seem to be in place here. First of all, we want to emphasize the complementary character of our semi-analytic approach (and consequently of the code IMPASS) and of the purely numerical method employed by the code TBCI. Indeed, i) IMPASS most easily handles cosine-like boundaries, while TBCI is more useful for square-wall boundaries; ii) IMPASS is the more accurate and the faster the smaller the depth of corrugations while, contrary to that, the speed of TBCI rapidly decreases with smaller depths; iii) IMPASS can be used for any bunch length, including point charges, while TBCI is not suitable for very small bunch lengths.

Next, the semi-analytic approach has a great advantage in the speed with which an estimate of the impedances can be obtained (e.g. a run of IMPASS on the computer CDC 875 takes only a fraction of a second of CPU time). On the other hand, it fails for more complicated boundaries, large depths (for example, for a chain of cavities), etc., at least in the present stage of its development. Another advantage of our approach is that it is easy to use it in general cases (e.g. for γ finite and $m > 2$), while TBCI is designed, for example, specifically for the ultra-relativistic limit. Certainly, more work is needed to improve and further develop the approach described in the present report.

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Fig. 1 Schematic layout of bellows geometry and coordinate system

$$f(y) = \frac{y I_0(y)}{I_1'(y)}$$

Fig. 2 Function $f(y)$ of Eq. (4.13)

Fig. 3 Coefficient $B_0^{(0)}$, which is proportional to the $m = 0$ longitudinal impedance Z_0 , for bellows with the boundary $w = 1 + \epsilon + (4/\pi)\epsilon \cos u$, as a function of the normalized frequency κ , for three values of the parameter η : (1) $\eta = 31.42$, (2) $\eta = 20.94$, (3) $\eta = 12.57$. Corrugation parameter $\epsilon = 0.06$.

Fig. 4 The same as in Fig. 3, but for $\epsilon = 0.09$.

Fig. 5 The same as in Fig. 3, but for $\epsilon = 0.12$.

Fig. 6 The coefficient $B_0^{(1)}$, which is proportional to the $m = 1$ longitudinal impedance Z_1 and to the transverse impedance Z_1 multiplied by κ , for bellows with the boundary $w = 1 + \epsilon + (4/\pi)\epsilon \cos u$, as a function of the normalized frequency κ , for three values of the parameter η : (1) $\eta = 31.42$, (2) $\eta = 20.94$, (3) $\eta = 12.57$. Corrugation parameter $\epsilon = 0.09$.

Fig. 7 The same as in Fig. 6, but for $\eta = 12.57$, and for three values of the corrugation parameter ϵ : (1) $\epsilon = 0.12$, (2) $\epsilon = 0.09$, (3) $\epsilon = 0.06$.

Fig. 8 The same as in Fig. 7, but for $\eta = 20.94$.

Fig. 9 The same as in Fig. 7, but for $\eta = 31.42$.

Fig. 10 Dependence of the transverse loss factor $K_{\perp}$ (in V/pCm) on the corrugation depth d (in mm) for 40 convolutions of bellows ($R = 39$ mm, $L = 3$ mm, $\sigma = 12$ mm). The solid line represents the results¹² obtained with the help of TBCI. The boundary was approximated by piece-wise straight lines. The broken line represents the results obtained with the help of IMPASS. The boundary is approximated by the function $w = 1 + \varepsilon + (4/\pi) \varepsilon \cos u$.

Fig. 11 The same as in Fig. 10, but as a function of the bellows period length L (in mm); $d = 4$ mm.

Fig. 12 Comparison of the results obtained with the help of TBCI (solid lines, square-wall boundary) and IMPASS [broken lines, boundary function $w = 1 + \epsilon + (4/\pi) \epsilon \cos u$]. The transverse loss factor $K_{\perp}$ (in V/pCm) per convolution of bellows is drawn as a function of the depth d (in mm), (1) $L = 10$ mm ($\eta = 12.57$), (2) $L = 6$ mm ($\eta = 20.94$), (3) $L = 4$ mm ($\eta = 31.42$).