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INSTANTONS IN THE QUANTIZED SU(2) VACUUM:

A LATTICE MONTE CARLO INVESTIGATION

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A B S T R A C T

After addressing the question of instanton stability on a space-time lattice, we investigate whether the full non-perturbative SU(2) gauge vacuum possesses an underlying (anti-)instanton structure. We present evidence that it does - although on the lattice only for $\beta \gtrsim 2.7$ - and that typical instanton sizes are of the order of the inverse mass gap: moreover they are remarkably stable with respect to quantum fluctuations, which encourages an optimistic attitude towards "semi-classical" calculations.

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Instantons¹⁾ are semi-classical objects²⁾ which have taught us that there is more to non-Abelian gauge theories than we first imagined (θ -vacua³⁾), and which have in addition provided (plausible) semi-classical resolutions of the U(1) problem⁴⁾ as well as models for chiral symmetry⁵⁾ and, more recently, supersymmetry⁶⁾, breaking. The question whether instantons play a significant rôle in the physics of a fully quantized non-Abelian gauge theory is thus an important one: however, like most questions concerning fluctuations on the typical hadronic scale, it does not have an analytic answer. In this letter we report on what we find when we examine the structure of vacuum "snapshots" obtained by numerically simulating⁷⁾ the lattice-regularized⁸⁾ SU(2) non-Abelian gauge theory. We shall focus on two questions: does the full non-perturbative vacuum actually contain any significant instanton structures, and how stable are such structures with respect to quantum fluctuations of the vacuum. First, however, some words of introduction will be useful, and a discussion of what happens to classical instanton stability when space-time is discretized to a lattice will be essential.

An instanton is the minimum (Euclidean) action configuration with unit winding number. It is self-dual

$$\int F \cdot \tilde{F} d^4x \equiv \int \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} d^4x = \int F \cdot F d^4x = \frac{8\pi^2}{g^2} \quad (1)$$

and can be detected by its core where the non-zero values of $F \cdot \tilde{F}$ and $F \cdot F$ are localized. The instanton action is scale-invariant at the classical level, as in Eq. (1), but acquires a size (ρ) dependence already at the perturbative one-loop level⁹⁾

$$S_I(\rho) = \frac{8\pi^2}{g^2(\rho)} \xrightarrow{\rho \rightarrow 0} \infty \quad (2)$$

which suppresses instantons on the small scale sizes where the semi-classical arguments are most reliable. This has the unfortunate consequence that instantons can only be physically important on those larger size scales where non-perturbative fluctuations are also important. Hence our motivation for resorting to lattice Monte Carlo techniques.

We regularize the SU(2) gauge theory on a hypercubic lattice in the standard [Wilson⁸⁾] fashion. Our lattice version of $F \cdot \tilde{F}$ is obtained by replacing $F_{\alpha\beta}$ in $\epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ by the $\alpha\beta$ -plane plaquette. This simple lattice $F \cdot \tilde{F}$ operator¹⁰⁾ has acquired a bad odour since its previous use^{11),12)} to calculate

$\langle (\Sigma F \cdot \tilde{F})^2 \rangle$ by Monte Carlo simulation produced results very small compared to the value expected from an analysis of the $\eta'-\pi$ mass splitting¹³⁾ [U(1) problem]. This failure is sometimes attributed to the fact that the (unsubtracted) lattice $F \cdot \tilde{F}$ operator receives cut-off perturbative contributions (appropriate to its dimension) which obscure the non-perturbative physical component. We find this explanation unconvincing: while it is true that deep in the continuum limit the perturbative piece dominates, at a relevant coupling such as $\beta \equiv 4/g^2 = 2.3$ the expected non-perturbative piece, if seen, would have been 0(10)-0(100) times larger than the perturbative piece! Moreover the observed distribution of $\Sigma F \cdot \tilde{F}$ is narrow and, typically, does not extend to non-zero integer values where instanton contributions would be expected to give peaks¹²⁾.

As a first step towards understanding what is going on we take the classical potential for an instanton of size ρ :

$$A_\mu(x) = - \frac{x^2}{x^2 + \rho^2} g^{-1} \partial_\mu g \quad ; \quad g = \frac{x_4 + i x_i \sigma_i}{(x^2)^{1/2}} \quad (3)$$

and discretize it on a lattice, by choosing the lattice link variables to be

$$U_\mu(n) = \exp \left[i (A_\mu(n) + A_\mu(n+\mu)) / 2 \right] \quad (4)$$

We now measure the total $F \cdot \tilde{F}$ (using our lattice operator) within a sphere of radius R about the instanton. We do this for various ρ and R . The results of this simple calculation are shown in Fig. 1a, and should be compared to the value one obtains for the continuum instanton in an infinite volume:

$$\sum_{\text{instanton}} F \cdot \tilde{F} = 32 \pi^2 \quad (5)$$

(which is the normalization we adhere to throughout this paper). There are two striking features: the very strong breaking of scale-invariance for $\rho \lesssim 0(a)$ (also evident for the action which is not shown here) and the fact that one requires a very large lattice to capture most of the topological charge of a large instanton. The first feature follows from the compact nature of the lattice degrees of freedom, and the second from the power-like fall-off of the instanton fields. This tells us that we should not have expected to see significant instanton contributions to $\langle (\Sigma F \cdot \tilde{F})^2 \rangle$ on small 4^4 lattices¹¹⁾. More generally, if we do not expect dynamical instantons to be larger than the typical non-perturbative length scale of the vacuum, ξ (the inverse mass-gap),

then whenever $\xi \lesssim 0(a)$ we should expect even large lattices to produce suppressed values of $\langle (\Sigma F \cdot \tilde{F})^2 \rangle$; as is the case for example at $\beta = 2.3^{12)}$.

Since the action vanishes for (off-site) pointlike instantons

$$S_I(\rho) \xrightarrow{\rho \rightarrow 0} 0 \quad (6)$$

it is clear that the classical lattice theory does not possess instantons in the exact sense that the continuum theory does. Similarly it does not possess non-trivial topology¹⁴⁾: topology and its stability is a dynamical phenomenon which the lattice theory recovers only sufficiently deep in the continuum limit, when the field strengths vary sufficiently smoothly between all neighbouring sites. In this same limit an instanton is very large (in units of a) and the lattice correction to its continuum action is infinitesimal. In particular the effect of one-loop perturbative corrections, Eq. (2), which favours larger instantons will be much more important than these classical lattice corrections.

One can pursue the semi-classical properties of lattice instantons further of course [see Ref. 15)], but at this point we prefer to turn directly to an associated, but physically more important, question: how stable is an instanton with respect to quantum fluctuations? Ideally we would like to measure this stability as a function of the length-scale of the fluctuations: in this letter we will only do so in a coarse sense. The basic idea of our procedure is simple: start with an instanton of size ρ , introduce quantum fluctuations into the system, then remove these fluctuations and see if the instanton is still there. We implement this procedure as follows: starting with a classical instanton of size ρ discretized on the lattice, perform several sweeps with the Monte Carlo at some bare coupling $\beta = 4/g^2$ and then, modifying the Monte Carlo by removing that part of the metropolis algorithm which sometimes accepts changes away from the minimum of the action (i.e., quantum fluctuations), perform some more sweeps to reach the local minimum of the action, and see whether the final gauge field configuration includes an instanton. Note that this last procedure will efficiently remove local fluctuations but will not, in finite (computer) time, "know" that a large lattice instanton can lose action, once it has become sufficiently small, by becoming completely pointlike. Note also that the Monte Carlo procedure produces local equilibrium quantum fluctuations first, and larger quantum fluctuations later: typically one requires $O(l^2)$ sweeps to equilibrate fluctuations on a length scale l (this follows from simple random walk arguments).

As a technical aside, we remark that our algorithm for locally minimizing the action is readily applicable to the general $SU(N)$ case. In the special case of $SU(2)$ a simpler algorithm can be constructed on the basis of the observation that the piece of the action involving the link, U , is of the form $\text{tr}\{U \cdot \Sigma\}$ where Σ is a multiple of an $SU(2)$ matrix. So if we sweep through the lattice choosing new links $\Sigma^+ / |\Sigma|$ we also drive the system to the minimum of the action.

To begin with, we take an almost pointlike instanton, $\rho = 0.1a$ (centred on a lattice site), and simply minimize the action of this gauge field configuration by systematically sweeping through the lattice with the modified metropolis algorithm described above. We do this on a 12^4 lattice with "zero-derivative" boundary conditions, and we plot in Fig. 1b the sum of $F \cdot \tilde{F}$ over all sites at a fixed time $t = 3, \dots, 10$ both for the starting classical configuration and for the configurations obtained after one and two minimization sweeps. We see that a small instanton is completely unstable. Now take a larger instanton, $\rho = 2a$, and apply quantum fluctuations to it, via the full Monte Carlo algorithm at $\beta = 2.3$, for five sweeps, and then remove these fluctuations with the modified metropolis for a further ten sweeps. The $F \cdot \tilde{F}$ profile (in time) is plotted in Fig. 1c for the discretized classical instanton, at the end of the five full Monte Carlo sweeps, and at the end of the action minimization. Strikingly, the instanton comes back! Lessons: small instantons cannot exist on the lattice, while large instantons are remarkably resilient - even at $\beta = 2.3$ where the local fluctuations are almost as strong as possible without falling out of the "continuum physics" regime. One can regard this as giving some basis for calculations where one uses instantons as action minima about which to calculate quantum fluctuations.

We now turn to the main question we wish to address in this letter: to what extent are instantons present in the full non-perturbative vacuum? We begin with our lattice-discretized vacuum at a bare coupling $\beta \equiv 4/g^2 = 2.3$ which corresponds to a lattice spacing more or less as large as possible without tipping over from "continuum" to strong coupling physics. We work on a 10^4 lattice with "zero-derivative" boundary conditions and initially produce a randomized configuration by running a few sweeps at $\beta = 0.5$. We follow this by $O(40)$ Monte Carlo sweeps at $\beta = 2.3$ to produce a typical vacuum "snapshot" at that coupling. We now freeze out the quantum fluctuations with 40 more sweeps of our modified metropolis algorithm: the final "frozen" gauge-field configuration to which we stabilize we label $\{A\}$. In Fig. 2 we show the values that $\Sigma F \cdot \tilde{F}$ (summed over the interior 8^4 sublattice of our 10^4 lattice) takes

during the course of the above procedure. To see what is the $F\tilde{F}$ structure of $\{A\}$ we print the location of every site where $|F\tilde{F}| > 2$. This gives us the (anti-)instanton, $(\bar{I})I$, structures schematically outlined in the square arrowed in Fig. 2; also shown are the co-ordinates of these (anti-)instantons, and rough estimates of sizes. The crucial question now is to decide how much is this nice structure intrinsic to our original $\beta = 2.3$ vacuum snapshot, and how much is it a random product of a particular action minimization path from a highly fluctuating gauge field configuration? To answer this question we try a different minimization path by using a different sequence of (pseudo-)random numbers: this changes both the sequence of sites (randomly chosen) when sweeping through the lattice, and the local changes one makes at these sites. The result of this second minimization is the configuration $\{B\}$, in Fig. 2, whose (anti-)instanton structure is outlined in the same figure. We see that $\{A\}$ and $\{B\}$ are very different. (Most of the structure overlaps the boundary in both cases simply because even on a 10^4 lattice most sites are close to the boundary.) Conclusion: at $\beta = 2.3$ the vacuum contains no real instanton structure.

We now follow a similar procedure at $\beta = 2.7$: but this time, as a variation, on an 8^4 lattice with periodic boundary conditions. We equilibrate with 60 sweeps and freeze for a further 60 sweeps: this brings us to $\{A\}$ in Fig. 3a. Now, as a further variation, perform another 20 Monte Carlo sweeps at $\beta = 2.7$, freeze for 60 to give $\{B\}$, and repeat to give $\{C\}$. In Fig. 3b we show the $F\tilde{F}$ profile of $\{C\}$ in each of two co-ordinates separately (it is very similar for the other two): we find a perfect instanton! And we find that $\{A\}$ and $\{B\}$ are almost identical to $\{C\}$. So we see that at $\beta = 2.7$ not only do typical vacuum configurations have a clean instanton structure, but that these structures are very stable against quantum fluctuations. Note, moreover, that the size of the instanton is of the order of the inverse mass gap: it is not merely a semi-classical object.

We finish with two more points of interest. First in Fig. 4a we plot

$$\langle Q^2 \rangle = 10^{-3} \cdot \left(\frac{4}{L}\right)^4 \cdot \langle (\sum F\tilde{F})^2 \rangle_{L \text{ lge of } L} \approx \text{ind.} \quad (7)$$

for various lattice sizes, L , at $\beta = 2.3$. We see that 4^4 and 5^4 lattices are too small to contain any (pseudo-)stable instanton structures - 6^4 is a minimum size. This is in contrast to the size dependence of $\langle Q^2 \rangle$ as measured on the fluctuating vacuum. Secondly we show $\langle Q^2 \rangle$ for the frozen and fluctuating vacua as a function of β (all taken on periodic 8^4 lattices). $\langle Q^2 \rangle$ has a magnitude

which is in the right ball-park - as predicted from analysis of the U(1) problem¹³⁾ - and a β dependence which confirms our earlier finding that one does not have continuum instanton physics for $\beta < 2.7$.

In summary: We have presented [sketchily: see Ref. 15) for details and further developments] some first steps in an attempt to understand the rôle of instantons in the fully quantized non-Abelian vacuum. Our evidence is that relatively large (and hence physically important) instantons are indeed there, although on a lattice one will have to work at $\beta > 2.7$ to see them. (This is a challenging but accessible coupling.) The stability of instantons with respect to vacuum quantum fluctuations is striking and might support an optimistic approach to the use of instantons as a calculational device. Finally we remark that it would be interesting to marry instantons and topology^{14),16)} in this kind of analysis.

NOTE ADDED

When the work reported upon in this paper was completed I learned that similar work is in progress elsewhere¹⁷⁾, using a heat bath analogue of our action minimization algorithm, and focussing on the global topology rather than the topological charge density as herein.

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FIGURE CAPTIONS

- Figure 1 (a) Sum of $F \cdot \tilde{F}$ within a sphere centred on a classical discretized instanton: as a function of sphere radius and instanton size, ρ .
(b) Sum of $F \cdot \tilde{F}$ at fixed times on a 12^4 lattice, for a $\rho = 0.1a$ classical instanton, and for the gauge-field configurations obtained by sweeping to the minimum action once and a second time.
(c) As in (b) but for $\rho = 2a$ on a 10^4 lattice, and for configurations obtained by five Monte Carlo sweeps at $\beta = 2.3$ and by a further 10 sweeps to the minimum of the action.
- Figure 2 Sum of $F \cdot \tilde{F}$ on a 10^4 lattice during Monte Carlo sweeps at $\beta = 2.3$ and subsequent action minimization sweeps, by two paths leading to configurations $\{A\}$ and $\{B\}$ whose respective (anti-)instanton structures are outlined in adjacent squares.
- Figure 3 Sum of $F \cdot \tilde{F}$ on a 10^4 lattice during a sequence of Monte Carlo sweeps at $\beta = 2.7$ interspersed with action minimizations. Shown is the $F \cdot \tilde{F}$ profile in both x and y co-ordinates, of the final "frozen" lattice configuration.
- Figure 4 Dependence of $\langle Q^2 \rangle = (4/L)^4 \langle (\Sigma F \cdot \tilde{F})^2 \rangle \cdot 10^{-3}$ on (a) lattice size at $\beta = 2.3$ for fluctuating (o) and frozen (•) gauge configurations, and (b) on β for an 8^4 lattice.

- Figure 1 -

- Figure 2 -

- Figure 3 -

- Figure 4 -