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σ -MODELS, DUALITY TRANSFORMATIONS AND SCALAR POTENTIALS

IN EXTENDED SUPERGRAVITIES

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A B S T R A C T

We develop a technique to compute scalar potentials in extended supergravities in any dimension on a pure group-theoretical basis. The fermionic supersymmetry variations, which determine the potential, only depend on boosted structure constants of a gauge subgroup $K \subset G$ of the isometry group G of the scalar manifold G/H . These variations follow from the Bianchi identities of the underlying gauge superalgebra. We apply this method to construct the coupling of $N = 3$ matter to supergravity.

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Extended supergravity theories have the unique feature that a scalar potential can exist only if a subgroup K of G is gauged, G being the full isometry group of the scalar manifold [1,2,3]. This is in contrast with $N=1$ supergravity where a scalar potential can be obtained even in the absence of any gauge coupling [4].

For $N \geq 3$ extended supergravities the scalar manifold has always the structure of a coset manifold G/H where G is some non-compact group and $H = SU(N) \times H'$ is a maximal compact subgroup.

For $N > 4$ H' can be at most $U(1)$, while for $N=3,4$ $\dim H'$ is arbitrary depending on the number of the matter multiplets coupled to supergravity. Since matter multiplets for $N > 2$ extended supergravity are just vector multiplets the group G is also the group of duality transformations on the vector field strengths and their duals.

According to a general argument due to Gaillard and Zumino [5] G must be a subgroup of $Sp(2k)$ where k is the number of vector fields. For $N=3$ $k=3+n$, for $N=4$ $k=6+n$ where n is the number of vector multiplets. From the dimension of the scalar manifold and the structure of the isotropy group H one can conclude that natural candidates are [6]

$$\frac{SU(3, m)}{SU(3) \times SU(m) \times U(1)} \quad \text{for } N=3 \text{ and} \quad \frac{SO(6, m)}{SO(6) \times SO(m)} \times \frac{SU(1,1)}{U(1)}$$

for $N=4$ Supergravity. The coupling of $N=4$ matter to supergravity has been recently worked out [2] and the coupling of $N=3$ matter to supergravity is derived here. The full $N=3$ Lagrangian will appear elsewhere [7].

In the present paper we use the group-manifold approach [8] as a key technique for deriving the scalar potential of any such theories without any need of the Noether procedure which has been heavily used to derive all previous couplings in $N > 2$ extended supergravities. This is possible because the group manifold technique allows to compute the field transformation rules under local supersymmetry variations through the Bianchi identities of the underlying gauge superalgebras.

To be more specific let us consider the set of curvature two-forms

of the N-extended supergravity algebra:

$$R^a = \mathcal{D}V^a - i \bar{\Psi}^A \wedge \gamma^a \Psi_A \quad (1)$$

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b \quad (2)$$

$$\mathcal{P}_A = \mathcal{D}\Psi_A \quad (3)$$

$$F_{AB} = dA_{AB} - \bar{\Psi}_A \wedge \Psi_B - \bar{\Psi}^A \wedge \Psi_B \quad (4)$$

where V^a , Ψ_A and A_{AB} are respectively the vierbein, (chiral) gravitino and central charge one-forms. Here $a, b, \dots$ refer to Lorentz indices and $A, B, \dots$ to $SU(N)$ indices.

If the supergravity multiplet contains scalar fields, these belong to the four-fold antisymmetric representation of $SU(N)$ while the scalar fields of the vector multiplets belong to the two-fold antisymmetric representation of $SU(N)$. Scalar fields simultaneously occur in the supergravity and matter multiplets only for $N=4$, in which case the scalar manifold is reducible. In $N=3$ the scalar fields are in the fundamental $SU(3)$ representation and the scalar manifold is the irreducible Kähler-Grassmann manifold $\frac{SU(3, n)}{SU(3) \times SU(n) \times U(1)}$, n being the number of vector multiplets.

Although our approach is general, we confine ourselves to the $N=3$ case since this is the only coupling which was not previously derived by the Noether method.

The general procedure will consist of introducing geometrical objects associated with the scalar manifold, corresponding to its vierbein and curvatures. The main point, however, is that the coordinates of the manifold are superfields since they provide a mapping of the superspace (whose anholonomic reference frames are the vierbein and the gravitino) into the coset manifold G/H .

Let us discuss the particle field content of the $N=3$ matter coupled theory. The $N=3$ Supergravity multiplet field content is given by the connection one-forms V^a , Ψ_A , A_{AB} introduced in eqs. (1)-(4) and an additional spin $\frac{1}{2}$ zero-form χ which is an $SU(3)$ singlet [9,10].

The n vector multiplets consist of n connection one-forms A^i , a triplet of spin $\frac{1}{2}$ λ^i , a singlet spin $\frac{1}{2}$ λ^i and a triplet of (complex) scalar fields z_A^i . As already discussed, the interaction of $N=3$ supergravity with n vector multiplets gives rise to a σ -model with isometry $SU(3,n)$. We can therefore introduce coset representatives [11] $L_\Lambda^Z(z)$ which associate to a point z of G/H an $SU(3,n)$ matrix L belonging to the equivalence class corresponding to z .

Since the self-dual $3+n$ field strengths

$$F_{\mu\nu}^{\hat{}} (F_{\mu\nu}^c = \epsilon^{AB} F_{\mu\nu AB}, F_{\mu\nu}^i) \quad (5)$$

are assigned to the fundamental representation of G , L is a $(3+n) \times (3+n)$ matrix. In $SU(3,n)$ we choose as $SU(3,n)$ invariant metric the matrix

$$J_{\Lambda\Sigma} = J^{\Lambda\Sigma} = \begin{pmatrix} \delta_{AB} & 0 \\ 0 & -\delta_{ij} \end{pmatrix} \text{ so that we have} \quad L^\dagger J L = J \quad (6)$$

As well known, the isometry group G is only an invariance of the equations of motion and not of the action, because G acts with duality transformations on the field strengths [12]. The largest symmetry of the Lagrangian is the non-compact subgroup $SO(3,n) \subset SU(3,n)$, which does not mix the electric and magnetic fields associated to the $3+n$ vectors.

Up to this point the $3+n$ vectors gauge an abelian group $U(1)^{3+n}$. However it is possible to gauge non-trivial subgroup K of $SO(3,n)$ of dimension $3+n$. In this paper we consider only semisimple groups K . Other situations will be discussed elsewhere.

The subgroup K to be gauged has to fulfil the following requirements. The fundamental representation D of $SU(3,n)$ must split as

$$D \xrightarrow{K} \text{adj} \oplus \text{adj} \quad (7)$$

Furthermore K must preserve the metric J , which therefore can be identified with the Killing metric of K . This amounts to saying that the structure constants $f_{\Lambda\Sigma}^{\Gamma}$ of K become fully antisymmetric by lowering the upper index with the metric $J_{\Lambda\Sigma}$:

$$f_{\Lambda\Sigma\Gamma} = f_{\Lambda\Sigma}^{\Gamma'} J_{\Gamma\Gamma'} = f_{[\Lambda\Sigma\Gamma]} \quad (8)$$

The only possibilities for K are readily found to be the semisimple groups

$$\begin{aligned} K &= SO(3) \times K_m \\ K &= SO(3,1) \times K_{m-3} \end{aligned} \quad (9)$$

where K_n is a semisimple group of dimension n.

We now introduce the gauged Maurer-Cartan one-forms

$$\Omega_\Lambda^\pi = (L^{-1})_\Lambda^\Sigma dL_\Sigma^\pi + g(L^{-1})_\Lambda^\Sigma \int_\Sigma^{\Omega^\pi} A_\Omega L_\pi^\pi \quad (10)$$

where g is the K-gauge coupling constant and for $g=0$ Ω reduces to the ordinary left-invariant form over G/H. For $g \neq 0$ Ω no longer satisfies the Maurer-Cartan equations and develops a non-vanishing $SU(3,n)$ curvature

$$d\Omega_\Lambda^\pi + \Omega_\Lambda^\pi \wedge \Omega_\pi^\pi = g(L^{-1})_\Lambda^\Sigma \int_\Sigma^{\Omega^\pi} \mathcal{F}_\Sigma L_\pi^\pi \quad (11)$$

where

$$\mathcal{F}_\Sigma = dA_\Sigma + \int_\Sigma^{\Omega^\pi} A_\Omega \wedge A_\Lambda \quad (12)$$

is the field strength of the gauge group K. Under $SU(3) \times SU(n) \times U(1)$ the $SU(3,n)$ connection Ω decomposes as follows :

$$\begin{aligned} \Omega_i^A &= P_i^A, \quad \Omega_A^i = P_A^i = (P_i^A)^* \\ \Omega_A^B &= \varphi_A^B - m \delta_A^B \varphi \\ \Omega_i^{\dot{j}} &= \varphi_i^{\dot{j}} + 3 \delta_i^{\dot{j}} \varphi \end{aligned} \quad (13)$$

where φ_A^B , $\varphi_i^{\dot{j}}$, φ and P_i^A are the $SU(3) \times SU(n) \times U(1)$ connections and the scalar vierbein respectively. In terms of the connection Ω the set of curvatures of the supersymmetry algebra given by (1-4) must be accordingly modified by appropriate covariantizations, since the gravitino and the vectors transform under H and G. The new set of curvatures

is given by

$$\begin{aligned}
 R^a &= DV^a - i\bar{\psi}^A \gamma^a \psi_A \\
 R^{ab} &= d\omega^{ab} - \omega^{ac} \omega_c^b \\
 \rho_A &\equiv \nabla \psi_A = DV_A + \rho_A^B \wedge \psi_B + p \phi \wedge \psi_A \\
 F_A &= F_A - \frac{1}{2} L_A^A \epsilon_{ABC} \bar{\psi}^B \wedge \psi^C - \frac{1}{2} (L^{-1})_A^{\pi} J_{\pi\pi} \epsilon^{ABC} \bar{\psi}_B \psi_C
 \end{aligned} \quad (14)$$

where p is the $U(1)$ chiral charge of the gravitino field.

Analogously we can write the covariant derivatives of the other spin $\frac{1}{2}$ fields $\chi, \lambda_{Ai}, \lambda^i$, also transforming under H :

$$\begin{aligned}
 \nabla \chi &= D\chi + q \phi \chi \\
 \nabla \lambda^i &= D\lambda^i - \phi_i^j \lambda^j - r \phi \lambda^i \\
 \nabla \lambda_{Ai} &= D\lambda_{Ai} + \phi_i^j \lambda_{Aj} + \phi_A^B \lambda_{Bi} + s \phi \lambda_{Ai}
 \end{aligned} \quad (15)$$

The $U(1)$ charges of the fermi fields are fixed to be $p=n/2, q=\frac{3}{2}n, r=3(1+\frac{n}{2}), s=3+\frac{n}{2}$ by requiring consistency of the superalgebra (14) (after expanding the connection one-form Ω_A^Σ in superspace differentials (V^a, ψ^A)). In our notations all spinor fields have the same chirality $\gamma_5 = +1$.

In the group manifold approach the Bianchi identities for the set of curvatures (14), (15) are supplemented by the rheonomy principle [8], which amounts to say that all fermionic components of the curvatures must be given as combinations of the bosonic ones, as well as of the fundamental fermionic fields $\lambda^i, \lambda_{Ai}, \chi$. The set of Bianchi identities for the curvatures given in (14) is:

$$DR^a + R^{ab} \wedge V_b + i\bar{\rho}^A \gamma^a \psi_A - i\bar{\psi}^A \gamma^a \rho_A = 0. \quad (16)$$

$$DR^{ab} = 0 \quad (17)$$

$$\nabla \rho_A + \frac{1}{4} R^{ab} \gamma_{ab} \psi_A - R_A{}^B \psi_B - \frac{M}{2} R_A \psi_A = 0 \quad (18)$$

$$\begin{aligned} \nabla F_A + \frac{1}{2} L_A{}^i \rho_i{}^A \epsilon_{ABC} \bar{\psi}^B \psi^C - \frac{1}{2} \rho_A{}^i (L^{-1})_i{}^\pi J_{\pi\tau} \epsilon^{ABC} \bar{\psi}_B \psi_C \\ - L_A{}^B \epsilon_{ABC} \bar{\psi}^B \rho^C - (L^{-1})_A{}^\pi J_{\pi\tau} \epsilon^{ABC} \bar{\psi}_B \rho_C = 0 \end{aligned} \quad (19)$$

with the additional constraints of torsion-free space: $R^a = 0$.

The full parametrization for the set of curvatures (14), (15) has been determined elsewhere [7]. Here we recall for convenience the terms induced by the gauge group K:

$$(\rho_A)_{\text{GAUGED}} = \rho_A + g S_{AB} \gamma_m \psi^B \wedge V^m \quad (20)$$

$$(\nabla \chi)_{\text{GAUGED}} = \nabla \chi + g U^B \psi_B \quad (21)$$

$$(\nabla \lambda_{Ai})_{\text{GAUGED}} = \nabla \lambda_{Ai} + g \mathcal{M}_{iA}{}^B \psi_B \quad (22)$$

$$(\nabla \lambda^i)_{\text{GAUGED}} = \nabla \lambda^i + g \mathcal{N}^{iA} \psi_A \quad (23)$$

$\rho_A, \nabla \chi, \nabla \lambda_{Ai}, \nabla \lambda^i$ are the parametrization of the ungauged curvatures.

The Bianchi identities (16) - (19) determine the fermionic shifts to be:

$$S_{AB} = -\frac{i}{8} (C_A{}^{PQ} \epsilon_{BQP} + C_B{}^{PQ} \epsilon_{APQ}) \quad (24)$$

$$U^B = -\frac{1}{4} C^B \quad (25)$$

$$\mathcal{N}_{iA}{}^B = -\frac{1}{2} C_i{}^{PQ} \epsilon_{APQ} \quad (26)$$

$$\mathcal{M}_{iA}{}^B = -C_{iA}{}^B + \frac{1}{2} C_{iH}{}^H \delta_A^B \quad (27)$$

in terms of the boosted structure constants:

$$C_A{}^{\pi\sigma} = (L^{-1})_A{}^{\lambda'} L_{\pi'}{}^\pi L_{\sigma'}{}^\sigma f_{\lambda'}{}^{\pi'\sigma'} \quad (28)$$

which are functions of the scalar fields (through L) and of the K structure constants.

In our notation, the basic supersymmetry relation [13] determining the scalar potential $W(z)$ is the matrix relation:

$$g^2 W(z) \delta_A^B = \frac{2}{3} \mu_A \mu^B + \frac{1}{6} \mathcal{N}_{iA} \mathcal{N}^{iB} + \frac{1}{6} \mathcal{M}_{iA}^M \mathcal{M}_{iM}^B + 2 S_{AM} S^{MB} \equiv T_A^B \quad (29)$$

Eq. (29) implies that the traceless part of the matrix T_A^B defined by the right-hand side vanishes as a consequence of the properties of the boosted structure constants $C_A^{\Sigma P}$. To show this, we define the following quantities:

$$X_A^B = C_A^{PQ} C_{PQ}^B, \quad Y_A^B = C_{AP}^B C^P + C_A^{BP} C_P \quad (30)$$

$$Z_A^B = C_A C^B, \quad W_A^B = C_{AP}^M C_M^{BP} \quad (C_P = C_{MP}^M)$$

$$\begin{aligned} \chi_A^B &= C_A^{iQ} C_{iQ}^B, \quad \eta_A^B = C_{Ai}^B C^i + C_A^{Bi} C_i \\ W_A^B &= C_i^{PB} C_{PA}^i \quad (C_i \equiv C_{Mi}^M) \end{aligned} \quad (31)$$

The matrices in (30), (31) are hermitean. The proof that $T_A^B - \frac{1}{3} \delta_A^B T_C^C = 0$ relies on the following two lemmas:

$$(Z_A^B - W_A^B - \frac{1}{2} X_A^B + Y_A^B)_{\text{TRACELESS PART}} = 0 \quad (32)$$

$$C_A^{\Lambda P} C_{\Lambda R}^B = - C_R^{\Lambda P} C_{A\Lambda}^B + C_A^{BP} C_{AR}^{\Lambda} \quad (33)$$

Eq. (33) is a consequence of the Jacobi identities for the K structure constants, and implies the matrix relation:

$$2 X_A^B - Y_A^B - 2 W_A^B = - (2 \chi_A^B - \eta_A^B + 2 W_A^B) \quad (34)$$

Eq. (32) follows from the SU(3) identity:

$$S_{AB} = - \frac{i}{4} (C_A^{PQ} \epsilon_{BPQ} + \epsilon_{ABM} C^M) \quad (35)$$

Tracing eq. (34) we also get

$$|C_i|^2 = |C_P|^2 + |C_A^{iP}|^2 - |C_i^{PQ}|^2 \quad (36)$$

Using eqs. (30) - (36) in the expression (29), we obtain the explicit form of the potential

$$12 g^2 W(z) = \frac{1}{2} |C_{iA}^B|^2 + \frac{1}{2} |C_i^{PQ}|^2 - (|C_A^{PQ}|^2 - |C_P|^2) \quad (37)$$

where the last term in the bracket is the contribution from the gravitino mass matrix $S_{AB} = S_{BA}$, $|S_{AB}|^2 = \frac{1}{8} (|C_A^{PQ}|^2 - |C_P|^2)$. Note that in absence of matter multiplets $C_{iA}^B = C_i^{PQ} = C_P = 0$, and we recover N=3 anti-de Sitter supergravity [10].

In extended supergravities the overall scalar potential can be written as a sum of squares of boosted structure constants of the gauge group K. It should be possible to find all the extrema of $W(z)$ by means of algebraic techniques, and thus determine all possible patterns of supersymmetry and gauge symmetry breaking. Also the possibility of gauging a non-semisimple group has to be considered and may lead to less constrained patterns of symmetry breaking.

Finally it is worth to mention that the technique we have used in the particular case of N=3 supergravity in d=4 dimensions can be applied to any supergravity model in any space-time dimension, and may be relevant for spontaneous compactification.

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