Science and Engineering Research Council
Rutherford Appleton Laboratory
CHILTON, DIDCOT, OXON, OX11 0QX

On Loop Space Formulation of Gauge Theories

CHAN Hong-Mo, Peter SCHARBACH and
TSOU Sheung Tsun

May 1985

RAL-85-043

RAL-85-043

© Science and Engineering

Research Council 1985

The Science and Engineering Research Council does not accept any responsibility for loss or damage arising from the use of information contained in any of its reports or in any communication about its tests or investigations.

On Loop Space Formulation of Gauge Theories

by

CHAN Hong-Mo, Peter SCHARBACH *

Rutherford Appleton Laboratory Chilton, Didcot, Oxon OX11 0QX
England

and

TSOU Sheung Tsun
Mathematical Institute, 24-29 St. Giles, Oxford, OX1 3LB
England

Abstract

In attempting to formulate gauge theories entirely in terms of loop variables, it is found convenient to work in the function space of parametrised loops with the loop space connection $F_\mu(C|s)$ as field variables. Equivalence to the conventional formulation in terms of the gauge potential $A_\mu(x)$ is ensured by imposing on $F_\mu(C|s)$ certain conditions implying the local absence of 'magnetic sources' or monopoles. These conditions are reminiscent of the Poincaré lemma in electromagnetism, and establish a one-one correspondence between $F_\mu(C|s)$ so constrained and $A_\mu(x)$ up to gauge equivalence. We use this approach to reformulate the action principle for pure gauge theories and to derive field equations from it completely in terms of loop variables. We believe that the formalism is useful in the theory of monopoles and may also find application in lattice calculations.

* Present Address: GEC Research Laboratories, Hirst Research Centre, East Lane, Wembley, Middlesex, England

1. Introduction

It is well known that the electromagnetic field tensor $f_{\mu\nu}$ though adequate for describing classical electromagnetism as originally conceived by Faraday and Maxwell, does not describe completely all electromagnetic effects on the wave function of an electron in the quantum theory. As clearly demonstrated by the Bohm-Aharonov experiment¹⁾, different physical conditions in some given region of space-time may correspond to exactly the same $f_{\mu\nu}$. On the other hand, the electromagnetic potential a_μ , which is usually employed as variables in this situation, overdescribes electromagnetism in that the same physical conditions can be equally described by different potentials a_μ . According, for example, to the analysis of Wu and Yang²⁾, what gives instead a complete description of all known electromagnetic effect and leaves no freedom which is intrinsically unobservable is the phase factor:

$$\phi(C) = \exp i e \oint_C a_\mu dx^\mu \quad (1.1)$$

over closed loops C .

The situation is similar also for nonabelian gauge theories except that $F_{\mu\nu}$ there is already inadequate for a complete description at the classical level³⁾, and that in the expression (1.1) for the phase factor, the quantities A_μ being noncommutative have to be path-ordered. Note that our convention is to denote abelian field quantities by small letters, and nonabelian quantities by capitals.

Theoretically therefore, it would seem rather appealing to describe gauge theories in terms of the phase factors (1.1) or their path-ordered nonabelian generalisations. 2),4)-15) However, a reformulation of gauge theories entirely in loop space is beset with considerable technical difficulties. The phase factors $\phi(C)$ are labelled by loops C in space-time which are much more numerous than the points x in space-time and the index μ which together label the gauge potentials $A_\mu(x)$. It follows therefore that as variables for describing gauge theories $\phi(C)$ form a highly redundant set which has to be severely constrained. However, it is a priori not clear exactly what constraints on ϕ are necessary to remove their intrinsic

redundancy, or how they can be imposed to formulate, for example, an action principle from which equations of motion can be derived.

We shall report in this paper our endeavours to solve these problems for a class of gauge theories of particular current interest. First, this necessitates a close examination of some geometrical properties of the space of loops, which is of course infinite dimensional. We find, for example, that it is vastly more convenient to work with parametrised loops (namely maps of the circle into space-time) rather than with the (directed) sets of points in space-time which they represent. The space of parametrised loops, being just a function space, allows loop derivatives and integrals to be defined as ordinary functional derivatives and integrals having familiar properties. Second, we find that it is much easier to write down constraint equations for the logarithmic derivatives¹⁰⁾ $F_\mu(C|s)$ of $\phi(C)$ rather than for $\phi(C)$ themselves. These logarithmic derivatives, as will be shown, form a kind of connection (or 'potential') for the space of parametrised loops. Third, we show that the constraint equations on $F_\mu(C|s)$ we constructed are complete in the sense that they imply the existence of the usual gauge potential $A_\mu(x)$ which is uniquely determined up to gauge transformations, removing thus entirely the redundancy of the loop space variables. This is our main result which may in some sense be regarded as a nonabelian generalisation of the Poincaré lemma in electromagnetism. Finally, with these constraints, we are then able to formulate the action principle for pure gauge theories directly in terms of the loop space variables.

It will be seen that this question of redundancy in loop variables is intimately related to the problem of monopoles in the theory. In electromagnetism, the Poincaré condition for the existence of a gauge potential $a_\mu(x)$ in any region of space is that the field tensor should satisfy there the Bianchi identity $*f_{\mu\nu,\lambda} = 0$, or that there should exist no magnetic source or monopole in that region. For nonabelian theories, the definition of a monopole charge becomes more complicated and is best expressed in terms of loop variables. What we then find is that it is again exactly the condition that monopoles be absent in some region which guarantees the existence of the gauge potential in that region, thereby removing the redundancy of the loop variables. Further, as in electromagnetism, the result is unaffected by the presence of monopoles elsewhere in space outside the particular region under consideration.

A formulation of gauge theories in loop space such as ours presented here, may have, besides purely aesthetic appeal, practical applications in developing the theory of monopoles. In fact, these considerations have led us to the equations of motion of the nonabelian monopole, to be presented in a separate paper. In contrast to the gauge potential $A_\mu(x)$, and also to the field tensor $F_{\mu\nu}(x)$ for nonabelian theories, loop variables such as $F_\mu(C, s)$ are patch-independent and may therefore be convenient in problems where patching is particularly cumbersome. In any case, since the minimum number of patches required for a system of m monopoles¹⁶⁾ increases as 2^m , the complications introduced by changing over to loop variables may become a price worth paying when the number of monopoles is large or indefinite. Further, loop variables being gauge invariant, it is conceivable that such formulations may also be useful in lattice calculations,^{16), 17)} where the present custom is to generate gauge configurations in terms of $A_\mu(x)$ at random, regardless of whether they are in fact gauge equivalent.

2. Parametrised Loop Space

Our first step is to label loops in space-time. It will be shown later in section 5 that it is sufficient for our purpose to consider only loops passing through a fixed point, say $P_0 = \{x_0^\mu\}$. Such a loop can be parametrized as follows:

$$C = \{\xi^\mu(s) : s = 0 \rightarrow 2\pi, \xi^\mu(0) = \xi^\mu(2\pi) = x_0^\mu\} \quad (2.1)$$

where ξ^μ are co-ordinates in ordinary 4-dimensional space-time X .

We define then the phase factor²⁾ (also called the holonomy element¹⁸⁾ or Wilson operator⁶⁾ over C in general as:

$$\Phi(C) = P_s \exp i e \int_0^{2\pi} ds A_\mu(\xi(s)) \frac{d\xi^\mu(s)}{ds}, \quad (2.2)$$

where P_s denotes ordering in s , with s increasing by convention from right to left. It is an element of the gauge group G , and can be considered as a finite version of the field (or curvature) tensor which is an element of its Lie algebra.

Now $\Phi(C)$ as defined remains unchanged under reparametrisation of the closed loop C so that there is no actual need to distinguish the different parametrisations. However, it is much more convenient to work with parametrised loops¹⁹⁾, which are just maps of the circle into space-time X , rather than the (directed) sets of points they represent. One immediate advantage is that a function of a parametrised loop, such as Φ in (2.2), is nothing but a functional of the functions $\xi^\mu(s)$ which we know how to manipulate.

For example, the loop derivative of any function $\Psi(C)$ can then be defined simply as the functional derivative:

$$\frac{\delta}{\delta \xi^\mu(s)} \Psi(C) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \{\Psi(C') - \Psi(C)\}, \quad (2.3)$$

with

$$C' = \{\xi'^\mu(s) = \xi^\mu(s) + \Delta \delta_\alpha^\mu \delta(s-s')\}. \quad (2.4)$$

[Should ambiguities arise in operations with the δ -function in (2.4) we shall first replace it as usual by a smooth function of finite width and then take the appropriate zero width limit]. Similarly, an integral over loops can also be defined simply as a functional integral.

In view of this we prefer henceforth to work with parametrised loops denoted by the symbol C or explicitly as the functions $\xi^\mu(s)$. The space of such loops, denoted by $\Omega^1 X$, is thus just an ordinary function space. On the few occasions when we have to consider the actual (directed) set of points in space-time which C represents, we shall refer to it as a 'point-set loop' denoted by the symbol $\mathcal{C}$.

The phase factor Φ then will henceforth be regarded as a functional of C or as a field in $\Omega^1 X$. However, by defining thus for every C a Φ which actually depends only on $\mathcal{C}$, one has introduced a further redundancy in the variables which eventually will also have to be removed.

3. Connection, Curvature and Holonomy

It follows from (2.2) that $\Phi(C)$ satisfies the composition law:

$$\Phi(C^{(2)} * C^{(1)}) = \Phi(C^{(2)}) \Phi(C^{(1)}). \quad (3.1)$$

where $C^{(2)} * C^{(1)}$ is the loop obtained by first going round $C^{(1)}$ and then $C^{(2)}$, suitably reparametrised. Further, under a gauge transformation $S(x)$, $\Phi(C)$ undergoes a constant gauge rotation:

$$\Phi(C) \rightarrow S(P_0) \Phi(C) S^{-1}(P_0), \quad (3.2)$$

which is not important for our considerations.

The loop derivative of $\Phi(C)$ is defined as in (2.3). However, $\Phi(C)$ being an element of the gauge group, only its logarithmic derivative is meaningful:

$$F_\mu(C|s) = \frac{1}{e} \Phi^{-1}(C) \frac{\delta}{\delta \xi^\mu(s)} \Phi(C), \quad (3.3)$$

which is an element of the corresponding Lie algebra. From (2.2) and (2.3) one then obtains that $F_\mu(C|s)$ is related to the gauge potential $A_\mu(x)$ as follows¹⁰:

$$F_\mu(C|s) = \Phi_C^{-1}(s, 0) F_{\mu\nu}(\xi(s)) \Phi_C(s, 0) \frac{d\xi^\nu}{ds}, \quad (3.4)$$

where

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ie [A_\mu(x), A_\nu(x)] \quad (3.5)$$

is the usual field tensor and

$$\Phi_C(s_2, s_1) = P_s \exp ie \int_{s_1}^{s_2} ds A_\mu(\xi(s)) \frac{d\xi^\mu}{ds} \quad (3.6)$$

is the parallel transport along C from the point $\xi^\mu(s_1)$ to the point $\xi^\mu(s_2)$ in X .

From (3.4) it is immediately apparent that $F_\mu(C|s)$ does not depend on that part of the loop C with parameter $> s$. [The asymmetry comes from the path-ordering convention in the definitions (2.2) and (3.3). Had we chosen the reverse convention, $F_\mu(C|s)$ would instead be independent of that part of the loop C with parameter $< s$.] This fact is reflected in our notation $\Phi(C|s)$ for the argument of F_μ which is henceforth to be so understood.

Further, from the antisymmetry of $F_{\mu\nu}$ it is seen in (3.4) that $F_\mu(C|s)$ has components only in directions transverse to that of the loop C at $\xi^\mu(s)$, namely that

$$F_\mu(C|s) \frac{d\xi^\mu(s)}{ds} = 0. \quad (3.7)$$

We shall refer to (3.7) as the transversality condition. It is equivalent to the statement that $\Phi(C)$ is invariant under reparametrisation of C .

Now $F_\mu(C|s)$ in (3.3) represents the change in the phase of $\Phi(C)$ under an infinitesimal variation in the loop C , i.e. a variation from one point C to a neighbouring point $C + \delta C$ in parametrised loop space $\Omega^1 X$. It may therefore be interpreted as a connection in $\Omega^1 X$ in the same sense that the gauge potential $A_\mu(x)$ is a connection in ordinary space-time X .

From the connection $F_\mu(C|s)$ one may then construct a curvature tensor in $\Omega^1 X$ as the covariant curl of $F_\mu(C|s)$ in complete analogy to (3.5):¹⁰

$$G_{\mu\nu}(C; s, s') = \frac{\delta}{\delta \xi^\nu(s')} F_\mu(C|s) - \frac{\delta}{\delta \xi^\mu(s)} F_\nu(C|s') + ie [F_\mu(C|s), F_\nu(C|s')]. \quad (3.8)$$

As usual, $G_{\mu\nu}(C; s, s')$ $\delta \xi^\mu(s) \delta \xi^\nu(s')$ represents the total change in phase under parallel transport in $\Omega^1 X$ around an infinitesimal closed loop. In particular, for $s=s'$, this loop sweeps over an infinitesimal closed surface in X surrounding the point $x = \xi(s)$ as depicted in Fig 1.

Evaluating (3.8) according to (2.3), one obtains^{13,14)}

$$G_{\mu\nu}(C; s, s') = \Phi^{-1}(s, 0) \{ D_\lambda F_{\mu\nu}(\xi(s)) + D_\mu F_{\nu\lambda}(\xi(s)) + D_\nu F_{\lambda\mu}(\xi(s)) \} \Phi \times \Phi(s, 0) \frac{d\xi^\lambda(s)}{ds} \delta(s-s') \quad (3.9)$$

where D_μ denotes the usual gauge covariant derivative with respect to the gauge potential $A_\mu(x)$. Equivalently, in terms of the dual field tensor:

$${}^*F_{\mu\nu}(x) = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \quad (3.10)$$

we have

$$G_{\mu\nu}(C; s, s') = -\phi^{-1}(s, 0) \{ \varepsilon_{\mu\nu\rho\sigma} D_{\alpha}^* F^{\sigma\alpha} \} \phi(s, 0) \frac{d\xi^{\rho}(s)}{ds} \delta(s-s'). \quad (3.11)$$

The Bianchi identity for $F_{\mu\nu}$ then implies that $G_{\mu\nu}$ is zero, except where there is a monopole at the point $x = \xi(s)$. The exception arises because at a monopole the field tensor $F_{\mu\nu}$ is not a gauge field expressible in terms of a gauge potential as in (3.5) so that the Bianchi identity no longer holds. In the case of an (abelian) magnetic monopole, $G_{\mu\nu}$ is proportional to its magnetic charge. Hence magnetic monopoles appear as sources of curvature in loop space¹⁴. The nonabelian case is more delicate, since then the covariant derivative in (3.12) involves A_{μ} which is not well-defined at the monopole position. We shall return to the question of nonabelian monopoles in the next section.

Just as the field tensor (3.5) (curvature tensor) representing the change in phase over an infinitesimal loop, is a differential version of the phase factor (2.2) (holonomy element) over a finite loop in ordinary space-time X , so is the curvature $G_{\mu\nu}$ the differential version of a holonomy element which can be similarly constructed in loop space $\Omega^1 X$.

Consider a continuous family of functions $\xi_t^{\mu}(s)$ labelled by the parameter t :

$$\Sigma = \{ \xi_t^{\mu}(s) : s = 0 + 2\pi, t = 0 + 2\pi \}, \quad (3.12)$$

satisfying:

$$\xi_t^{\mu}(0) = \xi_t^{\mu}(2\pi) = \xi_0^{\mu}, \quad t = 0 + 2\pi, \quad (3.13)$$

$$\xi_0^{\mu}(s) = \xi_{2\pi}^{\mu}(s) = \xi_0^{\mu}, \quad s = 0 + 2\pi, \quad (3.14)$$

where $\xi_t^{\mu}(s)$ for each t is a parametrised loop belonging to $\Omega^1 X$:

$$C_t = \{ \xi_t^{\mu}(s) : s = 0 + 2\pi \}. \quad (3.15)$$

For $t = 0$, 2π , C_t shrinks to the fixed point ξ_0^{μ} , according to (3.14). Hence as t varies from 0 to 2π , C_t describes a closed loop in loop space $\Omega^1 X$ passing through the fixed point in $\Omega^1 X$ which is the zero loop. The symbol Σ thus denotes a parametrised loop in $\Omega^1 X$. In ordinary space-time X , C_t sweeps over a closed 2-dimensional surface, as illustrated in Fig. 2. The set of points in X thus represented, together with their orientation, will be designated as Σ . The space of all Σ (i.e. parametrised loop space over $\Omega^1 X$) we denote by $\Omega^2 X$. It is also just a function space.

For the parametrised loop in $\Omega^1 X$ represented by Σ in (3.12), we construct then the phase factor (or Wilson operator, or holonomy element) as

$$\Theta(\Sigma) = P_t \exp i e \int_0^{2\pi} dt \int_0^{2\pi} ds F_{\mu}^{\nu}(C_t) \frac{\partial \xi_t^{\mu}(s)}{\partial t}, \quad (3.16)$$

where P_t denotes ordering in t , with t increasing again by convention from right to left. It is a close analogue of $\Phi(C)$ in (2.2) except that $F_{\mu}^{\nu}(C/s)$, being a connection in the infinite-dimensional loop space $\Omega^1 X$, has many more components than $A_{\mu}(x)$, requiring thus an extra label s in addition to μ , both of which have to be summed over.

Our discussion above of $G_{\mu\nu}$ indicates that the value of $\Theta(\Sigma)$ has to do with the presence or otherwise of monopoles inside the surface Σ in X . We shall examine this question further in the next section.

4. Monopoles

It is generally recognized that monopole charges in gauge theories have a topological origin. The presence (or indeed absence also) of a monopole charge at a point in space signifies a certain topological condition of the gauge field in the surrounding spatial region. It may thus be regarded as a constraint on the field variables describing the system, which is

maintained throughout all continuous variations of these variables. Here, we are particularly interested in finding out what form this constraint will take in terms of the loop variables introduced above.

One of the many equivalent definitions of a magnetic charge in electromagnetism is as follows. Consider the family of closed loops C_t in (3.15) represented by Σ in (3.12). For each value of t we define a phase factor $\phi(C_t)$ according to (1.1). This is an element of the gauge group $U(1)$. For $t=0, 2\pi$, C_t shrinks to a point so that $\phi(C_0)=\phi(C_{2\pi})=1$. As t varies from 0 to 2π therefore, $\phi(C_t)$ traces out a closed curve $\Gamma(\Sigma)$ in the group $U(1)$. Now, closed curves in $U(1)$ are divided into homotopy classes (equivalent classes under continuous deformation) labelled by the winding number n_Σ , which, apart from a numerical factor, is the quantised Dirac charge enclosed inside the 2-dimensional surface Σ in ordinary space. Its equivalence to the more conventional definition as an integral of magnetic flux over Σ is easily demonstrated.

One advantage of this topological definition is its immediate generalisation to nonabelian theories, as noted by Lubkin²⁰⁾, Wu and Yang²⁾, Coleman²¹⁾ and others. The phase factor $\phi(C_t)$ of (2.2) represents always an element of the gauge group G , so that as t varies from 0 to 2π , $\phi(C_t)$ again traces out a closed curve $\Gamma(\Sigma)$ in G . The homotopy class to which $\Gamma(\Sigma)$ belongs is automatically quantised (i.e. discrete) and conserved (i.e. invariant under continuous deformation of the field variables), and hence acceptable as the generalisation of the Dirac monopole charge.

The value that this generalised Dirac charge can take depends on the gauge group G , which in turn is determined by what particles and fields the theory contains. For example, for a pure Yang-Mills theory containing only gauge fields belonging to the adjoint representation of the $su(N)$ algebra, the gauge group is $SU(N)/Z_N$, and the Dirac monopole charge there may be labelled by $\zeta_r = \exp i2\pi r/N$, $r=0, 1, \dots, N-1$. On the other hand, for the more realistic class of theories with gauge group $U(N)$ (which include in particular the standard $U(3)=[SU(3) \times U(1)]/Z_3$ theory of gluons and colour triplet quarks with $1/3$ electric charges), the Dirac monopole charge can take any integer values; a monopole with charge n here may conveniently be regarded as carrying a monopole charge $\zeta = \exp i2\pi n/N$ of the colour group $SU(N)/Z_N$ as well as an ordinary magnetic charge $g = n/2e$ ²²⁾.

These facts are known already in the literature. For our purpose here, however, we shall need to express this abstractly defined generalised Dirac charge explicitly in terms of the loop variables. For the abelian theory, this is trivial. The winding number n_Σ can be expressed as $(2\pi)^{-1}$ times the total change in phase of $\phi(C_t)$ as t varies from 0 to 2π . We may therefore write:

$$\int dt \, ds \, f_\mu(C_t|s) \frac{\partial \phi_t^\mu(s)}{\partial t} = 2\pi g_\Sigma \quad (4.1)$$

with $g_\Sigma = m_\Sigma/2e$, where $f_\mu(C, s)$, defined as in (3.3), represents the differential change in phase of $\phi(C)$ as C varies.

Note that the phase of $\phi(C_t)$, $\phi(C_t)$ being an element of $U(1)$, may itself be regarded as a point in the covering space R of $U(1)$ whose image in $U(1)$ via exponentiation is $\phi(C_t)$. For t in the range $[0, 2\pi]$, this point traces a continuous curve in R so that the total change in the phase of $\phi(C_t)$ may also be regarded as the difference in R between the end-points of the curve so traced. In evaluating the phase of $\phi(C)$, however, account has to be taken of the fact that in the presence of a monopole, the potential a_μ is a patched quantity.

To see how this can be generalised to nonabelian theories, consider first the simple case of $SO(3)=SU(2)/Z_2$, where the monopole charge can take only two values, conveniently labelled by a sign $\pm$, with $+$ corresponding to the vacuum. We note first that the expression (2.2) for the phase factor $\phi(C)$ is subject to a slight ambiguity in interpretation. Suppose we represent as usual the gauge potential as $A_\mu = A_\mu^i(\sigma_i/2)$ where σ_i are the Pauli matrices, then the right hand side of (2.2) is an element of $SU(2)$, say $\tilde{\phi}(C)$. To obtain $\phi(C)$, which is in $SO(3)$, one ignores the sign of $\tilde{\phi}(C)$. However $\tilde{\phi}(C)$, being an element of the covering group $SU(2)$, is the nonabelian analogue of the phase of $\phi(C)$ in R in the discussion of the paragraph above.

Consider then again the family Σ of closed loops C_t as parametrised by (3.12)-(3.15) and depicted in Fig. 2. For t in the range $[0, 2\pi]$, $\tilde{\phi}(C_t)$ traces out a continuous curve in the covering group $SU(2)$. As in the abelian case, the difference in $SU(2)$ between the end-points of this curve gives the monopole charge enclosed inside the surface Σ in ordinary space, thus:

Although the above arguments are given explicitly only for an $SO(3)$ theory, it is evident that they can readily be extended to any pure Yang-Mills theories with gauge groups $SU(N)/Z_N$. The quantity $\Theta(\Sigma)$ defined in (3.17) when regarded as an element of the covering group $SU(N)$ gives always the monopole charge enclosed inside the surface $\mathcal{F}$, so that (4.2) applies, excepting that ζ_Σ now takes the values $\exp i2\pi r/N$, $r=0,1,\dots,N-1$.

With minor modifications, the arguments apply also to theories with gauge group $U(N)$, including in particular the standard theory of coloured quarks and gluons. We recall an observation made above that in a $U(N)$ theory, a monopole with charge n can be regarded as carrying both a 'colour' $SU(N)/Z_N$ monopole charge $\zeta = \exp i2\pi n/N$ and a 'magnetic' $U(1)$ monopole charge $g=n/2e$. One can then write for the first:

$$\Theta(\Sigma) = \zeta_\Sigma, \quad (4.5)a$$

and for the second,

$$\int dt ds f_\mu(C_t, s) \frac{\partial \zeta_\Sigma^\mu(s)}{\partial t} = 2\pi g_\Sigma, \quad (4.5)b$$

with $\zeta_\Sigma = \exp i2\pi n/N$ and $g_\Sigma = n/2e$, both relations to be satisfied simultaneously for the same n .

We believe that very similar considerations would apply also to theories with other gauge groups by making use of their covering groups although we have not studied the details in general, since the above examples already cover most cases of current interest.

We shall regard the expressions just derived for the monopole charge as topological constraints on the loop variables in accordance with our remarks at the beginning of this section. They are nonabelian generalisations of the statement made above in section 3 that monopoles are sources of the curvature $G_{\mu\nu}$ in loop space, only expressed here in a global form. In particular, in the absence of monopoles, (4.2) gives $\Theta(\Sigma) = 1$ for all Σ , which clearly implies $G_{\mu\nu}(C; s, s') = 0$ for all $(C; s, s')$ since the latter is merely a special case of the former when Σ is an infinitesimal loop in $\Omega^1 X$.

$$\Theta(\Sigma) = P_t \exp i e \int_\mu dt ds F_\mu(C_t) s = \zeta_\Sigma, \quad (4.2)$$

where $\Theta(\Sigma)$ is to be regarded as an element of $SU(2)$, not of $SO(3)$.

To see this, it suffices for illustration to evaluate $\Theta(\Sigma)$ for $\mathcal{F}$ enclosing a monopole charge in ordinary space. On $\mathcal{F}$, the gauge potential A_μ has to be patched. Suppose it is covered by two overlapping patches, in terms of the usual polar co-ordinates:

$$\begin{aligned} (N): \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi, \\ (S): \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi. \end{aligned} \quad (4.3)$$

The gauge potential is described by two functions $A_\mu^{(N)}$ and $A_\mu^{(S)}$ defined respectively in (N) and (S) and related by:

$$A_\mu^{(S)}(x) = Q(x) A_\mu^{(N)}(x) Q^{-1}(x) - \frac{1}{e} Q(x) \partial_\mu Q^{-1}(x) \quad (4.4)$$

where $Q(x)$ is a patching function which represents a closed curve in the gauge group $SO(3)$ belonging to the homotopy class - .

Consider again the closed loops C_t depicted in Fig. 2. At $t=0$, C_t shrinks to the point P_0 and the matrix $\tilde{Q}(C_0)$ is by definition 1. As t increases, C_t is gradually enlarged but can initially still be considered as lying entirely in (N) so that the matrix $\tilde{Q}(C_t)$ as calculated with $A_\mu^{(N)}$ varies continuously with t . However, at some value of t , C_t must pass through the south pole and can no longer remain entirely in (N) . One has therefore, at some value $0 < t_c < 2\pi$, to make a patching transformation from $A_\mu^{(N)}$ to $A_\mu^{(S)}$, at which point, according to (4.4), $\tilde{Q}(C_t)$ suffers a discontinuous change of sign. For $t > t_c$ the loop C_t can now be considered as lying entirely in (S) and $\tilde{Q}(C_t)$ again varies continuously, returning to the value 1 at $t=2\pi$ when $C_{2\pi}$ shrinks again to a point at P_0 . The total change in the matrix $\tilde{Q}(C_t)$ over the whole circuit is thus simply given by the jump at t_c , which is always -1 as anticipated by (4.2), independently of exactly where one chooses to perform the patching transformation.

about $(C; s, s')$. The converse is also true, namely that the differential condition $G_{\mu\nu}(C; s, s') = 0$ for all $(C; s, s')$ implies that all infinitesimal loops in $\Omega^1 X$ (and hence all finite loops also which can be constructed from them) must have trivial holonomy. An equivalent representation for (4.2) local in $\Omega^1 X$ exists also in the presence of monopole charges, but is more complicated. For the purposes of this paper, we find the global representation (4.2) more convenient.

Finally, we note in passing that although in the presence of a monopole both $A_\mu(X)$ and $F_{\mu\nu}(X)$ are patch-dependent quantities for a nonabelian theory, the loop variables $\Phi(C)$ and $F_\mu(C|s)$ are not. For example, $F_\mu(C|s)$ is given in terms of $F_{\mu\nu}(\xi(s))$ by (3.4). Although $F_{\mu\nu}(\xi(s))$ is patch-dependent, its patching transformation in (3.4) is exactly compensated by that in the parallel transports $\Phi_C(s, 0)$, leaving $F_\mu(C|s)$ invariant.

5. Removal of Redundancy

In the preceding sections it was found that the loop space connection $F_\mu(C|s)$ defined via (3.3) and (2.2) in terms of the gauge potential $A_\mu(x)$:

- (I) does not depend, for C parametrised as in (2.1), on $\xi^\nu(s')$ with $s' > s$, as signified in the notation $(C|s)$ of its argument;
- (II) has no transverse component, i.e. obeys (3.7);
- (III) satisfies a topological constraint, (4.1), (4.2) or (4.5) depending on the gauge group, for all Σ in $\Omega^2 X$.

We wish now to show the converse, namely that:

- (A) Given any $F_\mu(C|s)$ which satisfy (I), (II) and (III), one can construct, at every point x in X except on monopole worldlines, a gauge potential $A_\mu(x)$, unique up to gauge transformations and related to $F_\mu(C|s)$ via (2.2) and (2.3).

For nonabelian theories, this statement may be considered as a generalisation of the Poincaré lemma in electromagnetism. In the $U(1)$ theory it is well-known that the existence of a potential satisfying:

$$f_{\mu\nu} = \partial_\nu a_\mu - \partial_\mu a_\nu \quad (5.1)$$

implies the Bianchi identity:

$$\partial_\nu * f^{\mu\nu}(x) = 0; \quad (5.2)$$

conversely, for $f_{\mu\nu}(x)$ antisymmetric in μ and ν , the Bianchi identity also implies locally the existence of a potential $a_\mu(x)$ which is unique up to gauge transformations. Now, the condition that (5.2) holds at x is the statement that there is no magnetic source (or monopole) at x . Thus, by imposing on $f_{\mu\nu}(x)$ the condition that there is no monopole in some region of space, it is enough to guarantee the existence of a potential $a_\mu(x)$ there independently of whether there are monopoles outside that region.

For nonabelian theories, on the other hand, according to our considerations in the preceding section, the condition (III) contains the statement that except for the monopoles at the positions specified, there are no monopoles anywhere else in space. Hence, analogy to the Poincaré lemma suggests as in (A) that its imposition on $F_\mu(C|s)$ will also imply the existence of a gauge potential $A_\mu(x)$ for $F_\mu(C|s)$ at every point in X except at the said monopole positions. Of the other two conditions on $F_\mu(C|s)$ in (A), (II) for transversality, according to (3.4), is equivalent to the condition that $f_{\mu\nu}(x)$ in (5.2) be antisymmetric, while (I), having to do with path-ordering, is not expected to have an abelian analogue. Hence, beyond the replacement of a local condition by a global one for ensuring the absence of monopoles in a specified region, the contents of the Poincaré lemma for electromagnetism and of (A) are very similar.

We note that in the situation when there are no monopoles anywhere in space, then the integral condition (III) can be replaced in (A) by the differential condition $G_{\mu\nu}(C; s, s') = 0$ for all $(C; s, s')$, which is local in $\Omega^1 X$ as explained at the end of the last section. We find, however, that the integral form (III), as well as being more general, is more convenient both for verifying assertion (A) and for the applications so far attempted.

The assertion (A) establishes a one-one correspondence between the gauge equivalent classes of potentials $A_\mu(x)$ and the loop space connections $F_\mu(C|s)$ satisfying (I), (II) and (III). This means that one can replace

entirely the gauge-dependent $A_\mu(x)$ by the gauge-independent but constrained $F_\mu(C|s)$ as variables in describing the gauge theory. In other words, the original high degree of redundancy in $F_\mu(C|s)$ may be removed.

Finally, we remark that the knowledge, via (A), of $A_\mu(x)$ will allow one of course to construct phase factors $\phi(L)$ even for closed loops L not passing through the fixed point P_0 . For this reason, it is sufficient to include in our loop space $\Omega^1 X$ only loops passing through P_0 as parametrised in (2.1), as we have claimed in section 2.

The remainder of this section is devoted to verifying assertion (A) which unfortunately is somewhat lengthy and technical. To save on the notation, we shall work it out in detail only for the $SO(3)$ theory. It will be clear that an extension of the arguments to $SU(N)/Z_N$ and $U(N)$ theories is straightforward.

First, we show that the constraint (4.2) implies that $F_\mu(C|s)$ can be expressed in the form (3.3) in terms of some $\theta(C)$ in $SO(3)$. To do so, introduce in analogy to (3.6) the parallel transport:

$$\Theta_\Sigma(t_2, t_1) = P_t \exp ie \int_{t_1}^{t_2} dt \int_0^{2\pi} ds F_\mu(C_t|s) \frac{\partial \xi_t^\mu(s)}{\partial t}, \quad (5.3)$$

in loop space $\Omega^1 X$ from the 'point' C_{t_1} to the 'point' C_{t_2} along the 'loop' Σ . The condition (4.2) which for $SO(3)$ gives:

$$\Theta(\Sigma) = \pm I \quad (5.4)$$

for all 'closed loops' Σ in $\Omega^1 X$ then suggests that $\Theta_\Sigma(t, 0)$ for fixed 'end-point' C_t can depend on the 'path' Σ in $\Omega^1 X$ only through a sign. Explicitly, let C be any parametrised loop in space-time X , and

$$\Sigma(r) = \{\xi_t^\mu(r) : t = 0 \rightarrow 2\pi, s = 0 \rightarrow 2\pi\}, \quad (5.5)$$

$r=a, b$ be any two parametrised surfaces both passing through C , at $t=t_a$ and $t=t_b$ respectively. Define then $\Theta_\Sigma(r)(t_r, 0)$, $r=a, b$ as in (5.3). Change the integration variable t in these to respectively:

$$t' = \frac{\pi}{t} t ; \quad \xi_t^{(a)}(s) = \xi_{t'}^{(a)}(s), \quad (5.6)$$

$$t' = 2\pi - \frac{\pi}{t} t ; \quad \xi_t^{(b)}(s) = \xi_{t'}^{(b)}(s). \quad (5.7)$$

We have then:

$$\begin{aligned} & \Theta_\Sigma^{-1}(t_b, 0) \Theta_\Sigma(t_a, 0) \\ &= P_{t'} \exp ie \int_{\frac{\pi}{t}}^{2\pi} dt' \int_0^{2\pi} ds F_\mu(C_{t'}^b|s) \frac{\partial \xi_{t'}^{(b)\mu}(s)}{\partial t'} \times \\ & \quad \exp ie \int_0^{\frac{\pi}{t}} dt' \int_0^{2\pi} ds F_\mu(C_{t'}^a|s) \frac{\partial \xi_{t'}^{(a)\mu}(s)}{\partial t'}, \end{aligned} \quad (5.8)$$

which is $\Theta(\bar{\Sigma})$ for the parametrised surface $\bar{\Sigma}$

$$\bar{\Sigma} = \begin{cases} \bar{\xi}_{t'}^{(a)}(s) = \xi_{t'}^{(a)}(s) : t' = 0 \rightarrow \pi, \\ \bar{\xi}_{t'}^{(b)}(s) = \xi_{t'}^{(b)}(s) : t' = \pi \rightarrow 2\pi, \end{cases} \quad (5.9)$$

$\bar{\Sigma}$ being closed by construction. Hence, depending on the monopole charge enclosed

$$\Theta(\bar{\Sigma}) = \pm I, \quad (5.10)$$

or

$$\Theta_\Sigma(t_a, 0) = \pm \Theta_\Sigma(t_b, 0) \quad (5.11)$$

as expected.

Two $SU(2)$ elements differing only by sign, however, correspond to the same element in $SO(3)$. We may therefore denote the unique element in $SO(3)$ corresponding to $\phi^{-1}(t, 0)$ by $\phi(C_t)$ which depends only on the 'end-point' C_t but not otherwise on the 'path' Σ in loop space $\Omega^1 X$. Taking now the logarithmic derivative of this $\phi(C_t)$ according to (2.3) and (2.4) we obtain:

$$\begin{aligned} \phi^{-1}(C_t) \frac{\delta}{\delta \xi_t^\mu(s)} \phi(C_t) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} (p_t^{-1} \exp -ie \int_t^{t+\Delta} dt' \int_0^{2\pi} ds' F_\alpha(C_t | s') \frac{\partial \xi_t^\alpha(s')}{\partial t'} - 1) \\ &= -ie \int_0^{2\pi} ds' F_\alpha(C_t | s) \delta_\mu^\alpha \delta(s'-s), \\ &= -ie F_\mu(C_t | s) \end{aligned} \quad (5.12)$$

which is (3.3) as required.

Next, we show that the constraint (I) and (II) imply that the $\phi(C)$ so defined is parametrisation independent, and satisfies the composition law (2.1). That $\phi(C)$ is invariant under reparametrisation of C is obvious as a consequence of the transversality condition (II) on $F_\mu(C|s)$ and the relation (3.3) just derived. To prove (2.1), consider any two parametrised loops

$$C^{(1)} = \{ \xi^{(1)}(s_1) : s_1 = 0 + 2\pi \}, \quad (5.13)$$

$i = 1, 2$. For each, one can construct a parametrised surface,

$$\Sigma^{(i)} = \{ \xi_t^{(i)}(s_1) : t_1 = 0 + 2\pi, s_1 = 0 + 2\pi \}, \quad (5.14)$$

passing through $C^{(i)}$, say at $t_1 = \pi$, thus

$$C_\pi^{(1)} = \{ \xi_{t_1=\pi}^{(1)}(s_1) = \xi^{(1)}(s_1) : s_1 = 0 + 2\pi \}, \quad (5.15)$$

so that the matrix $\phi_{\Sigma^{(i)}}(\pi, 0)$ defined as in (5.3) corresponds to $\phi^{-1}(C^{(1)})$

in $SO(3)$, in accordance with our discussion above. Our strategy then is to construct a third surface

$$\Sigma^{(+)} = \{ \xi_t^{(+)}(s) : t = 0 + 2\pi, s = 0 + 2\pi \}, \quad (5.16)$$

passing through $C^{(2)} * C^{(1)}$ say at $t = \pi$, by a suitable reparametrisation of $\Sigma^{(1)}$ and $\Sigma^{(2)}$ so that $\phi_{\Sigma^{(+)}}(\pi, 0)$ defined as in (5.3) satisfies:

$$\phi_{\Sigma^{(+)}}(\pi, 0) = \phi_{\Sigma^{(1)}}(\pi, 0) \phi_{\Sigma^{(2)}}(\pi, 0). \quad (5.17)$$

For such a $\Sigma^{(+)}$, our result above will imply that $\phi(C^{(2)} * C^{(1)})$ in $SO(3)$ corresponding to $\phi_{\Sigma^{(+)}}^{-1}(\pi, 0)$ satisfies the composition law (2.1), which however, being a relation between quantities already shown to be Σ -independent, will acquire a general validity beyond our particular construction.

A parametrised surface with the required properties is:

$$\Sigma^{(+)} = \begin{cases} \xi_t^{(+)}(s) = \xi_0 : t = 0 + \frac{\pi}{2}, s = 0 + \pi, \\ \xi_t^{(+)}(s) = \xi_{t_2}^{(2)} (s_2 = 2s - 2\pi) : t = 0 + \frac{\pi}{2}, s = \pi + 2\pi, \\ \xi_t^{(+)}(s) = \xi_{t_1}^{(1)} (s_1 = 2s) : t = \frac{\pi}{2} + \pi, s = 0 + \pi, \\ \xi_t^{(+)}(s) = \xi_{t_2}^{(2)} (s_2 = 2s - 2\pi) : t = \frac{\pi}{2} + \pi, s = \pi + 2\pi. \end{cases} \quad (5.18)$$

This is illustrated in Fig. 3, where for $t = 0 + \frac{\pi}{2}$, $s = 0 + \pi$, $C_t^{(+)}$ remains at the fixed point $P_0 = \xi_0^\mu$ while for $s = \pi + 2\pi$, it sweeps over the surface $\mathcal{P}^{(2)}$ arriving for $t = \frac{\pi}{2}$ at a reparametrisation of the loop $C^{(2)}$. Then for $t = \frac{\pi}{2} + \pi$, $s = 0 + \pi$, $C_t^{(+)}$ sweeps over the surface $\mathcal{P}^{(1)}$ arriving for $t = \pi$ at a reparametrisation of the loop $C^{(1)}$, while for $s = \pi + 2\pi$ it remains unchanged at $C^{(2)}$. Thus for $t = 0 + \pi$, the whole loop $C^{(+)}$ sweeps over a surface composed of $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ arriving for $t = \pi$ at $C_{t=\pi}^{(+)} = C^{(2)} * C^{(1)}$ as we wanted.

To show that (5.17) is indeed satisfied, write first:

$$\Theta_{\Sigma^{(+)}(\frac{\pi}{2}, 0)} = \Theta_{\Sigma^{(+)}(\frac{\pi}{2}, \frac{\pi}{2})} \Theta_{\Sigma^{(+)}(\frac{\pi}{2}, 0)}. \quad (5.19)$$

The second factor is

$$\Theta_{\Sigma^{(+)}(\frac{\pi}{2}, 0)} = p_t \exp i e \int_0^{\frac{\pi}{2}} dt \int_0^{2\pi} ds F_{\mu}^{(+)}(s) \frac{\partial \xi_t^{(+)\mu}(s)}{\partial t}, \quad (5.20)$$

since for $s < \pi$, $\xi_t^{(+)\mu}(s)$ is by construction independent of t . Changing then the integration variables t and s in (5.20) to t_2 and s_2 according to (5.18) we have

$$\Theta_{\Sigma^{(+)}(\frac{\pi}{2}, 0)} = \Theta_{\Sigma^{(2)}(\pi, 0)}. \quad (5.21)$$

On the other hand, the first factor in (5.19) is

$$\Theta_{\Sigma^{(+)}(\frac{\pi}{2}, \frac{\pi}{2})} = p_t \exp i e \int_{\frac{\pi}{2}}^{\pi} dt \int_0^{\pi} ds F_{\mu}^{(+)}(s) \frac{\partial \xi_t^{(+)}(s)}{\partial t}, \quad (5.22)$$

since $\xi_t^{(+)\mu}(s)$ is by construction independent of t for $s > \pi$. However, by (I), $F_{\mu}^{(+)}(s)$ is independent of $\xi_t^{(+)\mu}(s')$ for $s' > s$ and can therefore be replaced in (5.22) by $\tilde{F}_{\mu}^{(+)}(s)$, where

$$\tilde{F}_{\mu}^{(+)} = \begin{cases} \tilde{\xi}_t^{(+)}(s) = \xi_0 : t = 0 \rightarrow \frac{\pi}{2}, s = 0 \rightarrow 2\pi, \\ \tilde{\xi}_t^{(+)}(s) = \xi_t^{(+)}(s) : t = \frac{\pi}{2} \rightarrow \pi, s = 0 \rightarrow \pi, \\ \tilde{\xi}_t^{(+)}(s) = \xi_0 : t = \frac{\pi}{2} \rightarrow \pi, s = \pi \rightarrow 2\pi. \end{cases} \quad (5.23)$$

Then changing the integration variables in (5.22) to t_1 and s_2 according to (5.18), we obtain

$$\Theta_{\Sigma^{(+)}(\frac{\pi}{2}, \frac{\pi}{2})} = \Theta_{\Sigma^{(1)}(\pi, 0)}. \quad (5.24)$$

Substituting now (5.21) and (5.24) into (5.19), we arrive at the desired result (5.17) which proves the composition law.

Finally, we show that the parametrisation independence of $\Phi(C)$ and the composition law (3.1) together imply that $\Phi(C)$ is the phase factor of a gauge potential $A_{\mu}(x)$ which is determined by $\Phi(C)$ uniquely up to gauge transformations²³.

Consider first the case when there is no monopole charge present in all space-time X . For each point x then in X , we choose a path γ_x joining the fixed point $P_0 = x_0^{\mu}$ to x in such a way that neighbouring points are joined to P_0 by neighbouring paths. For instance, we may take γ_x to be just the straight line joining P_0 to x . Hence for any point x' close to x and $\gamma_{x'}x$ being the straight line joining x to x' , the composition $\gamma_{x'}^{-1} * \gamma_{x'}x * \gamma_x$ forms a closed loop to which we can assign an element of the gauge group:

$$g(x', x) = \Phi(\gamma_{x'}^{-1} * \gamma_{x'}x * \gamma_x), \quad (5.25)$$

whose value is independent of how the closed loop is parametrised. Define then the gauge potential $A_{\mu}(x)$ corresponding to this choice of γ_x as the algebra-valued limit:

$$A_{\mu}(x) = \frac{1}{e} \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \{ g(x + \Delta \delta_{\mu}^{\alpha}, x) - 1 \}. \quad (5.26)$$

To show that this $A_{\mu}(x)$ so constructed does indeed give rise to a phase factor equal to $\Phi(C)$ as in (2.2) we note that by virtue of the composition law (3.1) and the definition (5.25) of $g(x', x)$, $\Phi(C)$ for any C can be written in the form:

$$\Phi(C) = p_s \prod_s g(\xi(s) + \frac{d\xi}{ds} ds, \xi(s)) \quad (5.27)$$

as illustrated in Fig. 4. Further, since by (5.26)

$$g(\xi(s) + \frac{d\xi}{ds} ds, \xi(s)) = \exp i e \left[A_{\mu}(\xi(s)) \frac{d\xi^{\mu}(s)}{ds} ds \right] \quad (5.28)$$

for ds infinitesimal, we obtain the expression (2.2) as required.

Given $\Phi(C)$, the gauge potential $A_\mu(x)$ defined in (5.26) depends only on the choice of the paths γ_x . Had we chosen another set of paths γ'_x joining again the fixed point P_0 to each point x in X , we would have obtained instead another potential $A'_\mu(x)$ derived similarly from:

$$g'(x', x) = \Phi(\gamma_{x'}^{-1} * \gamma_{x'} * \gamma'_x). \quad (5.29)$$

However, by the composition law (3.1), we may write:

$$\begin{aligned} \Phi(\gamma_{x'}^{-1} * \gamma_{x'} * \gamma'_x) \\ = \Phi(\gamma_{x'}^{-1} * \gamma_{x'}.) \Phi(\gamma_{x'}^{-1} * \gamma_{x'} * \gamma'_x) \Phi(\gamma_x^{-1} * \gamma'_x), \end{aligned} \quad (5.30)$$

as illustrated in Fig. 5. Then by defining

$$S(x) = \Phi(\gamma_x^{-1} * \gamma'_x), \quad (5.31)$$

we have

$$g'(x', x) = S(x')g(x', x)S^{-1}(x). \quad (5.32)$$

Equivalently, by (5.26), we have

$$A'_\mu(x) = S(x) A_\mu(x) S^{-1}(x) - \frac{i}{e} S(x) \partial_\mu S^{-1}(x), \quad (5.33)$$

which is just the usual gauge transformations on a gauge potential. In other words, we have shown that the gauge potential so defined is unique up to gauge transformations.

To see how the construction above for A_μ is to be generalised to the situation when there are monopoles present, it suffices for illustration to consider the case with a single monopole. The domain D we need to consider then is no longer all of space-time X but X minus the monopole world-line. This D not being contractible, it is no longer possible to define γ_x for all x in D in such a way that neighbouring points are joined to P_0 by neighbouring paths. We proceed then as follows. For each time, we cover

the space around the monopole by two patches N and S as in (4.3). The domain D is then covered by two patches obtained by taking the union of N and S respectively over all times, which we continue to denote by the same symbols. For each patch, we may now define a path γ_x for every point in the patch so that points which are neighbouring in the patch are joined to the fixed point P_0 by neighbouring paths. For instance, we may choose to define γ_x as follows. For any point x in the northern (southern) patch at the same time as the fixed point P_0 , namely $x^0 = x_0^0$, we take γ_x to be the straight line joining P_0 to a point $P_N(P_S)$ on the northern (southern) axis plus the straight line joining $P_N(P_S)$ to x . For any point x not at the same time as P_0 , i.e. $x^0 \neq x_0^0$, we extend the path further by another piece running parallel to the monopole world-line.

Now for any point x' close to x in the same patch, the straight line $\gamma_{x'}$ joining x' to x' lies also entirely in the patch. We can therefore repeat the arguments above to construct gauge potentials $A_\mu^{(N)}(x)$ and $A_\mu^{(S)}(x)$ for each of the patches separately. It remains then only to show that these potentials satisfy the proper patching conditions, so that together they represent a gauge potential over the whole domain D . To do this, consider some point x lying in the overlap region. The value of $A_\mu(x)$ constructed from (5.26) depends on whether x is considered as a point in the northern patch or as a point in the southern patch, since the choice of γ_x differs in the two patches. However, the result (5.33) relating $A_\mu^{(S)}(x)$ for two different choices of γ_x guarantees that $A_\mu^{(N)}(x)$ is related to $A_\mu^{(S)}(x)$ by a gauge transformation, thus;

$$A_\mu^{(N)}(x) = S(x) A_\mu^{(S)}(x) S^{-1}(x) - \frac{i}{e} S(x) \partial_\mu S^{-1}(x), \quad (5.34)$$

with

$$S(x) = \Phi(\gamma_{x'}^{-1} * \gamma_x^{(N)}). \quad (5.35)$$

In other words, $A_\mu(x)$ satisfies the proper patching conditions and is therefore well-defined as a global quantity over the whole domain.

Notice that this $A_\mu(x)$ is still related by (2.2) to $\Phi(C)$ for any C , only with the understanding now that if C lies partly in one patch and partly in the other, then in evaluating the integral (2.2) in the

overlapping region, the appropriate patching transformations have to be performed. Further, it is still uniquely determined by $\Phi(C)$ up to gauge transformations. This therefore completes our verification of assertion (A).

Although our arguments were presented explicitly only for a theory with gauge group $SO(3)$, they are seen to be readily generalisable. For pure Yang-Mills theories with gauge group $SU(N)/Z_N$, little more is required than replacing $\zeta_{\Sigma} = \exp i 2\pi/N$, $r=0,1,\dots,N-1$ and the Pauli matrices by the appropriate $N \times N$ matrices. For $U(N)$ theories on the other hand, one would have two sets of loop variables, the nonabelian set $F_{\mu}(C|s)$ and the abelian set $f_{\mu}(C|s)$, which are subject respectively to the conditions (4.5)a and (4.5)b. The construction of the abelian potential $s_{\mu}(x)$ from (4.5)b is trivial, being just a rephrasing of the Poincaré lemma, while the nonabelian potential $A_{\mu}(x)$ can be constructed using the above method from the condition (4.5)a.

6 Action Principle

The result in the preceding section allows one to replace $A_{\mu}(x)$ by $F_{\mu}(C|s)$ as variables for describing a gauge theory. In particular, one may now formulate the action principle for the theory in terms of the variables $F_{\mu}(C|s)$.

To do so, we note that the usual Yang-Mills action:

$$\mathcal{A} = -\frac{1}{16\pi} \int d^4x \operatorname{Tr} [F_{\mu\nu}(x)F^{\mu\nu}(x)], \quad (6.1)$$

for $F_{\mu\nu}$ given by (3.5) in terms of the potential A_{μ} , can be rewritten in terms of the loop space connection $F_{\mu}(C|s)$ as:

$$\mathcal{A} = -(4\pi\bar{N})^{-1} \int_0^{2\pi} ds \operatorname{Tr} [F_{\mu}(C|s)F^{\mu}(C|s)] \left[\frac{d\zeta_{\nu}(s)}{ds} \right]^{-1}, \quad (6.2)$$

where for C parametrised as in (2.1),

$$\int_0^{2\pi} ds \dots = \int_{s'} d^4\zeta(s'), \dots, \quad (6.3)$$

$$\bar{N} = \int_0^{2\pi} ds \int_{s' \neq s} d^4\zeta(s'), \quad (6.4)$$

both integrals being defined as functional integrals in the usual way.

To prove (6.2), substitute for $F_{\mu}(C|s)$ the expression (3.4) obtaining:

$$\mathcal{A} = -(4\pi\bar{N})^{-1} \int_0^{2\pi} ds \operatorname{Tr} [F_{\mu\nu}(\zeta(s))F^{\mu\lambda}(\zeta(s))] \left[\frac{d\zeta_{\nu}(s)}{ds} \right]^{-1} \frac{d\zeta_{\lambda}(s)}{ds} \quad (6.5)$$

$$\left[\frac{d\zeta_{\rho}(s)}{ds} \right]^{-1} \left[\frac{d\zeta_{\rho}(s)}{ds} \right]^{-1}.$$

Inserting in the integrand of (6.5)

$$\int d^4x \delta^4(\zeta(s)-x) = 1, \quad (6.6)$$

and integrating over $\zeta(s)$, we obtain

$$\mathcal{A} = -\int_0^{2\pi} ds \int_{s' \neq s} d^4\zeta(s') \int d^4x \operatorname{Tr} [F_{\mu\nu}(\zeta(s))F^{\mu\lambda}(\zeta(s))] \times \quad (6.7)$$

$$\left[\frac{d\zeta_{\nu}(s)}{ds} \right]^{-1} \left[\frac{d\zeta_{\rho}(s)}{ds} \right]^{-1} \left[\frac{d\zeta_{\rho}(s)}{ds} \right]^{-1} \left[\frac{d\zeta_{\lambda}(s)}{ds} \right]^{-1} \Big|_{\zeta(s)=x}.$$

Averaging now over all directions in space-time of the tangent vector $d\zeta^{\mu}(s)/ds$ to the loop C at $\zeta(s)=x$, we get:

$$\mathcal{A} = -(4\pi\bar{N})^{-1} \int_0^{2\pi} ds \int_{s' \neq s} d^4\zeta(s') \left\{ \frac{1}{4} \int d^4x \operatorname{Tr} [F_{\mu\nu}(x)F^{\mu\nu}(x)] \right\}, \quad (6.8)$$

which is (6.1) as required.

The classical action principle may then be formulated in loop space as the requirement that the action (6.2) be stationary against variations in $F_{\mu}(C|s)$, subject however to the constraints (I),(II) and (III) in section 5 being satisfied. The resulting equations of motion are exactly the Yang-Mills equations, as expected. Written in terms of loop variables, they are (10):

$$\frac{\delta}{\delta\zeta^{\mu}(s)} F_{\mu}(C|s) = 0, \quad (6.9)$$

to be satisfied for all $(C|s)$. The derivation of (6.9) from the action of (6.2) and the constraints (I), (II) and (III) requires some effort, and is summarized in the Appendix A.

If this formalism were to be extended to the quantum theory, then presumably Feynman integrals over the variables $F_\mu(C|s)$ will have to be performed, which however, by virtue again of assertion (A) in section 5, is allowed to range only over those values satisfying the constraints (I), (II) and (III). In appendix B, a suggestion is made on how these constraints may be imposed.

Again although the results in this section are given explicitly only for the gauge group $SO(3)$, they are seen to be trivially generalisable to all $SU(N)$ and $U(N)$ theories.

Finally, we remark that we have made the specific choice to use $F_\mu(C|s)$ as field variables, and to impose the constraints (I), (II) and (III) in the form given, as apparently the most convenient for our purpose of formulating the action principle and deriving equations of motion. In other problems it may happen that another choice of variables satisfying an equivalent set of constraints may be more appropriate. In particular, for lattice calculations done in the absence of monopole charges, it is probably much more convenient to impose the constraint (III) in the differential form $G_{\mu\nu}(C; s, s') = 0$. One may also consider using $\Phi(C)$ as variables rather than $F_\mu(C|s)$.

We are deeply indebted both to G. B. Segal for fruitful discussions with T.S.T., and to T.T. Wu for the same with C.H.M. and T.S.T. Two of us, C.H.M. and P.S. have also benefitted from conversations with G. Marchesini.

Appendix A. Yang-Mills Equations from Action Principle

The problem is to make the action $\mathcal{A}$ in (6.2) stationary against variations of the variables $F_\mu(C|s)$ subject to the constraints (I), (II) and (III) being satisfied.

The constraint (I) is already accounted for explicitly by the label $(C|s)$. The other two constraints will be imposed using the standard method of Lagrange multipliers. We introduce algebra-valued multipliers $\lambda(C|s)$ for (II) and Λ_Σ for (III). Notice that for (III), $\Theta(\Sigma)$ in (4.2) is a group element so that only its logarithmic derivative is meaningful. Hence, under a variation $\Delta F_\mu(C|s)$, we obtain from (4.2) the algebraic condition:

$$\begin{aligned} \Theta^{-1}(\Sigma) \Delta \Theta(\Sigma) &= ie \int dt \Theta_\Sigma^{-1}(t, 0) \int ds \Delta F_\mu(C|s) \frac{\partial \xi_t^\mu(s)}{\partial t} \Theta_\Sigma(t, 0) \\ &= 0. \end{aligned} \quad (A.1)$$

For $\mathcal{A}$ to be stationary under (II) and (III), we require:

$$\begin{aligned} \Delta \mathcal{A} + ie \int d\Sigma \int dt d\tau \text{Tr} [\Lambda_\Sigma \Theta_\Sigma^{-1}(t, 0) \Delta F_\mu(C|s) \Theta_\Sigma(t, 0)] \frac{\partial \xi_t^\mu(s)}{\partial t} \\ + \int dC \int ds \text{Tr} [\lambda(C|s) \Delta F_\mu(C|s)] \frac{d\xi_t^\mu(s)}{ds} = 0, \end{aligned} \quad (A.2)$$

for all variations $\Delta F_\mu(C|s)$, where

$$\int d\Sigma \dots = \int \prod_s \prod_t d^4 \xi_t(s) \dots \quad (A.3)$$

is again defined just as a functional integral.

Putting to zero then the coefficient in (A.2) of $\Delta F_\mu(C|s)$ for all μ and $(C|s)$, we have the Euler-Lagrange equations:

$$\begin{aligned} (2\pi N)^{-1} F^\mu(C|s) \rho(s) \left[\frac{d\xi_s^\nu(s)}{ds} \frac{d\xi_s^\mu(s)}{ds} \right] \\ = ie \int d\Sigma \int dt \Theta_\Sigma(t, 0) \Lambda_\Sigma \Theta_\Sigma^{-1}(t, 0) \frac{\partial \xi_t^\mu(s)}{\partial t} \delta(C_t - C) \Big|_s \\ + \lambda(C|s) \frac{d\xi_t^\mu(s)}{ds} \rho(s), \end{aligned} \quad (A.4)$$

where

$$\rho(s) = \int_{s' > s} d^4 \xi(s'), \quad (A.5)$$

$$\delta(C_t - C) \Big|_s = \pi \int_{s' < s} \delta^4(\xi_t(s') - \xi(s')). \quad (A.6)$$

The multiplier $\lambda(C|s)$ can be eliminated using the constraint (II), i.e. (3.7), giving

$$F^\mu(C|s) = (2\pi\bar{n}) \int \rho^{-1}(s) \left[\frac{d\xi^\nu(s)}{ds} \frac{d\xi_\nu(s)}{ds} \right] \times \int \delta\Sigma/dt \, \Theta_\Sigma(t,0) \Lambda_\Sigma^{-1} \Theta_\Sigma^{-1}(t,0) \frac{\partial \xi_t^\mu(s)}{\partial t} \delta(C_t - C) \Big|_s$$

- long. comp.,

(A.7)

where "long. comp." represents the component of the preceding term along the tangent $d\xi_t^\mu(s)/ds$ of the loop C at the point s .

In order to arrive at the Yang-Mills equation, we need further to eliminate Λ_Σ from (A.7). Take then the 'local loop divergence': $\delta/\delta\xi_t^\mu(s)$ (summed over μ) of (A.7). The only factor dependent on $\xi_t^\mu(s)$ in the integrand of both terms on the right is $\delta(C_t - C) \Big|_s$. Write $\delta/\delta\xi_t^\mu(s)$ on this factor as $-\delta/\delta\xi_t^\mu(s)$ and integrate by parts with respect to $d^4\xi_t(s)$. We obtain:

$$\frac{\delta}{\delta\xi_t^\mu(s)} F^\mu(C|s) = (2\pi\bar{n}) \int \rho^{-1}(s) \left[\frac{d\xi^\nu(s)}{ds} \frac{d\xi_\nu(s)}{ds} \right] \times \int \delta\Sigma/dt \, \Theta_\Sigma(t,0) \left\{ \frac{\delta}{\delta\xi_t^\mu(s)} \Lambda_\Sigma^{-1} \Theta_\Sigma^{-1}(t,0) \frac{\partial \xi_t^\mu(s)}{\partial t} \delta(C_t - C) \Big|_s \right. \\ \left. - \text{long. comp.} \right\} \quad (A.8)$$

where by virtue of (3.3) and (A.7), the terms obtained by differentiating $\Theta_\Sigma(t,0) = \Phi^{-1}(C_t)$ and its inverse with respect to $\xi_t^\mu(s)$ give together ie $[F^\mu(C|s), F_\mu(C|s)]$ and vanished.

Next, we remark that we have introduced above one Λ_Σ for each parametrised surface Σ . The conditions (4.2), or equivalently (A.1), however, are not independent for all Σ 's. If it so happens that two conditions (4.2) are identical for two different Σ 's, then the corresponding Λ_Σ should also be put equal.

We notice now that for $\Theta(\Sigma)$ and $\Theta_\Sigma(t,0)$ defined as in (3.16) and (5.3) in terms of $F_\mu(C|s)$ the following identity is satisfied,

$$\int \delta\Sigma \, \Theta_\Sigma(t,0) \frac{\delta}{\delta\xi_t^\mu(s)} \left\{ \Theta^{-1}(\Sigma) \Delta\Theta(\Sigma) \right\} \Theta_\Sigma^{-1}(t,0) \frac{\partial \xi_t^\mu(s)}{\partial t} \delta(C_t - C) \Big|_s = 0. \quad (A.9)$$

To see this, write first

$$\frac{\delta}{\delta\xi_t^\mu(s)} \left\{ \Theta^{-1}(\Sigma) \Delta\Theta(\Sigma) \right\} = \Theta^{-1}(\Sigma) \Delta \left\{ \frac{\delta}{\delta\xi_t^\mu(s)} \Theta(\Sigma) \Theta^{-1}(\Sigma) \right\} \Theta(\Sigma), \quad (A.10)$$

where, in close analogy to (3.4), we have

$$\frac{\delta}{\delta\xi_t^\mu(s)} \Theta(\Sigma) \Theta^{-1}(\Sigma) \\ = - \int ds' G_{\mu\nu}(C_t; s, s') \frac{\partial \xi_{t'}^\nu(s')}{\partial t} \Big|_{\Theta_\Sigma^{-1}(2\pi, t)}, \quad (A.11)$$

$G_{\mu\nu}(C; s, s')$ being the curvature in loop space defined in (3.8). Substituting into the right of (A.9), one obtains:

$$\int \delta\Sigma \, \Theta_\Sigma^{-1}(2\pi, t) \Delta \left\{ - \int ds' G_{\mu\nu}(C_t; s, s') \frac{\partial \xi_{t'}^\nu(s')}{\partial t} \frac{\partial \xi_t^\mu(s)}{\partial t} \right\} \times \\ \delta(C_t - C) \Big|_s \Theta_\Sigma^{-1}(2\pi, t) \Theta_\Sigma(2\pi, t), \quad (A.12)$$

where only the factors $\partial \xi_t^\mu(s)/\partial t$ and $\partial \xi_{t'}^\nu(s')/\partial t$ in the integrand depend on $\xi_t^\mu(s)$ for $t' < t$. Averaging then for $t' < t$ over all directions of the tangent $\partial \xi_t^\mu(s)/\partial t$ to the surface Σ , we may make the following replacement:

$$\frac{\partial \xi_t^\mu(s)}{\partial t} - \frac{\partial \xi_t^\nu(s')}{\partial t} + \frac{1}{4} g^{\mu\nu} \delta(s-s'), \quad (\text{A.13})$$

which, by antisymmetry of the curvature $G_{\mu\nu}(C; s, s')$ under $\mu \leftrightarrow \nu$ and $s \leftrightarrow s'$, gives (A.9) as required. By analogy to (3.7) to which it has some similarity, (A.9) reflects, we believe, the parametrisation independence of the quantity $\Theta(\Sigma)$.

Now, the identity (A.9) means that the following constraint on $F(C|s)$:

$$\int \delta \Sigma \Theta_\Sigma(t, 0) \{ \Theta^{-1}(\Sigma) \Delta \Theta(\Sigma) \} \Theta_\Sigma^{-1}(t, 0) \delta(C - C) \Big|_s = 0, \quad (\text{A.14})$$

which may be obtained from a subset of (A.1) by first conjugating with $\Theta_\Sigma(t, 0)$, then making the appropriate sum, is in fact identical to another constraint obtained similarly from another subset of (A.1) in which Σ is displaced at (t, s) along the surface Σ in the direction of t , thus,

$$\Theta^{-1}(\Sigma) \Delta \Theta(\Sigma) + \Theta^{-1}(\Sigma) \Delta \Theta(\Sigma) + \frac{\delta}{\delta \xi_t^\mu(s)} \{ \Theta^{-1}(\Sigma) \Delta \Theta(\Sigma) \} \frac{\partial \xi_t^\mu(s)}{\partial t} = 0. \quad (\text{A.15})$$

This in turn implies that their corresponding Lagrange multipliers are also equal, or that

$$\int \delta \Sigma \Theta_\Sigma(t, 0) \left\{ \frac{\delta}{\delta \xi_t^\mu(s)} \Lambda_\Sigma \right\} \Theta_\Sigma^{-1}(t, 0) \frac{\partial \xi_t^\mu(s)}{\partial t} - \delta(C - C) \Big|_s = 0. \quad (\text{A.16})$$

Here we take $\partial \xi_t^\mu(s)/\partial t$ by convention to be the derivative from below. If one takes instead $\partial \xi_t^\mu(s)/\partial t$ to be the derivative from above, the same result is obtained by inverting the ordering of the factors in (A.1) and in all subsequent formulas.

Hence, we obtain that the first term on the right of (A.8) vanishes. Very similar arguments applied to the second term 'long. comp.' in (A.8) shows that it too vanishes. Thus, we arrive finally at:

$$-\frac{\delta}{\delta \xi_t^\mu(s)} F^\mu(C|s) = 0 \quad (\text{A.17})$$

for all $C(s)$, which is Yang-Mills equation in loop space. Indeed, in terms of the more familiar field variables in space-time, (A.17) by (3.4) may be rewritten as:

$$\Phi_C^{-1}(s, 0) [D_{\mu\nu}^\mu(\xi(s))] \Phi_C(s, 0) \frac{d\xi^\nu(s)}{ds} = 0, \quad (\text{A.18})$$

which, being valid for all C and all s , is identical to the statement that $D_{\mu\nu}^\mu F_{\mu\nu} = 0$ for all points x in space-time.

Appendix B. Constraints in Feynman Integrals

Constraints (I) and (II) are readily imposed. For (III), we note that for any element ω in $SU(2)$:

$$\omega = \exp i \vec{x} \cdot \vec{\sigma}, \quad (\text{B.1})$$

σ_i being Pauli matrices, the following identity holds, being a generalisation of the familiar periodic δ -function:

$$\sum_{l=-\infty}^{\infty} \frac{1}{2} \text{Tr} [\omega^l] = 2\pi \sum_{n=0}^{\infty} \delta(|x| - 2n\pi), \quad (\text{B.2})$$

where the R.H.S. vanishes except for $|x| = 2n\pi$, which is exactly the necessary and sufficient condition for $\omega = I$. One may therefore formally incorporate the condition (III) into the Feynman integral by inserting in the integrand a factor of the type (B.2) for each Σ , resulting in Feynman integrals of the form:

$$W \equiv \int \prod_{(C|s), \alpha} \delta F_{\alpha}(C|s) \mu[F_{\rho}(C|s)] \exp i \mathcal{A} X$$

$$\prod_{(C|s)} \delta(F_{\sigma}(C|s) \frac{d\mathcal{E}_{\sigma}^{\alpha}(s)}{ds}) \prod \{ \sum_{\Sigma} \sum_{\lambda_{\Sigma}=-\infty}^{\infty} \frac{1}{2} \text{Tr} [(\theta(\Sigma) \zeta_{\Sigma}^{-1})^{\lambda_{\Sigma}}] \} , \quad (B.3)$$

where $\mu[F_{\mu}(C|s)]$ is a measure yet to be determined.

In the absence of monopoles, (III) can also be imposed in the differential form by just inserting a δ -function in $G_{\mu\nu}(C;s,s')$ for each $(C;s,s')$.

References

- 1) Y. Aharanov and D. Bohm, Phys. Rev. 115, (1959) 485;
R.G. Chambers, Phys. Rev. Letters 5, (1960) 3.
- 2) Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D12, (1975) 3845.
- 3) Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D12, (1975) 3843.
- 4) I. Biaynicki-Birula, Bull. Acad. Pol. Sci 11, (1963) 135.
- 5) S. Mandelstam, Phys. Rev. 175, (1968) 1580.
- 6) K.G. Wilson, Phys. Rev. D10, (1974) 2445.
- 7) Chen Ning Yang, Phys. Rev. Letters 33, (1974) 445.
- 8) G. 't Hooft, Nucl. Phys. B138, (1978) 1.
- 9) Yu. M. Makeenko and A.A. Migdal, Phys. Letters 88B, (1979) 135;
Nucl. Phys. B188, (1981) 269.
- 10) A.A. Migdal, Ann. Phys. (N.Y.) 126, (1980) 279.
- 11) A.M. Polyakov, Phys. Letters 82B, (1979) 247; Nucl. Phys. B164, (1979) 171.
- 12) S. Mandelstam, Phys. Rev. D19, (1979) 2391.
- 13) R. Giles, Phys. Rev. D24, (1980) 2160.
- 14) Chaohao Gu and Ling-Lie Chau Wang, Phys. Rev. Letters 45, (1980) 2004.
- 15) Tsou Sheung Tsun, Proceedings of the Trieste Meeting on Monopoles in Quantum Field Theory (ed. N.S. Craigie et al., World Scientific, 1981), 283.
- 16) Other related work:
R. Minlos and Ya. Sinai, Mat. Sbor. 18, (1967).
Y. Nambu, Phys. Letters 80B, (1979) 372.
J.L. Gervais and A. Neveu, Phys. Letters 80B, (1979) 255.
E. Corrigan and B. Hasselacher, Phys. Letters 81B, (1979) 181.
T. Eguchi and Y. Hosotani, Phys. Letters 96B, (1980) 349.
M. Lüscher, K. Szymanzik and P. Weisz, Nucl. Phys. B173, (1980) 365.
R. Gambini and A. Trias, Phys. Rev. D22, (1980) 1380.
R.A. Brandt, A. Gocksch, M.A. Sato and F. Neri, Phys. Rev. D26, (1982) 3611.
B. Durhuus and J.M. Leinaas, Physica Scripta 25, (1982) 504.
C. di Bartolo, F. Neri, R. Gambini, and A. Trias, Nuovo Cim. Letters 38, (1983) 497.

- 16) Tu Tung-Sheng, Wu Tai-Tsun and Yang Chen-ning, *Scientia Sinica* 21, (1978) 317.
- 17) D. Förster, *Phys. Letters* 87B, 87 (1979); *Nucl. Phys.* B170, (1979) 91.
T. Eguchi, *Phys. Letters* 87B, (1979) 91.
A. Jevicki and B. Sakita, *Phys. Rev. D* 22, (1980) 467.
J.B. Kogut, *Rev. Mod. Phys.* 55, (1983) 775.
A. Jevicki, O. Karim, J.P. Rodrigues and H. Levine, *Nucl. Phys.* B213 169 (1983); B230, (1984) 299.
- 18) For topics related to differential geometry, see e.g. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol.1, (Interscience Publishers, 1963, New York).
- 19) For a rigorous treatment, see e.g. J.P. Serre, *Ann. Maths.* 54, (1951) 425; G. B. Segal, private communication.
- 20) E. Lubkin, *Ann. Phys. (N.Y.)* 23, (1963) 233.
- 21) S. Coleman, *Erice School*, (1975) 297.
- 22) See e.g. Chan Hong-Mo and Tsou Sheung Tsun, *Nucl. Phys.* B189, (1981) 364 and references therein.
- 23) In the absence of monopoles, this result seems to be well known; see e.g. the parallel treatment in ref. 11) and 12). Here we follow the arguments outlined to one of us (TST) by G. B. Segal.

Figure Captions

- Fig. 1. : Illustration for the curvature $G_{\mu\nu}$ in loop space.
- Fig. 2. : Illustration for the holonomy $\theta(\Sigma)$ in loop space.
- Fig. 3. : Illustration for proving the composition law.
- Fig. 4. : Illustration for constructing the potential A_μ .
- Fig. 5. : Illustration for constructing the potential A_μ from different sets of paths γ and γ' .

FIG 1

FIG 2

FIG 3

FIG 4

FIG 5