RAL-85-044

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May 1985

RAL-85-044

Science and Engineering

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Action Principle and Equations of Motion for Nonabelian Monopoles

by

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EnglandAbstract

Monopoles in gauge theories have an intrinsic interaction with the gauge field. The definition of a monopole as a topological charge implies a certain constraint coupling the gauge field to the co-ordinates of a particle carrying that charge. Hence, even starting with the free action, the constraint will give equations of motion in which field and particle interact. Applied to electromagnetism, this procedure gives the Maxwell and Lorentz equations. In this paper, we apply the same idea to nonabelian monopoles to deduce their equations of motion which are otherwise unknown. To surmount certain technical difficulties connected with patching, loop space techniques are developed to solve the variational problem. A closed set of equations are obtained, which are analogous to the Maxwell and Lorentz equations, and bear also a formal resemblance to the Wong equations for a 'classical' point source of Yang-Mills fields.

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1. Introduction

Monopole charges are an intrinsic common attribute of gauge theories with compact gauge groups. The presence of a monopole charge at a point in space represents a certain topological configuration of the gauge field in the surrounding spatial region. When a monopole charge moves, the field around it will therefore have to rearrange itself accordingly. This means that the motion of the field is coupled to that of the monopole, or that there is an effective interaction between them. In other words, the definition of a monopole as a topological charge already specifies to a large extent its dynamics.

The observation above may be formulated more precisely as follows. Let $\dot{\phi}_F^0$ be the action of the free field and $\dot{\phi}_M^0$ that of a free particle, which are functionals of the gauge potential $A_\mu(x)$ and the particle co-ordinates $y^\mu(\tau)$ respectively. The requirement that

$$\dot{\phi}^0 = \dot{\phi}_F^0 + \dot{\phi}_M^0 \quad (1.1)$$

be stationary against all variations of $A_\mu(x)$ and $y^\mu(\tau)$ gives the equations of motions of a free particle and a free field. However, if one now specifies that the particle should carry a monopole charge, one imposes a topological constraint on the field around the particle, relating thus $A_\mu(x)$ to $y^\mu(\tau)$. Requiring that the action (1.1) be stationary under this constraint will no longer give the free equations of motion but those of an interacting system coupling the gauge potential to the particle co-ordinate.

When the procedure outlined above is applied to electromagnetism with:

$$\dot{\phi}^0 = -\frac{1}{16\pi} \int d^4x f_{\mu\nu}(x) f^{\mu\nu}(x) - M \int d\tau, \quad (1.2)$$

$$f_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x), \quad (1.3)$$

$$d\tau = \sqrt{dy^2}, \quad (1.4)$$

the following equations of motion result for a monopole carrying magnetic charge g ,

$$\partial_\nu f^{\mu\nu}(x) = 0. \quad (1.5)$$

$$M \frac{d^2 y^\mu(x)}{d\tau^2} = -g \int d\tau \dot{y}^\nu(\tau) \frac{dy^\mu(\tau)}{d\tau}, \quad (1.6)$$

$$\partial_\nu^* f^{\mu\nu}(x) = -4\pi g \int d\tau \frac{dy^\mu(\tau)}{d\tau} \delta^4(x-y(\tau)). \quad (1.7)$$

These are exactly what one would obtain by replacing e with g and $f_{\mu\nu}(x)$ with $f_{\mu\nu}^*(x) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}(x)$ in the familiar equations of motion of an electric charge e in the electromagnetic field. Indeed, since electromagnetism is dual invariant, one could if one chooses, 'derive' the equations of motion of an electric charge also in this way by considering it as an 'electric monopole' of the dual Maxwell field $f_{\mu\nu}^*(x)$. The derivation is an interesting alternative to the conventional approach in avoiding the explicit introduction of an interaction term in the action, and leads automatically and uniquely to the same results as the minimal coupling hypothesis.

In this paper, we are interested in applying the same procedure to monopole charges in $su(N)$ Yang-Mills theories. These theories being not dual invariant, the equations of motion can no longer be taken for granted to be the dual transforms of those of gauge field sources, as was the case in electromagnetism for g and e . Indeed, monopole charges here, defined in general as homotopy classes of closed curves in the gauge group, usually do not even take the same values as the charges of gauge field sources which are labelled by group representations. The procedure advocated above is therefore expected to lead to equations of motion which are otherwise unknown. If they exist, such equations ought in principle to have a similar significance to the Maxwell and Lorentz equations, namely (1.5) - (1.7), in view of the identical intrinsic manner in which both are deduced.

The programme, however, is not straightforward to carry out.

fits our purpose. It is closely related to $F_{\mu\nu}(x)$, but is both patch-independent and sufficient for a complete description of the theory. Moreover, the topological condition defining a nonabelian monopole charge is most conveniently given in terms of $F_{\mu}(C|s)$. (Note: P_s in (1.9) denotes ordering in s . In our convention, s increases from right to left and the derivative $d\xi^\mu(s)/ds$ is taken to be the derivative from below).

As variables, unfortunately, $F_{\mu}(C|s)$ form a highly redundant set, being much more numerous than the original variables $A_{\mu}(x)$. It can be shown, however, that by a sort of miracle, this redundancy is exactly removed by the topological constraint defining the monopole charge in a manner reminiscent of the Poincaré lemma in the abelian theory. As a result, one is able to solve the variational problem posed by the constrained action principle giving equations analogous to the Maxwell and Lorentz equations of electromagnetism. The solution, however, requires from us a considerable development of loop space techniques, which are reported in a separate paper.⁷⁾

The equations so obtained, being in terms of unfamiliar loop variables, are unfortunately rather hard to interpret. We have noted however an intriguing formal relation between them and the Wong equations⁸⁾ for a classical point source of Yang-Mills fields which offers some hints for future investigations.

These results are then further extended to theories with gauge groups $U(N) = (SU(N) \times U(1))/Z_N$ which include in particular the standard electroweak theory, and the chromodynamic theory combined with electromagnetism. Indeed, we believe that the principle advocated here, as well as some of the techniques used in its implementation, is applicable to deriving equations of motion for gauge theory monopoles in general, and perhaps also for topological charges of other types.

The topological constraint defining a monopole is usually stated as a patching condition on the gauge potential, which depends on the position of the monopole. As its position varies, the patching has to change as well, leading to a seemingly intractable variational problem. The technique used in ref. 1) for its solution in the abelian theory resulting in (1.5) - (1.7) relies on the theory's dual invariance and cannot be directly extended to the nonabelian case.

Our strategy is as follows. We begin with the observation that for the abelian theory, due to the particular form of the topological constraint, the variational problem can be reformulated in terms of the field tensor $f_{\mu\nu}(x)$ instead of the gauge potential $a_{\mu}(x)$ as the field variable. Since $f_{\mu\nu}(x)$ here, in contrast to $a_{\mu}(x)$, is patch-independent the technical difficulty noted above no longer arises so that the variational problem can be solved directly without relying on dual invariance, and leads to the same result.

Inspired by this example, we seek therefore an analogous variable for the nonabelian case. The field tensor $F_{\mu\nu}(x)$ is not suitable since for nonabelian theories it is also a patch-dependent quantity and does not in any case describe the theory completely.⁵⁾ We find, however, that the loop variable:⁶⁾

$$F_{\mu}(C|s) = \frac{1}{e} \Phi^{-1}(C) \frac{\delta}{\delta \xi^\mu(s)} \Phi(C), \quad (1.8)$$

the logarithmic derivative of the Dirac phase factor (or Wilson operator)

$$\Phi(C) = P_s \exp ie \int_0^{2\pi} ds A_{\mu}(\xi(s)) \frac{d\xi^\mu(s)}{ds} \quad (1.9)$$

for a loop C parametrised as

$$C = \{\xi^\mu(s) : s = 0 \rightarrow 2\pi, \xi^\mu(0) = \xi^\mu(2\pi) = \xi^\mu_0\} \quad (1.10)$$

2. The Abelian Theory

As explained in the introduction, the problem is to derive equations of motion by requiring the action $\mathcal{A}^0$ in (1.2) to be stationary against variations of the gauge potential $a_\mu(x)$ and the particle co-ordinate $Y^\mu(\tau)$ subject to the constraint that the particle at $Y(\tau)$ should carry a monopole charge $g = 1/2e$ of the gauge field $f_{\mu\nu}(x)$. The constraining condition is usually stated as a patching condition on the gauge potential $a_\mu(x)$. Thus covering as usual the spatial region around the monopole by two patches (N) and (S), say, the patching condition reads:

$$a_\mu^{(N)}(x) - a_\mu^{(S)}(x) dx^\mu = 4\pi g, \quad (2.1)$$

for any integral over a closed loop in the overlapping region.³⁾

To solve this problem, we note first that the patching condition (2.1) is exactly equivalent to the equation (1.7), as can be readily shown using Stokes' and Gauss' Theorems.¹⁾ Second, we remark that except along the monopole worldline the condition (1.7) implies that:

$$\partial_\nu f^{\mu\nu}(x) = 0, \quad x \neq Y(\tau) \quad (2.2)$$

which, by the Poincaré lemma, guarantees that for any antisymmetric $f_{\mu\nu}(x)$, there exists locally a gauge potential $a_\mu(x)$ satisfying (1.3) which is unique up to gauge transformations. This means that there is established a one-one correspondence between the gauge equivalent sets of $a_\mu(x)$ and the antisymmetric tensors $f_{\mu\nu}(x)$ satisfying the constraint (1.7). In other words, although $f_{\mu\nu}(x)$ are more numerous than $a_\mu(x)$ and therefore redundant as field variables, their redundancy is removed by the constraint (1.7). Thus, so long as (1.7) is satisfied, one may replace $a_\mu(x)$ by $f_{\mu\nu}(x)$ as variables in describing the system.

The field tensor $f_{\mu\nu}(x)$ being gauge invariant, is also patch-independent, so that the variational problem now becomes straightforward in terms of $f_{\mu\nu}(x)$ as variables. Introducing then the Lagrange multiplier $\lambda_\mu(x)$ for the constraint (1.7), we form the auxiliary

action:

$$\mathcal{A}^0 = \mathcal{A}^0 - \int d^4x \lambda_\mu(x) \left\{ \partial_\nu f^{\mu\nu}(x) + 4\pi g \int d\tau \frac{dY^\mu(\tau)}{d\tau} \delta^4(x - Y(\tau)) \right\}, \quad (2.3)$$

which we require to be stationary under all variations of $f_{\mu\nu}(x)$ and of $Y^\mu(\tau)$, obtaining in each case the Euler-Lagrange equations:

$$f^{\mu\nu}(x) = 4\pi \left\{ \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \left(\partial_\sigma \lambda_\rho(x) - \partial_\rho \lambda_\sigma(x) \right) \right\}, \quad (2.4)$$

$$M \frac{d^2 Y^\mu(\tau)}{d\tau^2} = 4\pi g \left\{ \partial_\nu \lambda_\mu(Y(\tau)) - \partial_\mu \lambda_\nu(Y(\tau)) \right\} \frac{dY^\nu(\tau)}{d\tau}. \quad (2.5)$$

The first, (2.4), implies the Maxwell equation (1.5), and further allows $\lambda_\mu(x)$ to be eliminated from the second equation (2.5) giving the Lorentz equation (1.6).

We have thus recovered exactly the same set of equations of motion as we anticipated. At no stage, however, have we made use of the dual invariance of the theory. Hence this looks like a procedure which can, we hope, be extended to the nonabelian case.

3. The Yang-Mills Theory Formulated in Loop Space

The analogous action to (1.2) for an $su(N)$ Yang-Mills theory is:

$$\mathcal{A}^0 = - \frac{1}{16\pi} \int d^4x \operatorname{Tr} [F_{\mu\nu}(x) F^{\mu\nu}(x)] - M \int d\tau, \quad (3.1)$$

with

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ie [A_\mu(x), A_\nu(x)], \quad (3.2)$$

where to avoid future confusion, we have used capital letters to denote nonabelian field quantities, reserving small letters for abelian fields.

Monopole charges here take the values $\zeta_r = \exp i2\pi r/N$, $r = 0, 1, 2, \dots, (N-1)$, which label the homotopy classes of closed curves in the gauge group $SU(N)/Z_N$. The constraint we wish to impose, namely that the particle at $Y(\tau)$ should carry a monopole charge ζ , is usually stated as a patching condition on the gauge potential, e.g.,

$$A_\mu^{(S)}(x) = Q(x) A_\mu^{(N)}(x) Q^{-1}(x) - \frac{i}{e} (Q(x) \partial_\mu Q^{-1}(x)), \quad (3.3)$$

where $Q(x)$ is a patching function, single-valued in the overlapping region, which represents a closed curve in the gauge group belonging to the homotopy class labelled by ζ . The problem is again to derive equations of motion for this system by requiring ω^0 in (3.1) to be stationary against variations of $A_\mu(x)$ and $\psi^\mu(\tau)$ subject to this constraint.

Our strategy, as explained in the introduction, is to replace $A_\mu(x)$ as variables by an equivalent patch-independent set, in analogy to $f_{\mu\nu}(x)$ in section 2. Clearly, $F_{\mu\nu}(x)$ does not work, since in nonabelian theories, $F_{\mu\nu}(x)$ is not only insufficient to describe the theory completely but also is gauge noninvariant and therefore still patch-dependent. We propose instead the variables $F_\mu(C/s)$ defined in (1.8) - (1.10) which have the following advantages.

First, they are related simply to $F_{\mu\nu}(x)$ by 6)

$$F_\mu(C/s) = \Phi_C^{-1}(s, 0) F_{\mu\nu}(\xi(s)) \Phi_C(s, 0) \frac{d\xi^\nu(s)}{ds}, \quad (3.4)$$

where $\Phi_C(s_2, s_1)$ is the parallel transport defined as:

$$\Phi_C(s_2, s_1) = P_s \exp ie \int_{s_1}^{s_2} ds A_\mu(\xi(s)) \frac{d\xi^\mu(s)}{ds}. \quad (3.5)$$

This relation shows incidentally that $F_\mu(C/s)$ depends only on $\xi^\mu(s')$ for $s' \leq s$ (as meant to be understood in the notation (C/s) of its arguments), and satisfies the transversality condition:⁶⁾

$$F_\mu(C/s) \frac{d\xi^\mu(s)}{ds} = 0, \quad (3.6)$$

which reflects the antisymmetry of $F_{\mu\nu}(x)$.

Second, although $F_{\mu\nu}(x)$ is patch-dependent, $F_\mu(C/s)$ is not, the patching transformation on $F_{\mu\nu}(x)$ in (3.4) being exactly compensated by that on $\Phi_C(s, 0)$ and its inverse. In fact, suppose that P is a point in the overlap region and C is some parametrised loop (1.10) passing through P at $s = s_P$ such that for $s \leq s_P$ C lies entirely in the northern patch. Then, the value of $F_\mu(C/s)$ in the northern gauge is obtained simply by inserting the superscript (N) to $A_\mu(x)$, $F_{\mu\nu}(x)$, $\Phi_C(s, 0)$ and $F_\mu(C/s)$ everywhere in (3.4) and (3.5). On the other hand, if we choose to regard P as a point in the southern patch, $F_{\mu\nu}(x_P)$ will take the value $F_{\mu\nu}^{(S)}(x_P)$ which is related to the northern value by the transformation:

$$F_{\mu\nu}^{(S)}(x_P) = Q(x_P) F_{\mu\nu}^{(N)}(x_P) Q^{-1}(x_P), \quad (3.7)$$

and we must write instead,

$$F_\mu^{(S)}(C|s_P) = [\Phi_C^{(N)}(s_Q, 0)]^{-1} Q^{-1}(x_Q) [\Phi_C^{(S)}(s_P, s_Q)]^{-1} F_{\mu\nu}^{(S)}(x_P) \times \Phi_C^{(S)}(s_P, s_Q) Q_C^{(N)} \Phi_C^{(N)}(s_Q, 0) \frac{d\xi^\nu(s)}{ds}, \quad (3.8)$$

where Q can be any point in the overlap region. However, the parallel transports $\Phi_C^{(S)}(s_p, s_Q)$ are now to be calculated with the potential $A_\mu^{(S)}(x)$ in the southern gauge and satisfy the following patching condition:

$$\Phi_C^{(S)}(s_p, s_Q) = \Omega(x_p) \Phi_C^{(N)}(s_p, s_Q) Q^{-1}(x_Q), \quad (3.9)$$

which, if substituted into (3.8) will give $F_\mu^{(S)}(C, s)$ the same value as $F_\mu^{(N)}(C, s)$, as stated above.

Third, the condition that the particle at $Y(\tau)$ should carry the monopole charge ζ is conveniently stated in terms of $F_\mu(C, s)$. Explicitly, let:

$$\Sigma = \{\xi_t^\mu(s); s = 0 \rightarrow 2\pi, t = 0 \rightarrow 2\pi\}, \quad (3.10)$$

with

$$\xi_t^\mu(0) = \xi_t^\mu(2\pi) = \xi_0^\mu, \quad t = 0 \rightarrow 2\pi,$$

$$\xi_0^\mu(s) = \xi_{2\pi}^\mu(s) = \xi_0^\mu, \quad s = 0 \rightarrow 2\pi, \quad (3.11)$$

represent a parametrised 2-dimensional surface in space-time X , or a loop in parametrised loop space Q^1X , passing through an arbitrary fixed point $P_0 = \{\xi_0^\mu\}$. Construct in Q^1X the holonomy,

$$\Theta(\Sigma) = P_t \exp i e \int_0^{2\pi} dt \int_0^{2\pi} ds F_\mu(C_t|s) \frac{\partial \xi_t^\mu(s)}{\partial t}, \quad (3.12)$$

as an element of the covering group $SU(N)$, where P_t denotes ordering in t , with t increasing by convention from right to left, and

$$C_t = \{\xi_t^\mu(s), s = 0 \rightarrow 2\pi\}. \quad (3.13)$$

Then it can be shown that $\Theta(\Sigma)$ is just the monopole charge,⁷⁾ taking the possible values $\zeta_r = \exp i 2\pi r/N$, which is enclosed inside the surface $\tilde{\Sigma}$ in X parametrised by Σ in (3.10). Hence the patching condition (3.3) on $A_\mu(x)$ can be restated as the requirement on $F_\mu(C, s)$ that $\Theta(\Sigma)$ satisfies:

$$\Theta(\Sigma) = \zeta_\Sigma, \quad (3.14)$$

where,

$$\zeta_\Sigma = \begin{cases} \zeta, & \text{if } \tilde{\Sigma} \text{ encloses } Y(\tau) \\ 1, & \text{otherwise.} \end{cases} \quad (3.15)$$

In general, if $\tilde{\Sigma}$ winds around the monopole more than once, then $\zeta_\Sigma = \zeta^n$ where n is the winding number $S^2 \rightarrow S^2$, which can take also negative values.

This condition is the equivalent of the Gauss Law (1.7) of the abelian theory. Although it may seem at first sight an unduly cumbersome generalisation, one is easily convinced after a few attempts that in view of the nonabelian nature of the theory here, one cannot express ζ locally in terms of the gauge potential as one did for g in the abelian case. In fact, besides (3.14), we know of no other expression which gives explicitly in terms of field variables the charge of a nonabelian monopole, which has been defined so far only abstractly as a homotopy class. The expression is particularly elegant when $\Theta(\Sigma)$ is considered as the holonomy in loop space Q^1X corresponding to $F_\mu(C, s)$ as the connection.

Fourth, although $F_\mu(C, s)$ are much more numerous than $A_\mu(x)$ and are therefore highly redundant in general as field variables, here it so

happens that their redundancy is exactly removed just by the constraint (3.14) we wish to impose. Indeed, it has been shown that from any $F_\mu(C|s)$ transverse to C and satisfying (3.14), a gauge potential $A_\mu(x)$ unique up to gauge transformations can be constructed for every point x in space-time not lying on the monopole worldline.⁷⁾ In other words, a one-one correspondence is established between the new variables $F_\mu(C|s)$ so constrained and the original variables $A_\mu(x)$ up to gauge equivalence. The situation is an almost exact parallel to that in the abelian theory of section 2 where it was the Gauss Law (1.7) (equivalent to (3.14) here) which permitted the replacement of $a_\mu(x)$ by the antisymmetric tensor $f_{\mu\nu}(x)$ (equivalent to transverse $F_\mu(C|s)$ here) as field variables.

In view of these observations, we propose to reformulate our problem in terms of $F_\mu(C|s)$ and $\gamma^\mu(\tau)$ as variables. The action then reads:⁷⁾

$$\mathcal{A}^0 = - (4\pi N)^{-1} \int_0^{2\pi} ds \operatorname{Tr} \left[F_\mu(C|s) F^\mu(C|s) \right] \left[\frac{d\xi_\alpha(s)}{ds} \frac{d\xi_\alpha(s)}{ds} \right]^{-1} \quad (3.16)$$

$$- M/d\tau,$$

with

$$\int \delta C \dots = \int \prod_{s'} d^4 \xi(s'), \dots, \quad (3.17)$$

$$\bar{N} = \int_0^{2\pi} ds \int \prod_{s' \neq s} d^4 \xi(s'), \quad (3.18)$$

both being ordinary functional integrals. The action principle is now stated as the requirement that $\mathcal{A}^0$ in (3.16) be stationary against variations in (transverse) $F_\mu(C|s)$ and $\gamma^\mu(\tau)$ subject to the condition (3.14) being satisfied for all parametrised surfaces Σ .

Both the notation and general loop space concepts used in the preceding paragraphs have been developed at length in another paper⁷⁾ to which our reader is referred for further details.

4. Constrained Variations of the Dynamical Variables

In order to solve the variational problem proposed in the preceding section, we need to "differentiate" the constraining equation (3.14) with respect to variations $\Delta F_\mu(C|s)$ of the field variables and $\Delta \gamma^\mu(\tau)$ of the particle co-ordinates. Since both the quantities $\Theta(\Sigma)$ and ζ_Σ in (3.14) are group elements only their logarithmic derivatives are meaningful, being elements of the corresponding Lie algebra.

For the quantity $\Theta(\Sigma)$ which depends only on $F_\mu(C|s)$, we have then:

$$\Theta^{-1}(\Sigma) \Delta \Theta(\Sigma) = ie \int dt ds \Theta_\Sigma^{-1}(t, 0) \Delta F_\mu(C, t, s) \Theta_\Sigma(t, 0) \frac{\partial \xi_\Sigma^\mu(s)}{\partial t}, \quad (4.1)$$

where $\Theta_\Sigma(t, 0)$ is the parallel transport along the loop Σ in $\Omega^1 X$:

$$\Theta_\Sigma(t_2, t_1) = P_t \exp ie \int_{t_1}^{t_2} dt \int_0^{2\pi} ds F_\mu(C, t, s) \frac{\partial \xi_\Sigma^\mu(s)}{\partial t}. \quad (4.2)$$

For the quantity ζ_Σ , defined only by (3.15) as a function of $\gamma(\tau)$, the problem is more subtle. Consider, for a certain Σ say, the logarithmic differential $\zeta_\Sigma^{-1} \Delta \zeta_\Sigma$ due to a variation $\Delta \gamma(\tau)$ at $\tau = \tau_0$. If $\Delta \gamma(\tau_0)$ does not cross the surface Σ in X , ζ_Σ is unchanged giving thus $\zeta_\Sigma^{-1} \Delta \zeta_\Sigma = 0$. But if $\Delta \gamma(\tau_0)$ crosses Σ in X , then $\zeta_\Sigma^{-1} \Delta \zeta_\Sigma$ must take a value $i\pi\kappa$, where κ satisfies the condition:

$$\exp i\pi\kappa = \zeta. \quad (4.3)$$

Thus $\pi\kappa$ represents the change in phase of $\Theta(\Sigma)$ as $\Delta \gamma(\tau_0)$ pokes through the surface Σ , and can depend on Σ as well as the point (t, s) on Σ at which it is poked through by $\Delta \gamma(\tau_0)$. What is κ , and how does it depend on Σ , t and s ?

To see this, we observe that the same change in phase can also be obtained in the passive mood by keeping $Y(\tau)$ fixed at τ_0 but shifting the whole surface Σ parallelly by an amount $\Delta Y(\tau_0)$ in the opposite direction. Hence,

$$\zeta_{\Sigma}^{-1} \Delta \zeta_{\Sigma} = - \int dt ds \, \theta^{-1}(\Sigma) \frac{\delta}{\delta \xi_{\Sigma}^{\mu}(s)} \theta(\Sigma) \Delta Y^{\mu}(\tau_0). \quad (4.4)$$

The quantity $\theta(\Sigma)$ being the holonomy in loop space $Q^1 X$ and closely analogous to $\Phi(C)$ in ordinary space-time X , its logarithmic derivative satisfies also a relationship in terms of the curvature similar to that satisfied by

$$ie F_{\mu} (C; s) = - \Phi^{-1}(C) \frac{\delta}{\delta \xi_{\Sigma}^{\mu}(s)} \Phi(C) \text{ in terms of } F_{\mu\nu}(x) \text{ in } (3.4). \text{ Thus: } \quad (7)$$

$$- \theta^{-1}(\Sigma) \frac{\delta}{\delta \xi_{\Sigma}^{\mu}(s)} \theta(\Sigma)$$

$$= ie \int ds' \, \theta_{\Sigma}^{-1}(t, 0) G_{\mu\nu}(C_t; s, s') \theta_{\Sigma}(t, 0) \frac{\delta \xi_{\Sigma}^{\nu}(s')}{\delta t}, \quad (4.5)$$

where

$$G_{\mu\nu}(C; s, s') = \frac{\delta}{\delta \xi_{\Sigma}^{\nu}(s')} F_{\mu}(C; s) - \frac{\delta}{\delta \xi_{\Sigma}^{\mu}(s)} F_{\nu}(C; s') + ie [F_{\mu}(C; s), F_{\nu}(C; s')] \quad (4.6)$$

is the curvature⁶⁾ in $Q^1 X$; it is the analogue of $F_{\mu\nu}(x)$ in X , only carrying two extra continuous indices (s, s') which reflect the loop space's additional ∞ - dimensionality.

As usual, the quantity $eG_{\mu\nu}(C; s, s') \delta \xi_{\Sigma}^{\mu}(s) \delta \xi_{\Sigma}^{\nu}(s')$ represents the change in phase of $\Phi(C)$ as the point C in $Q^1 X$ moves around an infinitesimal closed circuit. In ordinary space-time, this circuit in $Q^1 X$ corresponds to a closed 2-dimensional surface, enclosing thus an elemental 3-dimensional volume, which is however, vanishing except when $s = s'$. Fig 1 is an

illustration of this surface in ordinary space when $s = s'$. If the volume enclosed does not contain the monopole, then the phase change is zero corresponding to the holonomy $\theta = 1$; but if it contains the monopole, then the phase change is κ for some κ satisfying (4.3), which corresponds to the holonomy $\theta = \zeta$, as required by the condition (3.14). We may therefore write:

$$G_{\mu\nu}(C; s, s') = \frac{\pi}{e} \int d\tau \kappa(C; s) \varepsilon_{\mu\nu\rho\sigma} \frac{d\xi_{\Sigma}^{\rho}(s)}{ds} \frac{dY^{\sigma}(\tau)}{d\tau} \delta^4(\xi(s) - Y(\tau)) \delta(s-s'), \quad (4.7)$$

with $\kappa(C; s)$ satisfying (4.3), where we note that by its definition in terms of $F_{\mu}(C; s)$ via $G_{\mu\nu}$ the quantity $\kappa(C; s)$ can depend on $C = \{\xi^{\mu}(s')\}$ only for $s' < s$, as we have indicated in its argument.

Substituting then (4.7) and (4.5) into (4.4), we obtain the following condition relating the variables $\Delta F_{\mu}(C; s)$ and $\Delta Y^{\mu}(\tau)$:

$$ie \int dt ds \, \theta_{\Sigma}^{-1}(t, 0) \Delta F_{\mu}(C_t; s) \theta_{\Sigma}(t, 0) \frac{\delta \xi_{\Sigma}^{\mu}(s)}{\delta t} = i\pi \int dt ds d\tau \, \theta_{\Sigma}^{-1}(t, 0) \kappa(C_t; s) \theta_{\Sigma}(t, 0) \varepsilon_{\mu\nu\rho\sigma} \frac{\partial \xi_{\Sigma}^{\nu}(s)}{\partial t} \frac{\partial \xi_{\Sigma}^{\rho}(s)}{\partial s} \frac{dY^{\sigma}(\tau)}{d\tau} \quad (4.8)$$

It is to be satisfied for all Σ in solving our variational problem.

We remark that although (4.7) was deduced as a consequence of (3.14) by restricting Σ , the closed loop in $Q^1 X$ occurring there, to infinitesimal circuits such as that depicted in Fig. 1, its validity for all $(C; s, s')$

will imply that (3.14) should also hold for all Σ since any finite loop in Q^1X can be constructed by the composition of such infinitesimal circuits. In other words, the conditions (3.14) and (4.7) are equivalent, the latter being just the differential version of the former. We may therefore in future use either one or the other as our topological constraint, depending just on which is more convenient for the purpose.

5. Deriving the Equations of Motion

Apart from the constraint (4.8), variations in $F_\mu(C|s)$ have to satisfy the transversality condition (3.6). Introducing then the algebra-valued Lagrange multipliers $\lambda(C|s)$ and Λ_Σ respectively for the constraints (3.6) and (4.8), we restate the variational problem as the requirement that

$$\begin{aligned} \Delta \mathcal{H} = & \Delta_{\text{ext}}^0 + ie \int \delta \Sigma [dt ds \text{Tr} [\Lambda_\Sigma \Theta_\Sigma^{-1}(t, 0) \Delta F_\mu(C|s) \Theta_\Sigma(t, 0)]] \frac{\partial \xi_t^\mu(s)}{\partial t} \\ & - i\pi \int \delta \Sigma [dt ds d\tau \text{Tr} [\Lambda_\Sigma \Theta_\Sigma^{-1}(t, 0) \kappa(C|s) \Theta_\Sigma(t, 0)]] \\ & \times \varepsilon_{\mu\nu\rho\sigma} \frac{\partial \xi_t^\nu(s)}{\partial t} \frac{\partial \xi_t^\rho(s)}{\partial s} \frac{dY^\sigma(\tau)}{d\tau} \Delta Y^\mu(\tau) \delta^4(\xi_t(s) - Y(\tau)) \\ & + \int \delta C [ds \text{Tr} [\lambda(C|s) \Delta F_\mu(C|s)]] \frac{d\xi_t^\mu(s)}{ds} \end{aligned} \quad (5.1)$$

should vanish for all variations $\Delta F_\mu(C|s)$ and $\Delta Y^\mu(\tau)$, where as for $\int \delta C$ in (3.17),

$$\int \delta \Sigma \dots = \int_{t,s} \Pi \Pi d^4 \xi_t(s) \dots \quad (5.2)$$

is defined as an ordinary functional integral.

Putting to zero in (5.1) the coefficient of $\Delta F_\mu(C|s)$ for all $(C|s)$ and μ , we have:

$$\begin{aligned} (2\pi\bar{N})^{-1} F_\mu(C|s) \rho(s) \left[\frac{d\xi_t^\alpha(s)}{ds} \right]^{-1} \\ = (ie) \int \delta \Sigma [dt \Theta_\Sigma(t, 0) \Lambda_\Sigma \Theta_\Sigma^{-1}(t, 0) \frac{\partial \xi_t^\mu(s)}{\partial t} \delta(C_t - C)]_s \\ + \lambda(C|s) \frac{d\xi_t^\mu(s)}{ds} \rho(s), \end{aligned} \quad (5.3)$$

with

$$\rho(s) = \int_{s' > s} d^4 \xi(s'), \quad (5.4)$$

$$\delta(C_t - C) \Big|_s = \Pi_{s' < s} \delta^4(\xi_t(s') - \xi(s')). \quad (5.5)$$

Putting further the coefficient of $\Delta Y^\mu(\tau)$ to zero for all τ and μ , we obtain:

$$\begin{aligned} N \frac{d^2 Y^\mu(\tau)}{d\tau^2} = i\pi \int \delta \Sigma [dt ds d\tau \text{Tr} [\Theta_\Sigma(t, 0) \Lambda_\Sigma \Theta_\Sigma^{-1}(t, 0) \kappa(C_t|s)]] \\ \times \varepsilon_{\mu\nu\rho\sigma} \frac{\partial \xi_t^\nu(s)}{\partial t} \frac{\partial \xi_t^\rho(s)}{\partial s} \frac{dY^\sigma(\tau)}{d\tau} \delta^4(\xi_t(s) - Y(\tau)). \end{aligned} \quad (5.6)$$

The multiplier $\lambda(C|s)$ can be eliminated from (5.3) using the transversality condition (3.6) on $F_\mu(C|s)$ giving:

$$\begin{aligned} F^\mu(C|s) = (2\pi\bar{N}) \text{ ie } \rho^{-1}(s) \left[\frac{d\xi_t^\alpha(s)}{ds} \frac{d\xi_t(s)}{ds} \right] \\ \times \int \delta \Sigma [dt \Theta_\Sigma(t, 0) \Lambda_\Sigma \Theta_\Sigma^{-1}(t, 0) \frac{\partial \xi_t^\mu(s)}{\partial t} \delta(C_t - C)]_s \\ - \text{long. comp.}, \end{aligned} \quad (5.7)$$

where 'long. comp.' represents the component of the preceding term along the tangent $d\xi^\mu(s)/ds$ to the loop C at the point s .

Now, it has already been shown ⁷⁾ that (5.7) implies the Yang-Mills equation which in loop space notation reads:

$$\frac{\delta}{\delta \xi^\mu(s)} F^\mu(C, s) = 0 \quad (5.8)$$

and is to be satisfied for all (C, s) .

Further, (5.7) allows the Lagrange multipliers Λ_Σ to be eliminated from the equation (5.6) as follows. We note first that the constraint (3.14) holding for all closed loops Σ in $\Omega^1 X$ implies that the $SU(N)$ element $\Theta_\Sigma(t, 0)$ for fixed C_t can depend on the path Σ only through a factor $\zeta_\Sigma \in Z_N$. It corresponds therefore to an element say $\Phi^{-1}(C_t)$ of the gauge group $SU(N)/Z_N$ which depends only on the end-point C_t but not otherwise on the path Σ in $\Omega^1 X$. Moreover, the logarithmic derivative of this $\Phi(C_t)$ by (4.2) is indeed $F(C_t, s)$, as required by (1.8) for the original $\Phi(C_t)$ so that all factors $\Theta_\Sigma(t, 0)$ in (5.6) and (5.7) can now be so replaced. Insert next in (5.6) the following identity:

$$\int \delta C \Big|_s \delta(C_t - C) \Big|_s \delta^4(\xi(s) - Y(\tau)) = \delta^4(\xi_t(s) - Y(\tau)), \quad (5.9)$$

where

$$\int \delta C \Big|_s \dots = \int \prod_{s' \leq s} d^4 \xi(s') \dots \quad (5.10)$$

We obtain, making use of the δ -function defined in (5.5) and of the note on eq. (1.9) at the end of that paragraph,

$$M \frac{d^2 Y^\mu(\tau)}{d\tau^2} = i\pi \int ds \delta C \Big|_s \text{Tr} [\kappa(C; s) \{ \int \delta \Sigma dt \Phi^{-1}(C_t) \Lambda_\Sigma \Phi(C_t) \frac{\delta Y^\nu(s)}{\delta t} \delta(C_t - C) \Big|_s \}]$$

$$\epsilon^{\mu}_{\nu\rho\sigma} \frac{d\xi^\rho(s)}{ds} \frac{dY^\sigma(\tau)}{d\tau} \delta^4(\xi(s) - Y(\tau)), \quad (5.11)$$

where we notice that the quantity inside the curly brackets is, apart from some numerical factors, just the first term on the right of eq. (5.7).

Noting further that the second term 'long. comp.' in (5.7), being directed along the loop C at point s , will give zero when contracted with $\epsilon^{\mu}_{\nu\rho\sigma} d\xi^\rho(s)/ds$ in (5.11), we can rewrite (5.11) as:

$$M \frac{d^2 Y^\mu(\tau)}{d\tau^2} = (2Ne)^{-1} \int ds \delta C \epsilon^{\mu}_{\nu\rho\sigma} \text{Tr} [\kappa(C; s) F^\nu(C; s)]$$

$$\times \frac{d\xi^\rho(s)}{ds} \frac{dY^\sigma(\tau)}{d\tau} \left[\frac{d\xi^\alpha(s)}{ds} \frac{d\xi_\alpha(s)}{ds} \right]^{-1} \delta^4(\xi(s) - Y(\tau)), \quad (5.12)$$

where we have used (5.4) to change $\delta C \Big|_s$ into δC . The multiplier Λ_Σ has thus been eliminated.

Collecting together the Euler-Lagrange equations obtained from the variational principle and the imposed topological constraint, we have then the complete equations of motion for our field-monopole system in terms of the variables $F_\mu(C/s)$ and $Y^\mu(\tau)$ as follows:

$$(I) \quad \frac{\delta}{\delta \xi^\mu(s)} F^\mu(C;s) = 0,$$

$$(II) \quad M \frac{d^2 Y^\mu(\tau)}{d\tau^2} = (2\tilde{N}e)^{-1} \int ds \delta C^\mu_{\nu\rho\sigma} \text{Tr}[\kappa(C;s) F^\nu(C;s)] \frac{d\xi^\rho(s)}{ds} \frac{dY^\sigma(\tau)}{d\tau}$$

$$+ \left[\frac{d\xi^\alpha(s)}{ds} \frac{d\xi^\beta(s)}{ds} \right]^{-1} \delta^4(\xi(s) - Y(\tau)),$$

$$(III) \quad \frac{\delta}{\delta \xi^\nu(s')} F_\mu(C;s) - \frac{\delta}{\delta \xi^\mu(s)} F_\nu(C;s') + ie [F_\mu(C;s), F_\nu(C;s')] \\ = \frac{\pi}{e} \int d\tau \kappa(C;s) \varepsilon_{\mu\nu\rho\sigma} \frac{d\xi^\rho(s)}{ds} \frac{dY^\sigma(\tau)}{d\tau} \delta^4(\xi(s) - Y(\tau)) \delta(s-s'),$$

where (II) and (III) are to be satisfied for the same $\kappa(C;s)$ which solves eq. (4.3), and the transversality (3.6) of $F_\mu(C;s)$ is understood.

These equations correspond respectively to the Maxwell equation (1.5), the Lorentz equation (1.6), and the Gauss Law (1.7) of the abelian case. Indeed, a comparison with section 2 will show that there is an almost exact parallel in the derivation of the two sets of equations.

6. Extension to Other Gauge Groups

The results of sections 3-5 were obtained for monopoles in a pure $su(N)$ Yang-Mills theory whose gauge group is $SU(N)/Z_N$. The theories of present practical interest such as chromodynamics, however, are not pure Yang-Mills theories but contain also matter fields belonging usually to the fundamental representation, implying thus gauge groups $SU(N)$, not $SU(N)/Z_N$. Now, the group $SU(N)$, being simply-connected, has only one homotopy class of closed

curves, namely the trivial class which includes the vacuum, and admits therefore no monopoles in the sense defined above. Hence, it might appear at first sight that the discussion in the preceding sections is not of immediate practical interest. This, of course, is not true for the following reason. Quarks in chromodynamics, for instance, in addition to being a colour triplet, carries also electric charges equal to $-2/3$ or $1/3$ times that of the electron, and must therefore couple to the Maxwell field. The overall gauge group including both the Maxwell and Yang-Mills degrees of freedom is then no longer $SU(3)$ but $U(3) = SU(3) \times U(1)/Z_3$. A similar statement is true for the Glashow-Salam-Weinberg electroweak theory; in its standard form, the gauge group is $U(2) \times U(1)/Z_2$. Now the groups $U(N)$ are not simply-connected, and admits therefore, by the above reckoning, monopoles.⁹⁾ The only question is whether the results we have so far obtained can be readily adapted to these more realistic theories.

To answer this, we note first that homotopy classes in the group $U(N)$ are labelled by an integer n . A closed curve in $U(N)$ belonging to the class n winds n/N times around both the 'electromagnetic' $U(1)$ and the 'colour' $SU(N)$ subgroups, the latter corresponding in the usual convention to $\zeta = \exp i2\pi n/N$ of $SU(N)/Z_N$. Hence the statement that the particle at $Y(\tau)$ carries a $U(N)$ monopole charge n means not only that the Maxwell field has to satisfy (1.7) with $g=n/2e$ (the minimal electric charge being by convention e/N), but also that the Yang-Mills field satisfies eq. (3.14) with $\zeta = \exp i2\pi n/N$ simultaneously for the same n .

The equations of motion for the system are now to be derived by requiring the action:

$$S = - \frac{1}{4\pi N} \int dC ds \text{Tr} [F_\mu(C;s) F^\mu(C;s)] \left[\frac{d\xi^\alpha(s)}{ds} \frac{d\xi_\alpha(s)}{ds} \right]^{-1} \\ - \frac{1}{16\pi} \int d^4x f_{\mu\nu}(x) f^{\mu\nu}(x) - M \int d\tau \quad (6.1)$$

to be stationary against all variations of $f_{\mu\nu}(x)$, $F_\mu(C|s)$ and $Y^\mu(\tau)$ subject to the following constraints being satisfied:

$$\partial_\nu^* f^{\mu\nu}(x) = - (2\pi m/e) \int d\tau \frac{dY^\mu(\tau)}{d\tau} \delta^4(x-Y(\tau)), \quad (6.2)$$

and

$$\partial(\Sigma) = \zeta_\Sigma, \quad (6.3)$$

with $\zeta = \exp(-i2\pi m/N)$, if $\bar{p}$ encloses $Y(\tau)$,

$$\zeta_\Sigma = 1, \text{ otherwise} \quad (6.4)$$

for the same value of n in (6.2) and (6.4).

Repeating then exactly the arguments in sections 2 and 5 one easily arrives at the following equations respectively for the variations in $f_{\mu\nu}(x)$, $F_\mu(C,s)$ and $Y^\mu(\tau)$:

$$\partial_\nu f^{\mu\nu}(x) = 0, \quad (6.5)$$

$$\frac{\delta}{\delta F_\mu^H(s)} F^\mu(C,s) = 0, \quad (6.6)$$

$$M \frac{d^2 Y^\mu(\tau)}{d\tau^2} = (2\pi e)^{-1} \int ds \int d\sigma \epsilon^{\mu\nu\rho} \text{Tr}[\kappa(C,s) F^\nu(C,s)] \frac{d\xi^\rho(s)}{ds} \frac{dY^\sigma(\tau)}{d\tau} \\ + \left[\frac{d\xi^\alpha(s)}{ds} \frac{d\xi_\alpha(s)}{ds} \right]^{-1} \delta^4(\xi(s)-Y(\tau)) \\ - \frac{n}{2e} f^{\mu\nu}(Y(\tau)) \frac{dY_\nu(\tau)}{d\tau}, \quad (6.7)$$

where $\kappa(C,s)$ satisfies again (4.3) and (4.7) and we have used e' to denote the nonabelian coupling to avoid confusion.

One sees from these equations that for n being a multiple of N (when $\kappa(C,s)$ is by definition zero), a $U(N)$ monopole behaves exactly as an

ordinary magnetic monopole in an electromagnetic field. For other values of n , on the other hand, a $U(N)$ monopole acts as if it carries a magnetic charge as well as a monopole charge of the pure $su(N)$ Yang-Mills theory as discussed in sections 3 and 5. Although this conclusion comes as no surprise, it is nevertheless interesting to see it emerge as a consequence of our action principle in spite of the fact that $U(N)$ monopoles are characterised by only one winding number.

We believe that the considerations in this paper can readily be adapted also to theories with other gauge groups although we have not studied the extension in detail seeing that $SU(N)^{1/2}_N$ and $U(N)$ theories already cover many cases of current interest.

7. Remarks and Conjectures

Perhaps the most attractive feature in the equations of motion proposed above for nonabelian monopoles is that they have been derived from a 'free' action with only the defining condition of the monopole charge as constraint. Since the monopole charges are themselves an intrinsic topological attribute of the gauge theory, it would seem that the whole development is founded on a geometrical basis.

The derivation was given explicitly only for a single monopole moving in the gauge field. However, it is seen that an extension of the arguments to more than one monopole is straightforward. Indeed, a bonus of working with the patch-independent variables $f_{\mu\nu}(x)$ and $F_\mu(C,s)$ adopted for solving the variational problem is that we no longer have to contend with the complications of a number of patches increasing at least as fast as 2^m for an m monopole system.

On the other hand, a serious drawback of the loop variables is of course their unfamiliarity which makes equations in them hard to interpret. For this reason, we have not succeeded as yet in understanding much about the

physical content of the equations we have derived.

We have noted, however, an interesting formal relationship of these equations to some more familiar ones which may offer a hint for future investigations. This arises as follows. Whenever a potential $A_\mu(x)$ satisfying (1.8) - (1.10) exists, we have eq. (3.4) as well as the following relations between loop space and ordinary space-time field quantities: ⁶⁾¹⁰⁾

$$\frac{\delta}{\delta \xi^\mu(s)} \mathcal{P}^\mu(C|s) = \Phi_C^{-1}(s,0) D_\mu \mathcal{P}^{\mu\nu}(\xi(s)) \Phi_C(s,0) \frac{d\xi_\nu(s)}{ds}, \quad (7.1)$$

$$G_{\mu\nu}(C; s, s') = -\Phi_C^{-1}(s,0) \varepsilon_{\mu\nu\rho\sigma} D_\alpha^* \mathcal{P}^{\rho\sigma}(\xi(s)) \Phi_C(s,0) \frac{d\xi_\alpha(s)}{ds} \delta(s-s'). \quad (7.2)$$

These equations fail, however, on the monopole worldline since the quantities $A_\mu(x)$, $F_{\mu\nu}(x)$ and $\mathcal{P}_{\mu\nu}(x)$ are all patched around the monopole and lose their usual meaning at the monopole position.

Suppose we choose to ignore these last, very essential, complications and, in spite of them, apply these formulas to the equations of motion derived in section 5, we shall obtain formally in place of (I) and (III),

$$D_\nu \mathcal{P}^{\mu\nu}(x) = 0 \quad (7.3)$$

and

$$D_\nu \mathcal{P}^{\mu\nu}(x) = + \frac{\pi}{e} \int d\tau [\Phi_C(s,0) \kappa(C|s) \Phi_C^{-1}(s,0)] \xi(s)=x \quad (7.4)$$

$$\times \frac{dY^\mu(\tau)}{d\tau} \delta^4(x-Y(\tau)),$$

where we have used the fact that (I) and (III) are to be satisfied for all C , s , and s' . The quantity inside the square brackets in (7.4) can depend on C only through the point $\xi(s)$, and since it is to be evaluated only at $x=Y(\tau)$ by virtue of the δ -function, we shall replace it by the symbol $K(\tau)$. Thus (7.4) becomes:

$$D_\nu \mathcal{P}^{\mu\nu}(x) = \frac{\pi}{e} \int d\tau K(\tau) \frac{dY^\mu(\tau)}{d\tau} \delta^4(x-Y(\tau)). \quad (7.5)$$

Introducing now this symbol $K(\tau)$ together with (3.4) into the equation (II), one obtains:

$$M \frac{d^2 Y^\mu(\tau)}{d\tau^2} = (8\pi e)^{-1} \int ds \delta C \varepsilon^\mu{}_{\nu\rho\sigma} \text{Tr}[K(\tau) F^{\nu\rho}(\xi(s))] \frac{dY^\sigma(\tau)}{d\tau}$$

$$\times \delta^4(\xi(s)-Y(\tau)), \quad (7.6)$$

where we have averaged over all directions of the tangent vector $d\xi^\mu(s)/ds$ to the loop C at s . Integrating then over C and recalling the definition (3.18) of the normalisation factor $\bar{N}$, we have

$$M \frac{d^2 Y^\mu(\tau)}{d\tau^2} = (4e)^{-1} \text{Tr}[K(\tau) \mathcal{F}^{\mu\nu}(Y(\tau))] \frac{dY_\nu(\tau)}{d\tau}. \quad (7.7)$$

One sees then that the three equations so obtained, namely (7.3), (7.5) and (7.7) are very similar in appearance to the Wong equations for a 'classical' point source of a Yang-Mills field.⁸⁾ Indeed, if from the Wong equations, one makes the replacements $\mathcal{F}_{\mu\nu}(x) \rightarrow \mathcal{F}_{\mu\nu}(x)$, $I(\tau) \rightarrow K(\tau)$, and $e \rightarrow g = 1/4e$, one obtains exactly the equations quoted above. In other words these equations look like the dual transforms of the Wong equations, except that, rather intriguingly, the covariant differentiation appearing in them is still defined in terms of the potential $A_\mu(x)$, not in terms of some potential, $\bar{A}_\mu(x)$ say, of the dual field $\mathcal{F}_{\mu\nu}(x)$ as one might naively expect.

It should be stressed that this 'dual' relationship of (I), (II) and (III) with the Wong equations is at present only formal in that we are unable as yet to assign a meaning to the patch-dependent quantities $A_\mu(x)$, $F_{\mu\nu}(x)$ and $*F_{\mu\nu}(x)$ at the monopole position. Only for the abelian theory, which we may treat in the same manner if we wish to, does this relation become genuine, giving (1.5) - (1.7), since then both $f_{\mu\nu}(x)$ and $*f_{\mu\nu}(x)$ are patch-independent, and the potential does not appear in their covariant derivative.

Nevertheless, this formal relation suggests that there may perhaps be a similarity between the behaviour of a monopole charge and that of a point source in a Yang-Mills field somewhat analogous to the dual invariance interchanging magnetic and electric charges in electromagnetism. At first sight, this may seem surprising. Consider for example the gauge group $SO(3)$. In that case, the Wong equations imply the conservation of isotopic spin, which means of course that not only the length but also the direction of the isospin vector are conserved. Formally, the equations (7.3), (7.5) and (7.7) will do the same for $I(\tau)$ replaced by $K(\tau)$. However, we started with a monopole charge labelled only by a sign $\zeta = -I$, which means only a $K(\tau) = n\hat{\sigma}$ where n can be any odd integer and $\hat{\sigma}$ any isovector of unit length. Where does an isovector with length and direction both conserved come in? We think it probably arises as follows. The topological charge ζ is conserved simply because of continuity which requires no detail knowledge of the dynamics. Once the dynamics is defined, as for our case by the free action ω^0 in (3.16) and the constraint (3.14), then the symmetry in the problem can introduce additional conserved quantities. In other words, one may regard the monopole as carrying first a topological charge ζ which is conserved by continuity, and then via the induced dynamics a further conserved Noether charge, namely the isospin vector K . In a sense then, electromagnetism on which we base most of our intuition would in this case be a somewhat misleading example, being degenerate in that a monopole's topological charge happens here to be identical to its Noether charge - both labelled by the winding number n .

The conjectures of the last paragraph now raise an interesting question. In the abelian theory, we noted in section 1 that because of duality an electric charge, though usually taken to be a source of the Maxwell field $f_{\mu\nu}(x)$, can equally be regarded as a monopole of the dual field $*f_{\mu\nu}(x)$. One can then, if one so wish, recast the formulation of the theory in a more geometrical fashion without either introducing an interaction term in the action or making the conventional minimal coupling hypothesis. Now, in the nonabelian theory, for lack of dual invariance, sources and monopoles are presumably different objects. However, suppose it were true nevertheless that monopoles also carry a conserved Noether charge K , just like I for sources, can we then be sure that the particles we know, e.g. quarks, which we usually take to be gauge field sources, are not in fact monopoles? Naively, it would appear that monopoles - being just topological obstructions with dynamics induced by the topology as described above, and therefore in a sense more 'intrinsic' - may perhaps be considered a theoretically more appealing choice.

C.H.M. and T.S.T. are deeply indebted to Prof. Chen Ning Yang for prolonged and instructive conversations several years ago on related subjects, and to Prof. Tai Tsun Wu for a fruitful discussion on this problem at one stage.

References

- 1) Tai Tsun Wu and Chen Ning Yang, Phys. Rev. 14D, (1976) 437.
- 2) E. Lubkin, Ann. Phys. (N.Y.) 23, (1963) 233.
- 3) Tai Tsun Wu and Chen Ning Yang, Phys. Rev. 12D, (1975) 3845.
- 4) S. Coleman, Erice School, (1975) 297.
- 5) Tai Tsun Wu and Chen Ning Yang, Phys. Rev. 12D, (1975) 3843.
- 6) A.M. Polyakov, Nucl. Phys. B164, (1979) 171.
- 7) Chan Hong-Mo, P. Scharbach and Tsou Sheung Tsun, Rutherford Laboratory preprint RAL-85-043 (1985)
- 8) S.K. Wong, Nuovo Cim. 14A (1970) 689.
- 9) See e.g. Chan Hong-Mo and Tsou Sheung Tsun, Nucl. Phys. B189, (1981) 364 and references therein.
- 10) Chaohao Gu and Ling-Lie Chau Wang, Phys. Rev. Letters 45, (1980) 2004; Tsou Sheung Tsun, Proceedings of the Trieste Meeting on Monopoles in Quantum Field Theory (ed. N.S. Craigie et al., World Scientific, (1981)) 283.

Fig. 1. Illustration for $G_{\mu\nu}$ as an elemental 2-dimensional surface, enclosing here the monopole at Y .

FIG 1