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GAUGE COVARIANT LOCAL FORMULATION OF BOSONIC STRINGS

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ABSTRACT

A free bosonic string field action is constructed that is local and invariant under the infinite set of gauge transformations corresponding to the conformal group of the two-dimensional world sheet. This allows for a straightforward gauge covariant method of second quantizing string field theory. The method of extension to the interacting case is presented.

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1. INTRODUCTION

The classical bosonic string is described by the action¹⁾

$$A = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \left[\det \partial_\alpha x^\mu \partial_\beta x^\nu \delta_{\mu\nu} \right]^{1/2}$$

which is invariant under two-dimensional reparametrization of the world sheet. For the open string, $x^\mu(\sigma, \tau)$ is defined for $0 \leq \sigma \leq \pi$ with boundary conditions $\partial_\sigma x^\mu = 0$ at $\sigma = 0, \pi$. It is convenient to mathematically extend this range from $-\pi \leq \sigma \leq \pi$ by letting $x^\mu(-\sigma, \tau) = x^\mu(\sigma, \tau)$. The closed string is defined by $x^\mu(\sigma, \tau)$, $-\pi \leq \sigma \leq \pi$, periodic in σ with period 2π . As a result of this reparametrization invariance it is a constrained system. In the classical Hamiltonian formulation of the open string, the constraints may be written in the form

$$L_n = \frac{\pi\alpha'}{2} \int_{-\pi}^{\pi} d\sigma e^{-in\sigma} \left(p^\mu + \frac{1}{2\pi\alpha'} \frac{\partial x^\mu}{\partial \sigma} \right)^2, \quad (1.1)$$

where

$$p^\mu = \frac{\delta A}{\delta \dot{x}^\mu(\sigma)} \quad \text{and} \quad \dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau}.$$

The L_n 's obey the algebra

$$[L_n, L_m] = (n-m) L_{n+m}, \quad (1.2)$$

which is just that of the infinite dimensional conformal group of the two-dimensional world sheet.

To first quantize the system we adopt the usual correspondence between Poisson brackets and commutation relations and take

$$[x^\mu(\sigma), x^\nu(\sigma')] = 0, \quad [p^\mu(\sigma), p^\nu(\sigma')] = 0,$$

and

$$[x^\mu(\sigma), p^\nu(\sigma')] = i\hbar \delta^{\mu\nu} \delta(\sigma - \sigma') \quad (1.3)$$

The relations are represented by the operators

$$X^{\mu}(\sigma) = X^{\mu}(\sigma), \quad P^{\nu}(\sigma) = -i\hbar \frac{\delta}{\delta X^{\nu}(\sigma)} \quad (1.4)$$

The state of the string is described at a given τ by a functional of $x^{\mu}(\sigma)$, $\phi[x^{\mu}(\sigma)]$. Although the Hamiltonian vanishes, one may use the quantization methods of constraints discussed by Dirac²⁾.

In this paper, we shall give a method of second quantizing the string that is free from constraints and possesses gauge invariances corresponding to the infinite dimensional conformal group. For the open string, one of the infinite set of gauge transformations will be Yang-Mills transformations, while for the closed string they will contain general co-ordinate transformations. In this way, the gauge invariance of the string contains all previously known gauge symmetries, and in the gauge covariant action, one finds, at the linearized level, the Yang-Mills and Einstein actions.

Up until recently, the second quantization of strings has either been performed with constraints being present or carried out in a given gauge such as the light-cone gauge³⁾, where the constraints have in effect been solved. Quantization using BRS techniques of the linearized theory in a given gauge has been discussed in Ref. 4). It is possible that one can, in principle, discover all the properties of a theory by quantizing in a given gauge. However, without the ability to use all the wisdom acquired with second quantization, this may be difficult. An example is the computation of anomalies which are absent in the light cone gauge; however, a careful search of Lorentz invariance would show that it is violated when gauge anomalies are present. Also, the whole subject of non-perturbative semi-classical phenomena and spontaneous symmetry breaking has up to now only been developed in the gauge invariant framework. It is also possible that the finiteness properties of strings may become particularly apparent in a covariant formulation as they did in the case of supersymmetric theories.

To second quantize the system we suppress as usual the τ dependence in the wave functional $\phi[x^{\mu}(\sigma)]$ and require an action S which is used to weigh the Feynman path integral. The vacuum-to-vacuum amplitude is given generally by

$$\int \mathcal{D}\psi \exp \frac{i}{\alpha} S \quad (1.5)$$

where the action is of the generic form

$$S = \int \mathcal{D}x^\mu(\sigma) \mathcal{L}(\psi[x^\mu(\sigma)]) \quad (1.6)$$

+ gauge fixing + ghost + source terms

In a covariant formulation, S will be invariant under gauge invariances corresponding to the two-dimensional conformal group. We will present such an action for the linearized theory. For a linearized theory, the gauge invariances are Abelian. We present the strategy for building string interactions starting from the linearized theory. This is performed by a type of Noether technique, and one finds that the Abelian and rigid invariances of the rigid theory become knitted together. The knitting involves the three-Reggeon coupling.

The dependence of x^μ on σ in the open string can be written in the form

$$x^\mu(\sigma) = \sum_{n=-\infty}^{+\infty} e^{in\sigma} x_n^\mu \quad (1.7)$$

where $x^\mu(\sigma)$ is defined for $-\pi < \sigma < \pi$ and $x_{-n}^\mu = x_n^\mu$. The two-dimensional conformal group are generated by the Virasoro operators L_n which are defined by

$$L_n = \frac{1}{2} \pi \alpha' \int_{-\pi}^{\pi} d\sigma e^{-in\sigma} \hat{p}^\mu(\sigma) \hat{p}^\nu(\sigma) \delta_{\mu\nu} \quad (1.8)$$

where

$$\hat{p}^\mu(\sigma) = -i\alpha' \frac{\delta}{\delta x^\mu(\sigma)} + \frac{1}{2\pi\alpha'} \frac{\partial x^\mu}{\partial \sigma} \quad (1.9)$$

It is well known⁵⁾ that the on-shell states of the $D = 26$ bosonic open string are given by

$$(L_0 - 1)\psi = 0 \quad (1.10)$$

$$L_n \psi = 0 \quad \text{for } n > 0 \quad (1.11)$$

It has been shown⁶⁾ that this system possesses one tachyon at its lowest mass level, all other states having zero or positive masses and that all states are ghost free, i.e., have propagators with positive residues. A corresponding covariant projector was first written down in Ref. 7).

To analyze Eqs. (1.10) and (1.11), it is more convenient to introduce creation and destruction operators defined by the equation

$$\hat{P}^\mu(\sigma) = \frac{1}{\pi(2\alpha')^{1/2}} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu e^{in\sigma} \quad (1.12)$$

Using Eq. (1.7) we find that

$$\alpha_n^\mu = i \left(\left(\frac{\alpha'}{2} \right)^{1/2} \frac{\partial x_n^\mu}{\partial \tau} + n(2\alpha')^{-1/2} x_n^\mu \right) \quad (1.13)$$

From the reality of $P^\mu(\sigma)$ we find that $\alpha_{-n}^\mu = \alpha_n^{\mu+}$. The α 's commutation relations are

$$\begin{aligned} [\alpha_n^\mu, \alpha_m^\nu] &= 0 \\ [\alpha_n^\mu, \alpha_m^{\nu+}] &= n \delta_{n,m} \delta^{\mu\nu} \end{aligned} \quad (1.14)$$

for $n, m > 0$. The Virasoro operators⁵⁾ can be expressed in terms of the α_n 's by

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} \alpha_m^\mu \alpha_{n-m}^\mu \quad (1.15)$$

where L_0 is understood to be normal ordered. They obey the modified algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{26}{12} n(n^2-1) \delta_{n,-m}. \quad (1.16)$$

We find in particular that

$$L_0 = \frac{1}{2} \alpha_0^{\mu 2} + \sum_{m=1}^{\infty} \alpha_m^{\mu \dagger} \alpha_m^{\mu}, \quad (1.17)$$

$$L_1 = L_{-1} = \alpha_0^{\mu} \alpha_1^{\mu} + \sum_{m=1}^{\infty} \alpha_m^{\mu \dagger} \alpha_{m+1}^{\mu}, \quad (1.18)$$

where

$$\alpha_0^{\mu} = -i \left(\frac{\alpha'}{2} \right)^{1/2} \frac{\partial x^{\mu}}{\partial x^0}.$$

We may write the state of the string in occupation number basis in terms of the creation operators $\alpha_n^{\mu \dagger}$ by

$$\psi[X^{\mu}(\sigma)] = \left\{ \varphi(x) + A_{\mu}^{\dagger} \alpha_1^{\mu \dagger} + h_{\mu\nu} \alpha_1^{\mu \dagger} \alpha_1^{\nu \dagger} + \alpha_2^{\mu \dagger} A_{\mu}^2 + \dots \right\} |0\rangle. \quad (1.20)$$

The vacuum satisfies the equation

$$\alpha_n^{\mu} |0\rangle = 0, \quad n \geq 1. \quad (1.21)$$

Alternatively, one could change basis to diagonalize $x^{\mu}(\sigma)$, and this has already been considered in the light-cone formulation of interacting strings³⁾. The vacuum of Eq. (1.21) is of the form

$$|0\rangle = \prod_{n=1}^{\infty} c_n \exp\left(-\frac{n}{2\alpha'} x_n^{\mu} x_n^{\nu} \delta_{\mu\nu}\right) \quad (1.22)$$

The action of the $\alpha_n^{\mu \dagger}$ on $|0\rangle$ produces the well-known complete set of Hermite polynomials. In terms of component fields we find that Eq. (1.10) has as consequences

$$\begin{aligned} (\partial^2 + \frac{1}{\alpha'}) \varphi(x) = 0 &= [\partial^2 - (d-1) \frac{1}{\alpha'}] A_\mu^{(2)} \\ &= (\partial^2 - \frac{1}{\alpha'}) h_{\mu\nu} \dots \text{etc} \dots \end{aligned} \quad (1.23)$$

The action of Eq. (1.11) is

$$\partial^\mu A_\mu^{(1)} = 0 = \partial^\mu h_{\mu\nu} + A_\nu^{(2)} = 2 \partial^\mu A_\mu^{(2)} + h_\nu^\nu. \quad (1.24)$$

We now wish to find an action which does not have the constraints of Eq. (1.11) but instead possesses gauge invariances that allow the constraints of Eq. (1.11) to arise as a gauge choice. Consequently, we expect an infinite number of gauge invariances, one for every one of the constraints which generate the conformal group. This can be achieved mass level by mass level by successively releasing the constraints on ψ .

At the first level, we release the constraint $L_1 \psi = 0$ but ψ is still subject to

$$L_2 \psi = L_1^2 \psi = L_3 \psi = L_2 L_1 \psi \dots = 0. \quad (1.25)$$

Consider now the gauge transformation of ψ

$$\delta \psi = L_1 \Lambda_1, \quad (1.26)$$

where Λ_1 is subject to

$$L_1 \Lambda_1 = L_2 \Lambda_1 = 0. \quad (1.27)$$

Using the form of L_{-1} given in Eq. (1.18), we find that this invariance contains the transformation $\delta A_\mu^{(2)} = \partial_\mu \Lambda_1(x)$ which is the Abelian transformation expected for a linearized Yang-Mills theory.

An action invariant under $\delta \psi = L_{-1} \Lambda_1$ is given by

$$\frac{1}{2} (\psi, (L_0 - 1 - \frac{1}{2} L_{-1} L_1) \psi). \quad (1.28)$$

The equation of motion is given by

$$(L_0 - 1 - \frac{1}{2} L_{-1} L_1) \psi = 0. \quad (1.29)$$

Explicitly testing this, we find that

$$(L_0 - 1 - \frac{1}{2} L_{-1} L_1) L_{-1} \Lambda_1 = (L_0 L_{-1} - L_{-1} L_0) \Lambda_1 = 0, \quad (1.30)$$

since $L_1 \Lambda_1 = 0$.

Equation (1.29) was probably known to a few people in the old heyday of string theory, but has been rediscovered more recently^{8),9)}. The projector P of a string field onto the physical states of Eqs. (1.10) and (1.11) was given in Ref. 3). At lowest order it is given by

$$P = 1 - \frac{1}{2} L_{-1} \frac{1}{L_0} L_1 \quad (1.31)$$

We note that at this level the equation of motion is given by

$$(L_0 - 1) P \psi = 0 \quad (1.32)$$

It has been proposed recently^{9),10)} that Eq. (1.32) is the correct equation of motion of all levels of the string, and this speculation has been encouraged by the above coincidence for the spin one at the first level. In Refs. 9) and 10), P has been formally computed for all levels, and Eq. (1.32) is explicitly non-local at the second level and more and more so for higher levels. One's suspicions are further aroused by the fact that P can be constructed in an arbitrary space-time dimension, and that $D = 26$ is not particularly favoured. The clearest way to show that Eq. (1.32) is not the right generalization is to consider the first level of the closed bosonic oriented string. At this level, the covariant degrees of freedom are described by a single symmetric field $h_{\mu\nu}$. The Virasoro condition implies that it satisfies

$$\partial^2 h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} = 0. \quad (1.33)$$

These equations tell us that at this level the closed string contains only a spin two and a spin zero. The generalization of Eq. (1.32) for this level is

$$\partial^2 R_\mu{}^\rho R_\nu{}^\lambda h_{\rho\lambda} = 0, \quad (1.34)$$

where

$$R_{\mu}^{\rho} = \left(\delta_{\mu}^{\rho} - \frac{\partial_{\mu} \partial^{\rho}}{\partial^2} \right). \quad (1.35)$$

The reader immediately recognizes that this is a non-local equation, which does not admit a Hamiltonian formulation. Making it local by multiplication by ∂^2 leads to additional states.

In fact, it is known¹¹⁾ that there is no Lorentz invariant, gauge invariant, local action constructed from $h_{\mu\nu}$ alone, which describes both spin two and spin zero. The only way to achieve this is to introduce another, scalar field to describe the spin zero; this is the well-known Einstein + massless scalar action. As we shall see, rather than the spin one, the first level of the closed string illustrates the general pattern, namely, for a local formulation, the theory naturally requires supplementary fields, as we shall now demonstrate.

2. - GAUGE COVARIANT FORMULATION OF THE LINEARIZED STRING FIELD THEORY

2a) Open Strings

As our starting point, we take the action at the first level, Eq. (28)

$$\frac{1}{2} (\psi, (L_0 - 1 - \frac{1}{2} L_1, L_1) \psi), \quad (2.1)$$

which is invariant under the gauge transformations

$$\delta_1 \psi = L_1 \Lambda_1, \quad \delta_2 \psi = L_2 \Lambda_2 \quad (2.2)$$

where Λ_1, Λ_2 and ψ are subject to

$$L_1 \Lambda_1 = L_2 \Lambda_2 = 0, \quad L_1^2 \psi = L_2 \psi = L_3 \psi = L_2 L_1 \psi = \dots = 0 \quad (2.3)$$

We shall now find the covariant formulation level by level, systematically relaxing conditions (2.3). At the second level in 26 dimensions, the on-shell states are one massive pure "spin two" [$\equiv$ an $S(25)$ symmetric traceless tensor]. The theory of wave equations for particles of a given spin is a consequence of the theory of induced representations of the Poincaré group. It is a systematic procedure to find the projection operators onto the physical states of a given spin. In the study of these operators one finds the well-known wave equations. In fact the particles of spin zero, $\frac{1}{2}$ and 1 are the exception rather than the

rule, in that multiplying their projection operators by ∂^2 yields a local field equation. For spins greater than one, multiplication by ∂^2 leads to a non-local and so an unacceptable field equation. The procedure to be followed in these cases was worked out in Ref. 12) and it involves the introduction of supplementary fields. For example "pure spin two" is described on-shell by the field $h_{\mu\nu} = h_{\nu\mu}$ subject to the equations

$$\partial^\tau h_{\mu\nu} = h_{\mu\nu}^\tau = (\partial^2 - m^2) h_{\mu\nu} = 0. \quad (2.4)$$

The projector is well known¹³⁾. It involves terms of the form $\partial_\mu \partial_\lambda \partial_\rho \partial_\sigma / (\partial^2)^2$ and hence multiplication (∂^2) does not lead to a local field equation. In fact, there is no way to describe in a Lorentz covariant way only a massive spin two particle in terms of only $h_{\mu\nu} = h_{\nu\mu}$ subject to $h_\mu^\mu = 0$. The correct equations of motion involve the introduction of a supplementary field ϕ and are given by

$$\begin{aligned} (-\partial^2 + m^2) h_{\mu\nu} + (\partial_\mu \partial^\rho h_{\rho\nu} + \partial_\nu \partial^\rho h_{\rho\mu}) - 2 \frac{\delta_{\mu\nu}}{D} \partial^\rho \partial^\lambda h_{\rho\lambda} \\ = \frac{D-2}{D-1} (\partial_\mu \partial_\nu \phi - \frac{\delta_{\mu\nu}}{D} \partial^2 \phi), \end{aligned} \quad (2.5)$$

$$\partial^\tau \partial^\nu h_{\mu\nu} = (\partial^2 - \frac{D}{D-2} m^2) \phi,$$

where D is the dimension of space-time.

Indeed, these coupled equations lead to the desired result, namely

$$0 = \partial^\tau h_{\mu\nu} = \phi = (\partial^2 - m^2) h_{\mu\nu}. \quad (2.6)$$

The string field $\phi[x^\mu(\sigma)]$ contains at the second level $h_{\mu\nu}$ and $A_\mu^{(2)}$ [see Eq. (1.20)]; however, $A_\mu^{(2)}$, as we shall see, is gauged away and the above discussion implies that we require one extra supplementary field ϕ to implement the spin two equation. This field ϕ will be the lowest component of a supplementary string field $\phi^{(2)}[x^\mu(\sigma)]$.

We now release some of the constraints (2.3) of the first level, and ϕ and $\phi^{(2)}$ are now subject to

$$\begin{aligned} L_3 \psi &= L_2 L_1^2 \psi = L_1^3 \psi = \dots = 0, \\ L_1 \phi^{(2)} &= L_2 \phi^{(2)} = \dots = 0. \end{aligned} \quad (2.7)$$

The most general expression of the correct order is

$$\begin{aligned} (L_0 - 1 - \frac{1}{2} L_{-1} L_1 - \frac{1}{4} \gamma L_{-2} L_2) \psi \\ + (L_{-1}^2 + \frac{3}{2} \beta L_{-2}) \phi^{(2)} = 0 \end{aligned} \quad (2.8)$$

$$(L_{-1}^2 + \frac{3}{2} \beta L_{-2}) \psi = (a L_0 + b) \phi^{(2)} \quad (2.9)$$

The use of the same β in (2.7) and (2.8) is required by demanding that the equations of motion follow from an action. An alternative way of searching for a gauge invariance is to demand that L_1 on Eq. (2.8) should vanish when we use Eq. (2.9). Carrying this out and using the constraints of Eq. (2.7), we find

$$\begin{aligned} -\frac{1}{2} L_{-1} (L_{-1}^2 + \frac{3}{2} \gamma L_2) \psi + (4 L_{-1} L_0 + 2 L_{-1}) \phi^{(2)} \\ + \frac{9}{2} \beta L_{-1} \phi^{(2)} = 0 \end{aligned} \quad (2.10)$$

To eliminate ψ by Eq. (2.9) requires $\gamma = \beta$ and we find

$$-\frac{1}{2} L_{-1} (a L_0 + b) \phi^{(2)} + L_{-1} (4 L_0 + 2 + \frac{9}{2} \beta) \phi^{(2)} = 0 \quad (2.11)$$

Hence we conclude that $a = 8$ and $b = 4 + 9\beta$. Applying L_2 in a similar way fixes $\beta = 1$, and we find the equations of motion

$$\begin{aligned} \left(L_0 - 1 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{L_{-n} L_n}{n} \right) \psi + \left(L_{-1}^2 + \frac{3}{2} L_{-2} \right) \phi^{(2)} &= 0, \\ \left(L_{-1}^2 + \frac{3}{2} L_{-2} \right) \psi &= (8L_0 + 13) \phi^{(2)} \end{aligned} \quad (2.12)$$

This system of equations is in fact invariant under the gauge transformations

$$\begin{aligned} \delta_1 \psi &= L_{-1} \Lambda_1, \quad \delta_1 \phi^{(2)} = \frac{1}{2} L_{-1} \Lambda_1, \\ \delta_2 \psi &= L_{-2} \Lambda_2, \quad \delta_2 \phi^{(2)} = \frac{3}{2} \Lambda_2, \end{aligned} \quad (2.13)$$

with

$$\begin{aligned} L_2 \Lambda_1 &= L_1^2 \Lambda_1 = \dots = 0, \\ L_1 \Lambda_2 &= L_2 \Lambda_2 = \dots = 0. \end{aligned} \quad (2.14)$$

We stress that Eqs. (2.12) are Λ_2 invariant only for $D = 26$. It may be possible, with the introduction of further supplementary fields, to relax this condition. The corresponding action is given by

$$\begin{aligned} &\frac{1}{2} (\psi, (L_0 - 1 - \frac{1}{2} L_{-1} L_1 - \frac{1}{4} L_{-2} L_2) \psi) \\ &+ (\psi, (L_{-1}^2 + \frac{3}{2} L_{-2}) \phi^{(2)}) - \frac{1}{2} (\phi^{(2)}, (8L_0 + 13) \phi^{(2)}) \end{aligned} \quad (2.15)$$

This completes the analysis of the second level, and we now go to the third and fourth levels.

At the third level, we have as on-shell states, one massive vector, one antisymmetric second rank tensor and one "pure spin three". The latter is in

effect described by the field $\phi_{\mu\nu\rho} = \phi_{(\mu\nu\rho)}$ subject to $\partial^\mu \phi_{\mu\nu\rho} = 0 = \phi_{\mu\rho}^\mu$. This requires the supplementary fields ϕ_μ and ϕ to describe its propagation. The ϕ_μ field is supplied at this level by the second component field of $\phi^{(2)}$, i.e.,

$$\phi^{(2)} [x^\mu(\sigma)] = [\phi^{(2)}(x) + \alpha_1 x^\mu \phi_\mu^{(2)}(x) + \dots] |0\rangle \quad (2.16)$$

Hence, we only require one new supplementary string field $\phi^{(3)} [x^\mu(\sigma)]$ since the antisymmetric tensor and the vector do not require supplementary fields.

One can carry out the counting of states at the fourth level and one finds that one new supplementary string field is required $\phi^{(4)} [x^\mu(\sigma)]$.

At the fourth level the fields are subject to

$$\begin{aligned} 0 &= L_5 \psi = L_1^5 \psi = L_4 L_1 \psi \dots \text{etc.}, \\ 0 &= L_3 \phi^{(2)} = L_2 L_1 \phi^{(2)} = L_1^3 \phi^{(2)} \dots \text{etc.}, \\ 0 &= L_2 \phi^{(3)} = L_1^2 \phi^{(3)} \dots \text{etc.}, \\ 0 &= L_1 \phi^{(4)} \dots \text{etc.} \end{aligned} \quad (2.17)$$

The action is obtained as for the second level; one writes down all possible terms of the correct order and requires the result to be gauge invariant. The action is found to be

$$\begin{aligned} & \frac{1}{2} (\psi, (L_0 - 1 - \sum_{n=1}^4 \frac{1}{2n} L_{-n} L_n) \psi) + (\psi, (L_{-1}^2 + \frac{3}{2} L_{-2}) \phi^{(2)}) \\ & + (\psi, (L_{-2} L_{-1} + \frac{8}{3} L_{-3}) \phi^{(3)}) + (\psi, (L_{-3} L_{-1} + \frac{15}{4} L_{-4}) \phi^{(4)}) \\ & - \frac{1}{2} (\phi^{(2)}, (8L_0 + 13 + 2L_{-1} L_1 - L_{-2} L_2) \phi^{(2)}) \\ & - \frac{1}{2} (\phi^{(3)}, (8L_0 + \frac{128}{3}) \phi^{(3)}) \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2} (\phi^{(4)}, (12L_0 + \frac{207}{2}) \phi^{(4)}) \\
 & -3 (L_2 \phi^{(2)}, \phi^{(4)}) - 6 (L_1 \phi^{(2)}, \phi^{(3)}) \\
 & -2 (L_2 \phi^{(2)}, L_1 \phi^{(3)}) - 10 (L_1 \phi^{(3)}, \phi^{(4)}) .
 \end{aligned} \tag{2.18}$$

It is invariant under the gauge transformations

$$\begin{aligned}
 \delta_1 \psi &= L_1 \Lambda_1, \quad \delta_1 \phi^{(n+1)} = \frac{1}{2n} L_n \Lambda_1, \\
 \delta_2 \psi &= L_2 \Lambda_2, \quad \delta_2 \phi^{(2)} = \frac{3}{2} \Lambda_2 + \frac{1}{6} L_{-2} L_2 \Lambda_2, \\
 \delta_2 \phi^{(3)} &= \frac{1}{2} L_1 \Lambda_2 - \frac{1}{8} L_{-1} L_2 \Lambda_2, \quad \delta_2 \phi^{(4)} = \frac{1}{9} L_2 \Lambda_2.
 \end{aligned} \tag{2.19}$$

Further gauge transformations can be found by taking special values of Λ_1 and Λ_2 , say $\Lambda_1 = +L_{-2}\Lambda_3$, $\Lambda_2 = -L_{-1}\Lambda_3$ which gives $\delta\psi = L_{-3}\Lambda_3$ and expressions for $\delta\phi^{(n)}$. Since the linearized action (2.18) is quadratic in the supplementary fields, one could in principle integrate them out, recovering, up to that level, the projector of Ref. 7). In order to do this, however, one must invert the matrix of the quadratic form in the supplementary fields, a process which is already very painful at fourth level.

Before continuing to the fifth level, it is useful to reformulate the result at the fourth level by introducing a further supplementary field. This also allows us to give some sectors of the action to all levels. The ψ equation of motion is of the form

$$\begin{aligned} & (L_0 - 1 - \sum_{n=1}^{\infty} \frac{1}{2n} L_{-n} L_n) \psi \\ & + \text{supplementary field terms} = 0. \end{aligned} \quad (2.20)$$

This form is necessary in order to eliminate the $L_m L_0$ terms upon the multiplication by L_m . Acting with L_1 we find the result

$$\begin{aligned} & - \sum_{n=1}^{\infty} \frac{1}{2n} L_{-n} \left[L_1 L_n + \frac{n+2}{n+1} n L_{n+1} \right] \psi \\ & + \text{supplementary field terms} = 0. \end{aligned} \quad (2.21)$$

In order to eliminate the ψ term we require supplementary fields $\psi^{(n+1)}$ whose equations of motion are

$$\left(L_1 L_n + n \frac{n+2}{n+1} L_{n+1} \right) \psi = \text{supplementary fields}. \quad (2.22)$$

For the action of L_2 on Eq. (2.20), we find the result

$$\begin{aligned} & -\frac{3}{2} (L_1^2 + \frac{3}{2} L_2) \psi - \frac{1}{2} L_{-1} (L_1 L_2 + \frac{8}{3} L_3) \psi \\ & - \sum_{n=2}^{\infty} \frac{L_{-n}}{2n} \left(L_2 L_n + n \frac{n+4}{n+2} L_{n+2} \right) \psi \\ & + \text{supplementary field terms} = 0. \end{aligned} \quad (2.23)$$

We observe that the first two terms involve the same operators as occur for the application of L_1 . In fact, one can eliminate ψ from this equation using the identity

$$\begin{aligned} & L_2 L_n + \frac{n(n+4)}{n+2} L_{n+2} = \frac{2}{3} L_n (L_1^2 + \frac{3}{2} L_2) \\ & - \frac{2}{3} L_1 \left(L_1 L_n + n \frac{n+2}{n+1} L_{n+1} \right) \\ & - \frac{2}{3} \frac{n^2-2n-2}{n+1} \left[L_1 L_{n+1} + \frac{(n+1)(n+3)}{n+2} L_{n+2} \right] \end{aligned} \quad (2.24)$$

However, it is convenient to introduce a further set of supplementary fields $\chi^{(n)}$ ($n > 4$), the equation of motion for $\chi^{(n)}$ being

$$\left(L_2 L_{n-2} + \frac{n^2-4}{n} L_n \right) \psi = \text{supplementary field terms} \quad (2.25)$$

These equations follow from the action

$$\begin{aligned} & \frac{1}{2} \left(\psi, \left(L_0 - 1 - \sum_{n=1}^{\infty} \frac{1}{2n} L_{-n} L_n \right) \psi \right) \\ & + \sum_{n=1}^{\infty} \left(\psi, \left(L_{-n} L_{-1} + n \frac{n+2}{n+1} L_{-n-1} \right) \phi^{(n+1)} \right) \\ & + \sum_{n=2}^{\infty} \left(\psi, \left(L_{-n} L_{-2} + n \frac{n+4}{n+2} L_{-n-2} \right) \chi^{(n+2)} \right) \end{aligned} \quad (2.26)$$

plus supplementary terms alone. Varying ϕ we find the ϕ equation to be of the form

$$\begin{aligned} & \left(L_0 - 1 - \sum_{n=1}^{\infty} \frac{1}{2n} L_{-n} L_n \right) \psi \\ & + \sum_{n=1}^{\infty} \left(L_{-n} L_{-1} + n \frac{n+2}{n+1} L_{-n-1} \right) \phi^{(n+1)} \\ & + \sum_{n=2}^{\infty} \left(L_{-n} L_{-2} + n \frac{n+4}{n+2} L_{-n-2} \right) \chi^{(n+2)} = 0 \end{aligned} \quad (2.27)$$

From this equation we may read off the Λ_1 and Λ_2 gauge transformations to all orders. Substituting $\delta\psi = L_{-1} \Lambda_1$, we find

$$\begin{aligned} \delta_1 \psi &= L_{-1} \Lambda_1, \quad \delta_1 \phi^{(n+1)} = \frac{1}{2n} L_n \Lambda_1, \\ \delta_1 \chi^{(n)} &= 0 \end{aligned} \quad (2.28)$$

While substituting $\delta\psi = L_{-2} \Lambda_2$, we find

$$\begin{aligned}\delta_2 \phi^{(2)} &= \frac{3}{2} \Lambda_2, \quad \delta_2 \phi^{(3)} = \frac{1}{2} L_1 \Lambda_2, \\ \delta_2 \phi^{(n)} &= 0 \quad n \geq 4, \quad \delta \chi^{(n+2)} = \frac{1}{2n} L_n \Lambda_2.\end{aligned}\quad (2.29)$$

We could further simplify the above formulae by introducing two supplementary fields for each of the expressions $(L_1^2 + 3/2(L_2))\psi$ and $(L_1 L_2 + 8/3(L_3))\psi$. This corresponds to $\chi^{(2)}$, $\chi^{(3)}$ for the action of L_2 and $\phi^{(2)}$ and $\phi^{(3)}$ for the action of L_1 . The action, however, will only contain the combinations $\phi^{(2)} + \chi^{(2)}$ and $\phi^{(3)} + \chi^{(3)}$. This fact could be enforced by demanding that the action have the symmetries

$$\begin{aligned}\phi^{(2)} &\rightarrow \phi^{(2)} + \Omega^{(2)}, \quad \chi^{(2)} \rightarrow \chi^{(2)} - \Omega^{(2)}, \\ \phi^{(3)} &\rightarrow \phi^{(3)} + \Omega^{(3)}, \quad \chi^{(3)} \rightarrow \chi^{(3)} - \Omega^{(3)}.\end{aligned}\quad (2.30)$$

In this case the resulting gauge transformations are

$$\begin{aligned}\delta_1 \psi &= L_{-1} \Lambda_1, \quad \delta_1 \phi^{(n+1)} = \frac{1}{2n} L_n \Lambda_1, \\ \delta_1 \chi^{(n)} &= 0, \\ \delta_2 \psi &= L_{-2} \Lambda_2, \quad \delta_2 \phi^{(n)} = 0 \\ \delta_2 \chi^{(n+2)} &= \frac{1}{2n} L_n \Lambda_2, \quad n \geq 1 \\ \delta_2 \chi^{(2)} &= \frac{3}{2} \Lambda_2\end{aligned}\quad (2.31)$$

Henceforth we will regard $\chi^{(2)}$ and $\chi^{(3)}$ as being absent or gauged away by $\Omega^{(2)}$ and $\Omega^{(3)}$.

It is instructive to complete the squares in the action of Eq. (2.26). It becomes

$$\frac{1}{2} (\psi, (L_0 - 1) \psi) - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2n} (G^{(n)}, G^{(n)}) \quad (2.32)$$

+ supplementary fields terms ,

where

$$\begin{aligned} G^{(n)} = & L_n \psi - 2(n^2 - 1) \phi^{(n)} - 2n L_{-1} \phi^{(n+1)} \\ & - 2n L_{-2} \chi^{(n+2)} - 2(n^2 - 4) \chi^{(n)} \end{aligned} \quad (2.33)$$

Under Λ_1 all the $G^{(n)}$ are inert except $G^{(1)}$, while under Λ_2 they transform as

$$\begin{aligned} \delta_2 G^{(1)} &= L_{-2} L_1 \Lambda_2, \quad \delta_2 G^{(2)} = 4(L_0 + 1) \Lambda_2 - 2 L_{-1} L_1 \Lambda_2, \\ \delta_2 G^{(3)} &= -3 L_1 \Lambda_2, \end{aligned} \quad (2.34)$$

$$\delta_2 G^{(n)} = 0 \quad \text{for } n \geq 4.$$

Another useful combination is

$$\begin{aligned} \zeta^{(n+3)} = & n(n+2) \phi^{(n+3)} + (n+1) L_1 \phi^{(n+2)} \\ & - L_{n+1} \phi^{(1)} + 3(n+1) \chi^{(n+3)} \end{aligned} \quad (2.35)$$

This is also inert under Λ_1 and transforms under Λ_2 as

$$\delta_2 \zeta^{(4)} = L_1^2 \Lambda_2, \quad \delta_2 \zeta^{(n)} = 0, \quad n \geq 5 \quad (2.36)$$

It is now a simple matter to verify the invariance of the following action

at fourth level under the $\Lambda^{(1)}$ and $\Lambda^{(2)}$ gauge invariances

$$\begin{aligned} & \frac{1}{2} (\psi, (L_0 - 1) \psi) - \frac{1}{2} \sum_{n=1}^4 \frac{1}{2n} (G^{(n)}, G^{(n)}) \\ & - 2 (\phi^{(2)}, (L_0 + 1) \phi^{(2)}) + \frac{1}{2} (\xi^{(4)}, \xi^{(4)}) \\ & - 8 (\chi^{(4)}, (L_0 + 3) \chi^{(4)}) . \end{aligned} \quad (2.37)$$

The reader may wonder what the relation is between the above fourth level action and that given previously in Eq. (2.18). In fact, $\chi^{(4)}$ can be gauged away since the above action (2.37) possesses the additional symmetry

$$\begin{aligned} \delta \phi^{(2)} &= -2 L_{-2} \Omega^{(4)} , \quad \delta \phi^{(3)} = 2 L_{-1} \Omega^{(4)} \\ \delta \phi^{(4)} &= -\frac{4}{3} \Omega^{(4)} , \quad \delta \chi^{(4)} = 3 \Omega^{(4)} \end{aligned} \quad (2.38)$$

Upon gauging away $\chi^{(4)}$ we obtain the previous action of Eq. (2.18) and the previous transformations laws of Eq. (2.19) provided we make the necessary compensating transformations.

Having built up the above formalism, the fifth level is easily found. The action of Eq. (2.37) is Λ_1 and Λ_2 invariant provided we add the terms $12 (\chi^{(4)}, L_{-1} \chi^{(5)}) + 9 (\chi^{(5)}, \chi^{(5)}) + 4 (L_1 \chi^{(4)}, L_1 \chi^{(4)})$. The only Λ_1 and Λ_2 invariant term we may add to the action of Eq. (2.37) is

$$c (\xi^{(5)}, \xi^{(5)}) \quad (2.39)$$

as well as extending the sum of the G^n 's from 4 to 5. The coefficient c of this new term is easily found by demanding that the $\Omega^{(4)}$ symmetry inherits to the fifth level. One finds that $c = +2/3$.

The fifth level then has a new symmetry which is given by

$$\begin{aligned}
 \delta \phi^{(2)} &= -2 L_3 \Omega^{(5)} , \quad \delta \phi^{(3)} = 0 , \\
 \delta \phi^{(4)} &= 2 L_- \Omega^{(5)} , \quad \delta \phi^{(5)} = \frac{1}{2} \Omega^{(5)} \\
 \delta \chi^{(4)} &= 0 , \quad \delta \chi^{(5)} = 3 \Omega^{(5)}
 \end{aligned}
 \tag{2.40}$$

which allows one to gauge away $\chi^{(5)}$. This symmetry $\Omega^{(5)}$ and that of $\Omega^{(4)}$ are an invariance of the ψ equation of motion due to the identity (2.24).

The sixth level is presented in Appendix A.

We will now demonstrate that we can recover the equations (1.10) and (1.11) by a series of gauge choices. Of course, since one has an action invariant under arbitrary gauge transformation, we must choose a very specific gauge in order to arrive at Eqs. (1.10) and (1.11). Firstly, we gauge $\chi^{(n)} = 0$ using $\Omega^{(n)}$; we then use $\Lambda^{(2)}, \Lambda^{(3)}, \Lambda^{(4)} \dots$ to gauge away $\phi^{(2)}, \phi^{(3)}, \phi^{(4)} \dots$. Upon making the necessary compensating transformations, the resulting equations with $\phi^{(2)} = \phi^{(3)} = \phi^{(4)} = 0$ are still invariant under $\Lambda^{(1)}$. We may use $\Lambda^{(1)}$ to set $L_1 \psi = 0$. Substituting $L_1 \psi = \phi^{(2)} = \phi^{(3)} = \phi^{(4)} = \dots = 0$, we recover Eqs. (1.10) and (1.11). This demonstrates that the action constructed above does indeed describe the correct spectrum for the open bosonic string.

2b) Closed Strings

The closed string $x^\mu(\sigma)$ can be written in the form

$$X^\mu(\sigma) = \sum_{n=-\infty}^{+\infty} e^{in\sigma} x_n^\mu
 \tag{2.41}$$

where $x_n^\mu = x_{-n}^\mu$. We define the creation and destruction operators by

$$\hat{p}^\mu(\sigma) = -i \frac{\delta}{\delta X^\mu} + \frac{1}{2\pi\alpha'} \frac{\partial X^\mu}{\partial \sigma} = \frac{1}{\pi(2\alpha')^{1/2}} \sum_{n=-\infty}^{+\infty} e^{in\sigma} \alpha_n^\mu$$

$$\hat{p}^\mu(\sigma) = -i \frac{\delta}{\delta x^\mu} - \frac{1}{2\pi\alpha'} \frac{\partial x^\mu}{\partial \sigma} = \frac{1}{\pi(\alpha')^{1/2}} \sum_{n=-\infty}^{+\infty} e^{-in\sigma} \bar{\alpha}_n^\mu \quad (2.42)$$

The Virasoro operators are given by

$$L_n = \frac{1}{2} \pi \alpha' \int_{-\pi}^{\pi} d\sigma e^{-in\sigma} \hat{p}^\mu(\sigma)^2 = \frac{1}{2} \sum_{m=-\infty}^{+\infty} \alpha_m^\mu \alpha_{n-m}^\mu$$

$$\bar{L}_n = \frac{1}{2} \pi \alpha' \int_{-\pi}^{\pi} d\sigma e^{in\sigma} \hat{\bar{p}}^\mu(\sigma)^2 = \frac{1}{2} \sum_{m=-\infty}^{+\infty} \bar{\alpha}_m^\mu \bar{\alpha}_{n-m}^\mu \quad (2.43)$$

The α 's satisfy the relations

$$[\alpha_m^\nu, \alpha_n^{\mu\dagger}] = n \delta_{n,m} \delta^{\nu\mu}, \quad [\bar{\alpha}_m^\nu, \bar{\alpha}_n^{\mu\dagger}] = n \delta_{n,m} \delta^{\nu\mu}$$

for $n, m \geq 1$, (2.44)

$$[\bar{\alpha}_m^\nu, \alpha_n^\mu] = 0.$$

It is well known that the on-shell states of the bosonic oriented closed string are described by the equations

$$(L_0 + \bar{L}_0 - 2) \psi = 0,$$

$$(L_0 - \bar{L}_0) \psi = 0, \quad L_n \psi = \bar{L}_n \psi = 0, \quad n \geq 1 \quad (2.45)$$

At the first level, these equations have a massless "pure spin 2" and one spin zero. As discussed in the Introduction, these states can only be described in a local, gauge- and Lorentz-covariant way by the fields $h_{\mu\nu}$ and ϕ . Since the string field ϕ only contains $h_{\mu\nu}$ at the first level, we must add a supplementary string field $\phi^{(1)}$. The first level result is found in much the same way as for

the open string; we begin by relaxing some of the constraints on ψ , which is now subject to

$$(\bar{L}_0 - L_0) \psi = L_1^2 \psi = L_2 \psi = \bar{L}_1^2 \psi = \bar{L}_2 \psi = 0 \quad (2.46)$$

as well as

$$L_1 \phi^{(1)} = \bar{L}_1 \phi^{(1)} = 0$$

The action is found to be

$$\begin{aligned} & \frac{1}{2} (\psi, (L_0 + \bar{L}_0 - 2 - L_1, L_1 - \bar{L}_1, \bar{L}_1) \psi) \\ & + (\psi, L_1, \bar{L}_1, \phi^{(1)}) - \frac{1}{2} (\phi^{(1)}, (L_0 + \bar{L}_0) \phi^{(1)}) \end{aligned} \quad (2.47)$$

and is invariant under the transformations

$$\delta_1 \psi = L_1 \bar{\Lambda}_1 + \bar{L}_1 \Lambda_1, \quad \delta_1 \phi^{(1)} = L_1 \Lambda_1 + \bar{L}_1 \bar{\Lambda}_1 \quad (2.48)$$

where the gauge parameters Λ_1 and $\bar{\Lambda}_1$ are subject to

$$\begin{aligned} & (L_0 - \bar{L}_0 + 1) \bar{\Lambda}_1 = 0, \quad (L_0 - \bar{L}_0 - 1) \Lambda_1 = 0, \\ & L_1 \bar{\Lambda}_1 = 0, \quad \bar{L}_1 \Lambda_1 = 0, \\ & \bar{L}_2 \bar{\Lambda}_1 = \bar{L}_1 \bar{\Lambda}_1 = 0, \quad L_2 \Lambda_1 = L_1 \Lambda_1 = 0 \end{aligned} \quad (2.49)$$

The action of Eq. (2.47) at the first mass level can be evaluated in terms of the component fields $h_{\mu\nu}$ and ϕ , and is non other, modulo a trivial field redefinition, than the linearized Einstein plus scalar action.

The treatment of higher levels proceeds in essentially the same way as for the open string. We treat here only the next (second) level, where the "spin" can be as high as four. One first arranges the $\Lambda_1, \bar{\Lambda}_1$ gauge invariances, relaxing appropriately the conditions (2.49). One then constructs an action invariant under $\Lambda_1, \bar{\Lambda}_1$, introducing the necessary supplementary fields. After that, one imposes the new $\Lambda_2, \bar{\Lambda}_2$ invariances. The conditions (2.49) are now replaced by

$$\begin{aligned} 0 &= L_2 \bar{\Lambda}_1 = L_1^2 \bar{\Lambda}_1 = \bar{L}_2 \Lambda_1 = \bar{L}_1^2 \Lambda_1 = L_1 \bar{\Lambda}_2 = \bar{L}_1 \Lambda_2 \\ &= L_2 \bar{\Lambda}_2 = \bar{L}_2 \Lambda_2 \end{aligned} \quad (2.50)$$

Upon applying this procedure, one finds that five new supplementary fields (ϕ_2 , $\bar{\phi}_2$, ϕ_2' , χ_2 , $\bar{\chi}_2$) are needed at the second level, with transformation laws

$$\begin{aligned} \delta_1 \phi_2 &= L_1 \phi_1, \quad \delta_1 \bar{\phi}_2 = \bar{L}_1 \Lambda_1, \quad \delta_1 \chi_2 = L_2 \Lambda_1, \\ \delta_1 \bar{\chi}_2 &= \bar{L}_2 \bar{\Lambda}_1, \quad \delta_1 \phi_2' = 0, \quad \delta_2 \phi_2 = L_2 \bar{\Lambda}_2 + \bar{L}_2 \Lambda_2 \\ \delta_2 \bar{\phi}_2 &= 3 \bar{\Lambda}_2, \quad \delta_2 \bar{\phi}_2 = 3 \Lambda_2, \quad \delta_2 \chi_2 = 2 \bar{L}_1 \bar{\Lambda}_2 \quad (2.51) \\ \delta_2 \bar{\chi}_2 &= 2 L_1 \Lambda_2, \quad \delta_2 \phi_2' = L_2 \Lambda_2 + \bar{L}_2 \bar{\Lambda}_2 \end{aligned}$$

At this level, the invariant action is

$$\begin{aligned} &\frac{1}{2} (\psi, (L_0 + \bar{L}_0 - 2 - L_1, L_1 - \bar{L}_1, \bar{L}_1 - \frac{1}{2} L_2 L_2 - \frac{1}{2} \bar{L}_2 \bar{L}_2) \psi) + (\psi, L_1, \bar{L}_1, \phi_1) \\ &- \frac{1}{2} (\phi_1, (L_0 + \bar{L}_0) \phi_1) + \frac{1}{2} (\psi, \bar{L}_2 L_2 \phi_2') - \frac{1}{2} (\phi_2', (L_0 + \bar{L}_0 + 2) \phi_2') \\ &+ (\psi, (\bar{L}_1^2 + \frac{3}{2} \bar{L}_2) \bar{\phi}_2) + \frac{1}{2} (\psi, L_1, \bar{L}_2 \bar{\chi}_2) + \frac{1}{2} (\psi, \bar{L}_1, L_2 \chi_2) \\ &- (\phi_1, L_1, \bar{L}_1, \phi_2) - (\phi_1, \bar{L}_1, L_1, \bar{\phi}_2) - (\bar{\phi}_2, (L_0 + \bar{L}_0 + \frac{5}{4}) \bar{\phi}_2) \quad (2.52) \\ &+ \frac{1}{2} (\bar{\phi}_2, L_1, L_1, \bar{\phi}_2) - \frac{3}{2} (\bar{\phi}_2, L_1, \bar{\chi}_2) - (\phi_2, (L_0 + \bar{L}_0 + \frac{5}{4}) \phi_2) + \frac{1}{2} (\phi_2, \bar{L}_1, \bar{L}_1, \phi_2) \\ &- \frac{3}{2} (\phi_2, \bar{L}_1, \chi_2) - \frac{1}{4} (\bar{\chi}_2, (L_0 + \bar{L}_0 + 1) \bar{\chi}_2) - \frac{1}{4} (\bar{\chi}_2, L_1, L_1 \bar{\chi}_2) - \frac{1}{4} (\chi_2, (L_0 + \bar{L}_0 + 1) \chi_2) \\ &- \frac{1}{4} (\chi_2, \bar{L}_1, \bar{L}_1, \chi_2) - \frac{3}{2} (\phi_2', \bar{L}_2 \phi_2 + L_2 \bar{\phi}_2) + \frac{1}{4} (\bar{\chi}_2 L_1 - \phi_2 \bar{L}_2, L_1 \bar{\chi}_2 - \bar{L}_2 \phi_2) \\ &+ \frac{1}{4} (\chi_2 \bar{L}_1 - \bar{\phi}_2 L_2, \bar{L}_1 \chi_2 - L_2 \bar{\phi}_2) - \frac{1}{2} (\phi_2', (L_0 + \bar{L}_0 + 2) \phi_2') \end{aligned}$$

3. INTERACTIONS

In this section, we will demonstrate how to construct the interactions for the open bosonic string and carry out the first step of this procedure. To find the interacting theory from its linearized form, we will employ a generalization of the Noether method. In this method, one starts with the linearized theory which possesses Abelian local invariances and a similar number of rigid invariances. One then makes the rigid invariances local and finds a invariant non-linear action order by order in the coupling constant. This is achieved by knitting together the rigid and local Abelian invariances.

For the linearized theory given in the previous section, the linearized Abelian invariances are the transformations with parameters $\Lambda^{(1)}$, $\Lambda^{(2)}$, $\Lambda^{(3)}$, etc. The rigid invariances are the usual rigid rotations of the fields under the group G . In fact, there are an infinite set of such invariances, since we can make an independent rotation at every mass level. The first such invariance is the one which rotates all mass levels in the same way; it is given by

$$\delta\psi = [\Omega^{(1)}, \psi] \quad (3.1)$$

where $\Omega^{(1)}$ is independent of $x^\mu(\sigma)$. Let us denote the n th rigid rotation by $\Omega^{(n)}$. $\Omega^{(n)}$ is an infinitesimal antiHermitian matrix in the fundamental representation of the group acting on the Chan-Paton indices carried by the antiHermitian matrix ψ . The first step is to make this rigid invariance into a local one, which is most easily achieved by working in the operator formalism. We regard ψ as a ket $|\psi\rangle$ and promote $\Omega^{(1)}$ [$\Omega^{(n)}$] to also be a ket $|\Omega^{(1)}\rangle$ [$|\Omega^{(n)}\rangle$]. The variation of $|\psi\rangle$ is given by

$$\delta|\psi\rangle = \langle\Omega_c| \langle\psi| T'_1 + \langle\psi_c| \langle\Omega_c| T_2 \quad (3.2)$$

where T'_1 and T_2 are operators of the form

$$T_i = f_i(\alpha_a, \alpha_b, \alpha_c) |0_a\rangle |0_b\rangle |0_c\rangle \quad (3.3)$$

which are to be determined. The labels a , b and c correspond to the three "legs" of T_i which are illustrated in Fig. 1:

Fig 1

There also exist similar equations for the supplementary fields $|\phi^{(n)}\rangle$. In this sense, the method we are using represents a generalization of the usual Noether method.

Upon substituting the now-local variations of $|\psi\rangle$ of Eq. (3.2) and those for

$|\phi^{(n)}\rangle$ into the linearized action, we will obtain an expression which is linear in $\langle \Omega^{(n)} |$ and bilinear in $|\psi\rangle$ and $|\phi^{(n)}\rangle$. The action of the L_n 's which make up the linearized action does not, in general, obey the Liebnitz rule. Hence the T's will have to be very special operators, so that the action of the L_n 's produces further L_n 's on the other legs.

In the discussion which follows, we will only refer to $\Omega^{(1)}$, but the extension to $\Omega^{(n)}$ is in principle clear. The variation of the linearized action can then be written in the form

$$\langle \Omega^{(1)} | L_n S^{(n)} \quad (3.4)$$

where $S^{(n)}$ is an expression bilinear in $|\psi\rangle$ and $|\phi^{(n)}\rangle$. This variation of the linearized action can be cancelled by the addition of a term of order g to the action. The variation of this term at order g^0 results from only the variation of the fields under the Abelian invariances whose parameter $\Lambda^{(1)}$ is identified by

$$|\Lambda^{(1)}\rangle = \frac{1}{g} |\Omega^{(1)}\rangle \quad (3.5)$$

For example, a term of the form

$$\langle \Omega^{(1)} | L_1 S^{(1)} \quad (3.6)$$

is cancelled by the addition of the term of the form

$$\langle \psi | S^{(1)} \quad (3.7)$$

The variations of this new action at order g are then cancelled by adding new terms to the action and possibly new terms to the transformation laws. This is repeated until one obtains an action invariant to all orders in g .

From the light-cone formulation of the interacting string field theory and the geometrical interpretation of interactions such as splitting, joining and rearrangement of strings, one can expect at most quartic interactions for open strings and cubic ones for closed strings also in a gauge covariant formulation. However, the vertices will necessarily involve the supplementary fields $\phi^{(n)}$. The occurrence of interactions at least quadratic in $\phi^{(n)}$ results in the practi-

cal inability to integrate out the supplementary fields in the path integral. Such terms are inevitable; for example, there exists in the closed string sector the usual graviton-massless scalar coupling, even before taking the field theory limit, hence a non-zero $\phi^{(1)}\phi\phi^{(1)}$ interaction term. This shows that the non-local formulation of the free string theory of Refs. 9) and 10) can definitely not be generalized to the interacting case.

The above procedure can be carried out level by level, and we will now find the order g interactions at the first level. We begin with the action

$$\frac{1}{2} \langle \psi | (L_0 - 1 - \frac{1}{2} L_1, L_1) | \psi \rangle \quad (3.8)$$

Using the variation of $|\psi\rangle$ of Eq. (3.2), we find

$$\langle \psi_e | (L_0 - 1 - \frac{1}{2} L_1, L_1)_e \rangle \{ \langle \psi_a | \alpha_{-1} | T_1 \rangle + \langle \alpha_{-1} | \psi_c | T_2 \rangle \} \quad (3.9)$$

This must be expressible in the form

$$\langle \alpha_{-1} | L_1^a \rangle \langle \psi_e | \langle \psi_c | W \rangle \quad (3.10)$$

which we can cancel with the variation of the term

$$\frac{1}{3} g \langle \psi_a | \langle \psi_e \rangle \langle \psi_c | W \rangle \quad (3.11)$$

Comparing Eqs. (3.9) and (3.10), we require the operators T_1 and W to satisfy

$$\begin{aligned} L_1^a W = & (L_0 - 1 - \frac{1}{2} L_1, L_1)_e T_1 \\ & + (L_0 - 1 - \frac{1}{2} L_1, L_1)_c T_2 \end{aligned} \quad (3.12)$$

Long ago, a cyclically symmetric three-point vertex was written down¹⁴⁾ in the form

$$V = \exp - \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (\alpha_{-n}^a \alpha_{-m}^b + \alpha_{-n}^b \alpha_{-m}^c + \alpha_{-n}^c \alpha_{-m}^a) \frac{(-1)^m \Gamma(n)}{\Gamma(m+1) \Gamma(n-m+1)} |0\rangle \quad (3.13)$$

It is instructive to evaluate V, taking the external states to contain only the massless Yang-Mills vectors; one finds the result

$$\begin{aligned} & k_1 \cdot \varepsilon_3 \varepsilon_1 \cdot \varepsilon_2 + k_2 \cdot \varepsilon_1 \varepsilon_2 \cdot \varepsilon_3 + k_3 \cdot \varepsilon_2 \varepsilon_1 \cdot \varepsilon_3 \\ & + k_1 \cdot \varepsilon_3 k_2 \cdot \varepsilon_1 k_3 \cdot \varepsilon_2 \end{aligned} \quad (3.14)$$

The last term, of course, disappears in the zero slope limit. This vertex differs from one which is antisymmetric under the interchange of any two external lines by longitudinal terms which are:

$$\begin{aligned} & \varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_3^\rho \left[\frac{1}{2} k_1^\mu \delta_{\nu\rho} + \frac{1}{2} k_2^\nu \delta_{\mu\rho} + \frac{1}{2} k_3^\rho \delta_{\mu\nu} \right. \\ & - \frac{1}{2} (k_1^\mu k_3^\nu k_2^\rho + k_3^\rho k_2^\mu k_1^\nu + k_2^\nu k_1^\rho k_3^\mu) \\ & \left. + \frac{1}{2} k_1^\mu k_2^\nu k_3^\rho \right] \end{aligned} \quad (3.15)$$

We therefore take our operators to be of the form

$$\begin{aligned} W = & \left[1 + \alpha (L_{-1}^a + L_{-1}^b + L_{-1}^c) + \beta (L_{-1}^b L_{-1}^c + L_{-1}^a L_{-1}^b + L_{-1}^c L_{-1}^a) \right. \\ & \left. + \delta L_{-1}^a L_{-1}^b L_{-1}^c \right] V \end{aligned}$$

$$T_1 = \gamma (1 + \alpha' L_{-1}^c) V \quad (3.16)$$

$$T_2 = \varepsilon (1 + \alpha'' L_{-1}^b) V$$

These are the most general operators, taking into account the constraints for the first level written in the previous section:

$$L_1 |\Omega\rangle = L_2 |\psi\rangle = L_2^2 |\phi\rangle = \dots = 0 \quad (3.17)$$

Using the result¹⁵⁾

$$(L_1^a - L_0^a + L_{-1}^b - L_0^b + L_0^c) V = 0, \quad (3.18)$$

and the analogous results obtained by cyclically permuting a, b and c, we can evaluate L_1^a on W and fix the above coefficients. One finds

$$\begin{aligned} W = & \left[1 - \frac{1}{2} (L_{-1}^a + L_{-1}^b + L_{-1}^c) + \frac{1}{2} (L_{-1}^a L_{-1}^b + L_{-1}^b L_{-1}^c + L_{-1}^c L_{-1}^a) \right. \\ & \left. - \frac{1}{2} L_{-1}^a L_{-1}^b L_{-1}^c \right] V, \\ T_1 = & V, \\ T_2 = & -(1 - L_{-1}^b) V, \end{aligned} \quad (3.19)$$

where

$$T_1'(b, a, c) = T_1(c, b, a) \quad (3.20)$$

At this level, T_1' happens to be cyclically symmetric, and the permutation (3.20) is irrelevant. It may become relevant in next order.

Hence the order g result at the first level is

$$\begin{aligned} & \frac{1}{2} \langle \psi | L_{-1} - \frac{1}{2} L_{-1} L_{-1} | \psi \rangle \\ & - \frac{1}{3} g \langle \psi_a | \langle \psi_b | \langle \psi_c | W \end{aligned} \quad (3.21)$$

This action is invariant to order g^0 under

$$\begin{aligned} \delta |\psi_0\rangle = & \frac{1}{g} L_{-1}^b |\Omega_b\rangle + \langle \psi_a | \langle \Omega_c | T_1' \\ & + \langle \Omega_a | \langle \psi_c | T_2 \end{aligned} \quad (3.22)$$

In the literature of fifteen years ago on dual models, there appeared other three-Reggeon vertices. One, $\tilde{W}$, is just W with the opposite cyclic ordering of external lines. Both W and $\tilde{W}$ were obtained from the Sciuto¹⁶⁾ vertex S by applying appropriate twist operators Ω_{twist} , according to Fig. 2 and Refs. 14) and 15). It has been shown in Ref. 17) that one obtains in the zero slope limit the correct Feynman diagrams of a field theory by summing over twisted and untwisted internal lines of the string diagram. This result shows that one should include

all four vertices $W, \tilde{W}, S, \tilde{S}$ of Fig. 2 (suitably modified by L_n 's in the manner shown above for W) in the action and transformation laws. The relative coefficients must be determined by closure of the algebra of gauge transformations. From (3.22) and (3.19), it is obvious that the structure constants of this algebra are essentially given by the matrix elements of the three-Reggeon vertices. This will be dealt with in a future publication.

4. OUTLOOK

Some novel qualitative features emerge from the analysis of the previous section which we expect to carry to all orders in spite of technical complications. These features are best seen and explained by going to functional form. In such a form, under $\Lambda^{(n)}$, the gauge transformation of ψ may be written as

$$\begin{aligned} \delta\psi &= \frac{1}{2}\pi\alpha' \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} d\sigma e^{in\sigma} p^2(\sigma) \Lambda^{(n)} \\ &= \frac{1}{2}\pi\alpha' \int_{-\pi}^{\pi} p^2(\sigma) \Lambda d\sigma \end{aligned} \quad (4.1)$$

where we have set

$$\Lambda \equiv \Lambda[X^\mu(\sigma), \sigma] = \sum_{n=1}^{\infty} e^{in\sigma} \Lambda^{(n)}[X^\mu(\sigma)] \quad (4.2)$$

Equation (4.2) collects all $\Lambda^{(n)}$ gauge transformations into a single one by adding an explicit dependence on σ of the string field Λ .

The transformation of the supplementary fields $\phi^{(n)}$ under $\Lambda^{(1)}$ is

$$\delta\phi^{(n+1)} = \frac{1}{2n} L_n \Lambda^{(1)} \quad (4.3)$$

and we may rewrite this as

$$\delta\phi = \alpha' \pi^2 p^2 \Lambda^{(1)} \quad (4.4)$$

where

$$\phi \equiv \phi[X^\mu(\sigma), \sigma] = \sum_{n=1}^{\infty} e^{in\sigma} \frac{1}{2n} \phi^{(n+1)}[X^\mu(\sigma)] \quad (4.5)$$

There exist similar formulae for $\chi^{(n)}$ and $\Lambda^{(2)}$. Hence, the infinite set of $\Lambda^{(n)}$ gauge transformations will collect into one with parameter $\Lambda[x^\mu(\sigma), \sigma]$, and the infinite set of supplementary fields $\phi^{(n+1)}$ will be contained in $\phi[x^\mu(\sigma), \sigma]$.

It has been clear for some time that there must exist some new non-Abelian invariance principle that determines the structure of interactions in string theories. This invariance will hopefully follow from a new geometric principle and have a conceptually simple form. We note that the procedure outlined in the previous section on string interactions in principle determines it, and we remarked that the structure constants are intimately related to the matrix elements of the three-Reggeon vertex. We also find that the parameter of the gauge transformation is of the functional form $\Lambda[x^\mu(\sigma), \sigma]$.

In the action constructed above, there are arbitrarily high spins which are consistently coupled together in a gauge- and Lorentz-invariant way. Writing this out in terms of component fields would reveal how the consistency occurs and how the difficulties encountered previously are overcome.

Possessing the interacting action for all levels would allow for a straightforward quantization of the theory. One selects a gauge and introduces Faddeev-Popov ghosts in the normal way. At the linearized level, one would recover the results of Ref. 4) in an appropriate gauge.

Having introduced gauge invariances corresponding to the two-dimensional conformal group, one might wonder about their anomalies. For example, starting from bosonic open strings, one should find that closed strings are needed for consistency at the one-loop level. The $SO(32)$ anomaly cancellation of Ref. 18) must be one of an infinite series of cancellations arising from the gauging of the superconformal two-dimensional group; the application of the methods of this paper to the covariant formulation of superstrings is under investigation¹⁹⁾.

Phenomenological applications of superstrings require dimensional reduction by spontaneous compactification. Armed with covariant string equations of motion, one could look for classical exact solutions and examine their properties.

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APPENDIX A

We will now extend the results of Section 2 to the sixth excited level of the open string. The only new Λ_1 and Λ_2 invariant combinations apart from $G^{(6)}$ at this level are

$$\zeta^{(6)} = 15 \phi^{(6)} + 4 L_1 \phi^{(5)} - L_4 \phi^{(2)} + 12 \chi^{(5)} \quad (A1)$$

and

$$\zeta'^{(6)} = 5 \phi^{(6)} + 3 L_2 \phi^{(4)} - 2 L_3 \phi^{(3)} + 16 \chi^{(6)} + 6 L_1 \chi^{(5)} \quad (A2)$$

The only term we can add to the action which is only bilinear in derivatives and Λ_1 and Λ_2 invariant at this level is

$$\begin{aligned} \frac{1}{2} a (\zeta^{(6)}, \zeta^{(6)}) + b (\zeta^{(6)}, \zeta'^{(6)}) \\ + \frac{1}{2} c (\zeta'^{(6)}, \zeta'^{(6)}) \end{aligned} \quad (A3)$$

Applying the $\Omega^{(5)}$ symmetry of Eq. (2.38), we find that the action is invariant provided that

$$9b + 4a = 2 \quad \text{and} \quad 8b + 18c = 6 \quad (A4)$$

The $\Omega^{(4)}$ symmetry does require some corrections terms, which only affect this level, before it can be a symmetry. The correct $\Omega^{(4)}$ transformations are

$$\begin{aligned} \delta \phi^{(2)} &= -2 L_{-2} \Omega^{(4)} + \frac{1}{3} L_{-4} L_2 \Omega^{(4)}, \\ \delta \phi^{(3)} &= 2 L_{-1} \Omega^{(4)}, \quad \delta \phi^{(4)} = -\frac{4}{3} \Omega^{(4)} \\ \delta \phi^{(5)} &= -\frac{1}{3} L_{-2} L_2 \Omega^{(4)}, \quad \delta \phi^{(6)} = -\frac{2}{3} L_2 \Omega^{(4)}, \\ \delta \chi^{(4)} &= 3 \Omega^{(4)}, \quad \delta \chi^{(5)} = -\frac{1}{2} L_2 \Omega^{(4)} \end{aligned} \quad (A5)$$

and they form an invariance provided

$$a = \frac{1}{2} \quad \text{and} \quad b = \frac{1}{6} \quad (\text{A6})$$

The resulting action possesses a further symmetry which can be used to gauge away $\chi^{(6)}$. It is of the form

$$\begin{aligned} \delta \phi^{(2)} &= -2 L_{-4} \Omega^{(6)} - 2 L_{-2} \Omega', \quad \delta \phi^{(3)} = 2 L_{-1} \Omega', \\ \delta \phi^{(4)} &= -\frac{4}{3} L_{-1} \Omega', \quad \delta \phi^{(5)} = 2 L_{-1} \Omega^{(6)} \\ \delta \phi^{(6)} &= \frac{12}{5} \Omega^{(6)}, \quad \delta \chi^{(3)} = 3 \Omega', \\ \delta \chi^{(6)} &= 3 \Omega^{(6)} \end{aligned} \quad (\text{A7})$$

$$\text{where} \quad \Omega' = \frac{4}{13} (L_{-1}^2 + \frac{3}{2} L_{-2}) \Omega^{(6)}$$

We now give the complete action at sixth level which is invariant under $\Lambda^{(1)}$ $\Lambda^{(6)}$ and $Q^{(2)}$ $Q^{(6)}$:

$$\begin{aligned} A = & \frac{1}{2} (\psi, (L_0 - 1) \psi) - 2 (\phi^{(2)}, (L_0 + 1) \phi^{(2)}) \\ & - \frac{1}{2} \sum_{n=1}^6 \frac{1}{2n} (G^{(n)}, G^{(n)}) + \sum_{n=4}^6 \frac{1}{n-2} (\zeta^{(n)}, \zeta^{(n)}) \\ & + \frac{1}{12} (\zeta^{(6)}, \zeta^{(6)}) - 8 (\chi^{(4)}, (L_0 + 3 + \frac{1}{2} L_{-1} L_1) \chi^{(4)}) \quad (\text{A8}) \\ & + 12 (\chi^{(4)}, L_1 \chi^{(5)}) + 9 (\chi^{(5)}, \chi^{(5)}) \end{aligned}$$

One may expect the transformations of Eqs. (A5) and (A7) to simplify with the introduction of further gauged-away supplementary fields. The general pattern begins to emerge at this level and will be reported elsewhere.

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$$\delta |\psi_a\rangle = a \begin{array}{c} \text{---} \backslash \text{---} \\ \text{---} / \text{---} \\ T_2 \end{array} \begin{array}{c} |\Omega_b\rangle \\ |\psi_c\rangle \end{array} + a \begin{array}{c} \text{---} \backslash \text{---} \\ \text{---} / \text{---} \\ T_1 \end{array} \begin{array}{c} |\psi_b\rangle \\ |\Omega_c\rangle \end{array}$$

Fig. 1

Fig. 2