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ANOMALOUS NON-ABELIAN GAUGE THEORY IN TWO DIMENSIONS

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ABSTRACT

We consider the two-dimensional model of a $U(N)$ gauge field coupled to just the right-handed current of N massless fermions. We show, using its bosonized version, that this model, although anomalous and non-gauge invariant, formally yields a consistent unitary Lorentz-invariant quantum theory. As by-products, the commutators of the currents and of the Gauss-law generators are obtained explicitly.

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It has been shown^{1),2)} that the chiral Schwinger model in two dimensions, although anomalous and not gauge-invariant, is nevertheless consistent, unitary and Lorentz-invariant. The theory is exactly solvable^{1),3)} and its spectrum consists of a massive particle plus massless excitations. In its Hamiltonian form, the theory has second-class constraints which can be consistently imposed on the space of physical states acted on by a positive Hamiltonian²⁾.

In this paper we study the consistency, unitarity and Lorentz invariance of the corresponding non-Abelian theory in two dimensions. Our analysis will be similar to that in Ref. 2). We will work with the bosonized form of the action which is Lorentz-invariant and already reflects, at its classical level, the presence of the anomaly. We then set up the associated Hamiltonian and canonical brackets, paying due attention to the constraints of the system. It will be seen that these constraints can be imposed consistently. The resulting Hamiltonian is real and positive for $\alpha > 1$, where α refers to the coefficient of a counter term in the action (see below). Thus, formally one has a consistent constrained unitary quantum theory; however, unlike the U(1) case, here the theory is not exactly solvable. The question of whether or not higher order divergences spoil the good behaviour of the theory if treated perturbatively has not been investigated here, so our demonstration is in this sense formal and incomplete. (The divergence of the basic one-fermion-loop graph and the resulting anomaly are incorporated in our bosonized action. We are referring to higher order divergences above and beyond this.)

$$S_F = \int d^2x dt \sum_{j=1}^N i \bar{\Psi}^j \gamma^\mu \partial_\mu \Psi^j \quad (1)$$

is equivalent in its neutral sector to a bosonic system with action

$$S_B = \frac{1}{8\pi} \int d^2x dt \text{Tr} (\partial_\mu U \partial^\mu U^{-1}) + \frac{1}{12\pi} \int_V d^3y \epsilon^{ijk} \text{Tr} [U^{-1} \partial_i U U^{-1} \partial_j U U^{-1} \partial_k U] \quad (2)$$

Here U is a U(N) group element at each point and the second term has the Wess-Zumino form⁵⁾ integrated over a three-dimensional manifold V , whose boundary is compactified space-time.

The field equations arising from this action are

$$(\partial_0 - \partial_1) \rho_R = 0 = (\partial_0 + \partial_1) \rho_L \quad (3)$$

where

$$\rho_R \equiv \frac{i}{4\pi} U^{-1} (\partial_0 + \partial_1) U \quad ; \quad \rho_L \equiv \frac{i}{4\pi} U (\partial_0 - \partial_1) U^{-1} \quad (4)$$

The right and left fermionic currents are identified with⁴⁾

$$j_R^\mu = \frac{i}{4\pi} (g^{\mu\nu} + \varepsilon^{\mu\nu}) U^{-1} \partial_\nu U = \begin{pmatrix} \rho_R \\ -\rho_R \end{pmatrix}$$

and

$$j_L^\mu = \frac{i}{4\pi} (g^{\mu\nu} - \varepsilon^{\mu\nu}) U \partial_\nu U^{-1} = \begin{pmatrix} \rho_L \\ \rho_L \end{pmatrix} \quad (5)$$

The classical Hamiltonian corresponding to this action is

$$H_0 = \pi \int dx \left(\rho_R^a \rho_R^a + \rho_L^a \rho_L^a \right) \quad (6)$$

with Poisson Brackets [P.B.] given by

$$\begin{aligned} [\rho_R^a(x), \rho_R^b(y)] &= -f^{abc} \rho_R^c(x) \delta(x-y) + \frac{\delta^{ab}}{2\pi} \delta'(x-y) \\ [\rho_L^a(x), \rho_L^b(y)] &= -f^{abc} \rho_L^c(x) \delta(x-y) - \frac{\delta^{ab}}{2\pi} \delta'(x-y) \end{aligned}$$

and

$$[\rho_R^a(x), \rho_L^b(y)] = 0 \quad (7)$$

Here, a, b and c refer to the Lie-algebra indices $[\rho_{L,R}^a \equiv \text{Tr } \lambda^a \rho_{L,R}]$; λ^a are Hermitian generators of U(N); $\text{Tr } \lambda^a \lambda^b = \delta^{ab}$ and $[\lambda^a, \lambda^b] = if^{abc} \lambda^c$. This Hamiltonian and the associated current algebra for free massless fermions are well

known^{6),7)}. They can also be derived from the bosonized action S_B by using the corresponding symplectic form in phase space. One can also directly check that the Hamiltonian system in Eqs. (6) and (7) yields the same equations of motion (3) as did the action S_B .

Now let us add a $U(N)$ gauge field $A_\mu \equiv \lambda^a A_\mu^a$ to the system and couple it to just the right-handed current. We take the action of this system to be

$$S = S_B + \int dx dt \text{Tr} \left\{ -\frac{ie}{4\pi} (g^{\mu\nu} + \varepsilon^{\mu\nu}) U^{-1} \partial_\nu U A_\mu + \alpha \frac{e^2}{8\pi} A_\mu A^\mu - \frac{i}{4} F_{\mu\nu} F^{\mu\nu} \right\} \quad (8)$$

where S_B is the free action in (2). The first term in the brackets couples A_μ to the right current and the third term is the usual pure gauge field action. The second term $\alpha(e^2/8\pi)A_\mu A^\mu$, where α is a dimensionless parameter, is added on the same basis as in Ref. 1). We take it as representing polynomial ambiguities in the fermionic determinant, in the presence of A^μ . Recall that such a term also needs to be introduced to satisfy gauge invariance when the non-Abelian theory is gauged vectorially⁸⁾. In our chiral model, because of the anomaly, gauge invariance cannot be restored by such a term, but we will retain the freedom to introduce this term, and keep its coefficient α arbitrary for the present. In any event, Eq. (8) is the definition of our anomalous chirally-gauged theory. Regardless of the value of α , the action S is anomalous, but Lorentz-invariant.

The Euler-Lagrange equations arising from this action are:

$$\begin{aligned} D_\mu J_R^\mu &= \frac{e}{4\pi} (g^{\mu\nu} - \varepsilon^{\mu\nu}) \partial_\mu A_\nu \\ \text{and} \quad D_\mu F^{\mu\nu} &= e J_R^\nu - \frac{\alpha e^2}{4\pi} A^\nu \end{aligned} \quad (9)$$

where

$$J_R^\mu \equiv \frac{i}{4\pi} (g^{\mu\nu} + \varepsilon^{\mu\nu}) U^{-1} \partial_\nu U \quad (10)$$

and D_μ is the covariant derivative. Equations (9) imply

$$0 = D_\mu D_\nu F^{\mu\nu} = \frac{e^2}{4\pi} \left\{ (1-\alpha) \partial_\mu A^\mu - \varepsilon^{\mu\nu} \partial_\mu A_\nu \right\} \quad (11)$$

The right-hand side of (11) represents the anomaly in this model. It is not identically zero for any choice of α , but will vanish for any consistent solution of the coupled equations in (9).

To see if the action (8) can lead to a consistent quantum theory, let us set up the Hamiltonian system associated with it. As a prelude, notice that as compared to Eq. (5) of the non-interacting problem, we have used a different symbol (J_R^μ in place of j_R^μ) in Eq. (10) to denote the current $i/4\pi(g^{\mu\nu} + \epsilon^{\mu\nu})U^{-1}\partial_\nu U$. This is because the canonical status of $U^{-1}\partial_0 U$ has been altered by the presence of time derivatives in the interaction term in (8). If for the free theory (2) we define a canonical momentum $p \equiv i U^{-1}\partial_0 U$, then clearly for the interacting theory $(i U^{-1}\partial_0 U)_{\text{interact.}} = p + e(A_0 - A_1)$. Correspondingly, $J_R^\mu \equiv \begin{pmatrix} \tilde{p}_R \\ -\tilde{j}_R \end{pmatrix}$ is related to ρ_R by

$$\tilde{p}_R = \left\{ \frac{i}{4\pi} U^{-1}(\partial_0 + \partial_1)U \right\}_{\text{interact.}} = j_R + \frac{e}{4\pi} (A_0 - A_1) \quad (12)$$

The charges ρ_R^a , associated with $\psi^\dagger \lambda^a ((1-\gamma_5)/2)\psi$, have no gauge field content, whereas the $\tilde{\rho}_R^a$ do. The Hamiltonian associated with the action (8) is

$$H = \int dx \left\{ \pi \left(p_R^a p_R^a + j_L^a j_L^a \right) + e \left(p_R^a + \frac{e}{4\pi} (A_0^a - A_1^a) \right) (A_0^a - A_1^a) \right. \\ \left. - \frac{\alpha e^2}{8\pi} (A_0^a A_0^a - A_1^a A_1^a) + \left(\frac{1}{2} \epsilon^a \epsilon^a - (D_i \epsilon)^a A_0^a + \pi_0^a v^a \right) \right\} \quad (13)$$

Here π_0^a and E^a are canonical momenta conjugate to A_0^a and A_1^a respectively with familiar Poisson brackets

$$[A_0^a(x), \pi_0^b(y)] = [A_1^a(x), E^b(y)] = \delta^{ab} \delta(x-y) \\ [A_i^a, A_j^b] = [\pi_0^a, \pi_0^b] = 0, \dots \text{etc.} \quad (14a)$$

The charges $\rho_{R,L}^a$ are the same as in the non-interacting system (6) and (7). Their P.B. with each other are given in (7), and since they have no gauge field content,

$$[p_{L,R}^a, M] = 0 \quad \text{for} \quad M = A_0^b, A_1^b, E^b \text{ or } \pi_0^b \quad (14b)$$

As usual

$$\pi_0^a(x) = 0 \quad (15)$$

is a constraint. Equation (15) holds only "weakly" at this stage, in Dirac's terminology⁹⁾. It is not compatible with the P.B. in (14a). Corresponding to that, we have added to the Hamiltonian a term proportional to π_0^a , as permitted by constraint theory. Its coefficient $v^a(x)$ is undetermined at this stage.

The Hamiltonian (13) and the brackets in (7) and (14), as well as the relation (12) can all be derived systematically from the action (8). We omit such a derivation here. But the equivalence of this Hamiltonian to that action will be evident from the fact, shown below, that it yields the same field equations.

Let us proceed with the constraint analysis. The consistency of the constraint (15) under time evolution requires that

$$\begin{aligned} G^a(x) &\equiv [\pi_0^a(x), H] \\ &= (D_1 E)^a + \frac{e^2}{4\pi} \{(\alpha-1)A_0^a + A_1^a\} - e p_R^a \\ &= 0 \end{aligned} \quad (16)$$

This is the analogue of Gauss's law and is another set of constraints. Both (15) and (16) are second-class constraints for $\alpha \neq 1$. [The case $\alpha = 1$ is special and can be treated along the lines discussed in Ref. 2). We will limit ourselves here to $\alpha \neq 1$.] It can be verified that

$$[\pi_0^a(x), G^b(y)] = \frac{e^2}{4\pi} (1-\alpha) \delta^{ab} \delta(x-y) \quad (17a)$$

$$[G^a(x), G^b(y)] = e f^{abc} \left\{ G^c(x) + \frac{e^2}{4\pi} (A_1^c(x) + (1-\alpha)A_0^c(x)) \right\} \delta(x-y) \quad (17b)$$

and

$$[\pi_0^a(x), \pi_0^b(y)] = 0 \quad (17c)$$

Furthermore,

$$\begin{aligned} \partial_0 G^a(x) &= [G^a(x), H] \\ &= -ef^{abc} A_0^b G^c + \frac{e^2}{4\pi} \left\{ E^a + ef^{abc} A_1^b A_0^c + (1-\alpha)(\partial_1 A_1^a - v^a) \right\} \end{aligned} \quad (18)$$

Consistency of the constraint (16) under time evolution requires that (18) vanish. Hence

$$v^a(x) = \partial_1 A_1^a + \frac{1}{(1-\alpha)} \left(E^a + ef^{abc} A_1^b A_0^c \right) \quad (19)$$

Inserting this $v^a(x)$ into (13), we thus have an unambiguous Hamiltonian describing motion on the submanifold of phase space constrained by $\pi_0^a(x) = G^a(x) = 0$.

Note that the Hamiltonian (13), along with the P.B. in (7) and (14) yield the following equations of motion:

$$(D_0 E)^a = e \int_R \mathcal{L}^a + \frac{e^2}{4\pi} (A_0^a - (1+\alpha) A_1^a) \quad (20a)$$

$$\left\{ (D_0 - D_1) \left(p_R + \frac{e}{4\pi} (A_0 - A_1) \right) \right\}^a = \frac{e}{4\pi} (\partial_0 + \partial_1) (A_0 - A_1)^a \quad (20b)$$

and

$$\partial_0 A_0^a = v^a = \partial_1 A_1^a + \frac{1}{(1-\alpha)} \left(E^a + ef^{abc} A_1^b A_0^c \right) \quad (20c)$$

Equations (20a), (20b) and (16) together are precisely the same as the field equations (9) obtained from the action (8), when ρ_R is related to the components

of J_R^μ by Eq. (12). Equation (20c) is the same as Eq. (11) and shows explicitly that the anomaly of this model vanishes by virtue of the equations of motion. Note also that our P.B. in (17b) agrees with the commutator of gauge generators evaluated by Jo¹⁰⁾ directly from fermion-loop diagrams. Recall that the bosonized action contains fermionic radiative effects and the coefficient α represents regularization ambiguities. Jo's regularization corresponds here to the choice $\alpha = 2$. With this choice, our anomaly [the right-hand side of Eq. (11)] also agrees with Jo's result. Finally, using (7), (12) and (14b), we have

$$[\tilde{J}_R^a(x), \tilde{J}_R^b(y)] = -e f^{abc} \left(\tilde{J}_R^c - \frac{e}{4\pi} (A_0 - A_1)^c \right) + \frac{1}{2\pi} g^{ab} \delta'(x-y) \quad (21)$$

This also agrees with Jo's commutators for the interacting charge densities. (In comparing Jo's equations with ours, the usual factor of i must be inserted to convert our P.B.'s into commutators. Also, the chirality he gauges is opposite to ours.)

Equation (13), with (19) inserted, gives us an unambiguous Hamiltonian for this constrained system at the classical level. The system cannot be quantized as it stands, since the constraints (15) and (16) are of the second class. They do not commute with one another. But this can be repaired by replacing all P. B.'s with Dirac brackets⁹⁾. For our system, the Dirac bracket between any two observables P and Q is given in terms of the P.B. by

$$\begin{aligned} [P, Q]_D &= [P, Q] - \int dx [P, \pi_0^a(x)] K^{ab}(x) [\pi_0^b(x), Q] \\ &+ \frac{4\pi}{e^2(\alpha-1)} \int dx \{ [P, G^a(x)] [\pi_0^a(x), Q] - [P, \pi_0^a(x)] [G^a(x), Q] \} \end{aligned} \quad (22)$$

where

$$K^{ab}(x) \equiv \left(\frac{4\pi}{e^2(\alpha-1)} \right)^2 e f^{abc} \left\{ G^c(x) + \frac{e}{4\pi} (A_1^c(x) + (1-\alpha) A_0^c(x)) \right\}$$

The Dirac brackets in (22) are so designed that $[P, \pi_0^a]_D = [P, G^a]_D = 0$ for any P ; therefore, with respect to these brackets, both constraints (15) and (16) hold strongly. One can now quantize the theory by (i) replacing all Dirac brackets by

commutators after multiplying by i on their right-hand sides, and (ii) requiring that $\pi_0^a(x)$ and $G^a(x)$, which now form a commuting set, all vanish as operators on the space of physical states. Since A_0^a can now be eliminated using (16), and $\pi_0^a = 0$, the only independent canonical variables are A_1^a , E^a , ρ_R^a and ρ_L^a . Furthermore, it is clear from (22) that the Dirac brackets between these independent variables are the same as the corresponding P.B., since they all have vanishing P.B. with π_0^a . Finally, one can now insert the constraints (15) and (16) into the Hamiltonian and simplify it. It can be cast in the form

$$H = \int dx \left\{ \pi \left(\rho_R^a - \frac{e A_1^a}{4\pi} \right)^2 + \pi \rho_L^a \rho_L^a + \frac{1}{2} E^a E^a + \frac{e^2 (\alpha - 1)}{8\pi} (A_0^a A_0^a + A_1^a A_1^a) \right\} \quad (23)$$

where A_0^a is now just a symbol representing

$$\frac{4\pi}{e^2 (\alpha - 1)} \left\{ -(\mathcal{D}_1 E)^a + e \rho_R^a - \frac{e^2}{4\pi} A_1^a \right\}$$

We see that if $\alpha > 1$, the Hamiltonian is positive. Thus we have, for $\alpha > 1$, a Hamiltonian which is unambiguous, Hermitian, positive and explicitly shown in (23) in terms of the independent variables A_1^a , E^a , ρ_L^a and ρ_R^a , whose commutation rules can be read off from the P.B. relations (7) and (14). This is the basis for our claim that the anomalous chiral non-Abelian model in (8) leads to a consistent unitary theory, provided the underlying fermionic determinant is regularized in such a way that the coefficient α is greater than unity. The result is similar to the one proved earlier^{1),2)} for the Abelian case. But unlike the Abelian case, where one could diagonalize the Hamiltonian and completely solve the theory, here we cannot do so any more than one can presently solve even $[QCD]_2$. If perturbation theory were to be employed, then the divergences of higher order terms would need to be examined. We have not studied the question of whether they ruin this theory. Our demonstration is thus still formal, but one step less naive than using the parent fermionic action to draw conclusions.

Finally, one can ask whether similar results could conceivably hold for anomalous gauge theories in four dimensions. Certainly our derivation throws very little light on this question. We have used techniques special to two dimensions at several places. The issue of renormalizability has not been discussed here at all; however, this and earlier work^{1),2)} have shown that anomalous theories are not all necessarily inconsistent or non-unitary. Furthermore, we have seen in the two-dimensional cases that although the divergence of the gauge

current does not vanish identically, it does vanish by virtue of the bosonized equations of motion. If some similar feature holds in four dimensions, then together with the fact that a massive vector meson coupled to a conserved current is renormalizable in four dimensions, it offers some remote hope that Abelian chiral $U(1)$ gauge theory in four dimensions may be consistent and on-shell renormalizable.

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REFERENCES

- 1) R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54 (1985) 1219.
- 2) R. Rajaraman, Phys. Lett. 154B (1985) 305.
- 3) C.R. Hagen, Ann. Phys. (NY) 81 (1973) 67.
- 4) E. Witten, Com. Math. Phys. 92 (1984) 455.
- 5) A. D'Adda, A. Davis and P. Di Vecchia, Phys. Lett. 121B (1983) 335;
A. Polyakov and P. Wiegmann, Phys. Lett. 131B (1983) 121;
O. Alvarez, Nucl. Phys. B238 (1984) 61;
R.E. Comba-Saravi, F.A. Schaposnik and J.E. Solomin, Phys. Rev. D30 (1984) 1353.
- 6) S. Coleman, D. Gross and R. Jackiw, Phys. Rev. 180 (1969) 1359.
- 7) L.D. Faddeev, Phys. Lett. 145B (1984) 81.
- 8) P. Di Vecchia, B. Durhuus and J.L. Petersen, Phys. Lett. 144B (1984) 245;
D. Gonzales and A.N. Redlich, Phys. Lett. 147B (1984) 150.
- 9) P.A.M. Dirac, Can. J. Math 2 (1950) 129;
E.C.G. Sudarshan and N. Mukunda, "Classical Dynamics", Wiley, New York, (1974);
A. Hanson, T. Regge and C. Teitelboim, Acad. Nazion. Lincei, Rome (1976).
- 10) S. Jo, MIT preprints CTP 1236 and CTP 1264 (May 1985).