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EXCEPTIONAL COSET SPACES AND THE
SPECTRUM OF QUARKS AND LEPTONS

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Abstract

It is an interesting possibility that the effective low energy theory of quarks and leptons is a supersymmetric σ -model. We determine all complex structures of homogeneous Kähler manifolds of the form G/H with G being E_7 or E_8 and H containing as factors the GUT groups $SU(5)$, $SO(10)$ or E_6 . The most promising candidates for a grand unified theory are $SO(10)$ models with three generations of quarks and leptons and one mirror generation.

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Exceptional groups play a special role in particle physics: $E_3 = SU(3) \times SU(2)$, $E_4 = SU(5)$, $E_5 = SO(10)$ and E_6 are the non-Abelian gauge groups of the standard model of strong and electroweak interactions and its unified extensions, and the quark-lepton multiplets transform as their fundamental representations¹⁾. Furthermore it is a remarkable phenomenon that the generators of the coset spaces $E_{n+1}/E_n \times U(1)$ ($n = 3, \dots, 7$) transform with respect to the subgroup $E_n \times U(1)$ precisely as the quark-lepton representations (cf. the Table).

In the context of supersymmetric theories this suggests that a σ -model based on exceptional groups may be the appropriate low energy effective Lagrangian^{2),3)} of a more fundamental theory which will ultimately explain the generation puzzle of the standard model. A supersymmetric σ -model could result from a supersymmetric preon theory with spontaneously broken global symmetry²⁾ or from a higher dimensional supergravity theory after spontaneous compactification⁴⁾. Some supersymmetric σ -models based on E_7 ⁵⁾, E_6 ⁶⁾ and E_8 ^{7),8)} have already been constructed. The purpose of this letter is to provide a complete list of possible representations of Goldstone superfields for σ -models with $G = E_7, E_8$ and $H = H' \times K$, $H' = SU(5), SO(10), E_6$.

The complex scalar fields of a supersymmetric σ -model parametrize a Kähler manifold⁹⁾. Special Kähler manifolds with maximal isometry group are homogeneous spaces. G/H is a Kähler manifold if H is the centralizer of a torus of G ¹⁰⁾. This means that in order to find a Kähler manifold for a given group G one has to choose a $U(1)$ subgroup and to determine the maximal subgroup H of G which commutes with the chosen $U(1)$ factor. On a Kähler manifold G/H there exist one or more " G -invariant complex structures"¹¹⁾, i.e., ways of dividing the $2n$ coordinates of the manifold into a set of n holomorphic coordinates ϕ_i and a set of n antiholomorphic coordinates ϕ_i^* such that the metric is invariant under a realization of G separately on the holomorphic (antiholomorphic) coordinates by means of holomorphic (antiholomorphic) Killing vectors.

A G -invariant complex structure on G/H can be defined through a set of positive roots Ω with the following properties^{11),*)}: (1) $\Omega \cup \Sigma$ forms a system of positive roots of G , with Σ being a system of positive roots of H ; (2) Ω is closed, i.e., the sum of two roots of Ω is contained in Ω if it is a root in G . As the positive roots in Ω and their negative counterparts represent the

*) For an application with $G = E_8$, see C.L. Ong, Ref. 7).

generators of G/H , Ω specifies the representation of the holomorphic coordinates ϕ_i with respect to H , and thereby the quantum numbers of the chiral superfields of the supersymmetric σ -model with respect to the unbroken group.

Recently, Bordemann, Forger and Römer have performed a systematic analysis of homogeneous Kähler manifolds¹²⁾. One result of this investigation is a simple recipe for finding all homogeneous Kähler manifolds G/H . As H is the centralizer of a torus of G , one always has $H = H' \times U(1)^p$, with $p \geq 1$ and H' semi-simple. All Kähler manifolds $G/H' \times U(1)^p$ can be obtained by crossing out p dots from the Dynkin diagram for G ; the remaining Dynkin diagram gives H' , and the p $U(1)$ factors are generated by the elements of the Cartan subalgebra of G which are not contained in H' . This prescription for finding Kähler manifolds corresponds to a p -fold application of Dynkin's method for obtaining regular maximal non semi-simple subalgebras¹³⁾. The root systems Ω , which define the complex structures, can be obtained by selecting all roots of G/H which are positive with respect to any chosen $U(1)$ generator contained in $U(1)^p$. As a consequence, the complex structure is unique for $p = 1$, but in general not unique for $p \geq 1$.

Let us consider as an example the Kähler manifold $G(n,1,1) \equiv SU(n+2)/SU(n) \times U(1)^2$ ¹¹⁾ which corresponds to the Dynkin diagram of $SU(n+2)$ with two dots at one end crossed out (Fig. 1a). In order to obtain the complex structures it is convenient to decompose the adjoint representation of $SU(n+2)$ (A_{n+2}) with respect to $SU(n) \times U(1)^2$. Choosing as $U(1)$ charges $Q_1 = \text{diag}(1, \dots, 1, -n, 0)$ and $Q_2 = \text{diag}(1, \dots, 1, 0, -n)$ one has

$$A_{n+2} = [A_n^{0,0} + 2 \cdot 1^{0,0}] + \left[\tilde{n}^{1,n+1} + \tilde{n}^{n+1,1} + \tilde{1}^{n,-n} + \tilde{n}^{-1,-n-1} + \tilde{n}^{-n-1,-1} + \tilde{1}^{-n,n} \right] \quad (1)$$

The different complex structures can now be obtained by determining the different sets of roots Ω of $G(n,1,1)$ which are positive with respect to a chosen linear combination $\alpha Q_1 + \beta Q_2$.

As we are interested in different complex structures only up to complex conjugation we can restrict ourselves to root systems which contain $\tilde{1}^{n,-n}$, i.e., to charges of the form $Q_1 + \alpha(Q_1 + Q_2)$. Obviously there exist three regions for α (Fig. 1b) which lead to the following sets of positive roots:

$$(I) \quad -\frac{1}{n+2} < \alpha$$

$$\Omega_I = \tilde{n}^{1, n+1} + \tilde{n}^{n+1, 1} + \tilde{1}^{n, -n} \quad , \quad (2a)$$

$$(II) \quad -\frac{n+1}{n+2} < \alpha < -\frac{1}{n+2}$$

$$\Omega_{II} = \tilde{n}^{-1, -n-1} + \tilde{n}^{n+1, 1} + \tilde{1}^{n, -n} \quad , \quad (2b)$$

$$(III) \quad \alpha < -\frac{n+1}{n+2}$$

$$\Omega_{III} = \tilde{n}^{-1, -n-1} + \tilde{n}^{-n-1, -1} + \tilde{1}^{n, -n} \quad . \quad (2c)$$

We count only those complex structures as inequivalent whose root systems cannot be transformed into each other by complex conjugation and/or redefinition of the $U(1)$ charges. In the above example only Ω_I and Ω_{II} are inequivalent (cf. Fig. 1c); Ω_{III} can be transformed into Ω_I by complex conjugation and the exchange of Q_1 and Q_2 .

The existence of two inequivalent complex structures on $G(n,1,1)$ may appear rather surprising in view of the fact that the metric of the real manifold $G(n,1,1)$ is entirely determined up to the "decay constants" f_i . The two root systems Ω_I and Ω_{II} correspond to two inequivalent ways of dividing the $4n+2$ real coordinates of $G(n,1,1)$ into $2n+1$ holomorphic and $2n+1$ antiholomorphic coordinates, and we have checked explicitly that a nonlinear realization of $SU(n+2)$ in terms of holomorphic Killing vectors can be defined on coordinates transforming as Ω_I as well as Ω_{II} .

Kähler manifolds with exceptional groups can be treated in analogy to the example discussed above. In order to accommodate three quark-lepton families in a coset space G/H the dimension of G/H has to be sufficiently large. As each generation contains 15 Weyl fermions one has

$$n_G - n_H \geq 2 \cdot 15 \cdot 3 = 90 \quad . \quad (3)$$

Here the factor 2 corresponds to the case of "total pairing"^{*)}, in which each

^{*)} For a more complete discussion, see Refs. 2), 3), 15-17).

Weyl fermion is associated with two complex conjugate broken generators. In the case of "partial doubling" (see footnote on p. 3), where some of the Weyl fermions are connected with only one broken generator, Eq. (3) is a strict inequality.

If H contains the GUT group $E_6 = SU(5)$ a choice consistent with (3) is $G = E_7$. The Kähler manifold of smallest dimension with $G = E_7$ and $H = SU(5) \times K$ is $E_7/SU(5) \times SU(3) \times U(1)$ (Fig. 2a). In order to obtain the corresponding complex structure we need the decomposition of the adjoint representation of E_7 with respect to $SU(5) \times SU(3) \times U(1)$. From the $SU(6) \times SU(3)$ content¹⁴⁾

$$133 = (35, 1) + (1, 8) + (15, \bar{3}) + (\bar{15}, 3) \quad (4)$$

one immediately obtains

$$133 = [(24, 1)^0 + (1, 8)^0 + (1, 1)^0] + [(\bar{5}, 1)^{-6} + (\bar{5}, 3)^4 + (10, \bar{3})^2 + c.c.] \quad (5)$$

This yields as the unique set of positive roots of $E_7/SU(5) \times SU(3) \times U(1)$

$$\Omega = (5, 1)^6 + (\bar{5}, 3)^4 + (10, \bar{3})^2 \quad (6)$$

This complex structure has first been found by Kugo and Yanagida⁵⁾ who constructed explicitly the nonlinear realization of E_7 on holomorphic coordinates transforming as (6)^{*)}.

Unfortunately the complex structure (6) alone cannot be the basis of an $SU(5)$ grand unified theory because fermions transforming as (6) have an $SU(5)$ anomaly. For the Kähler manifolds $E_7/SU(5) \times SU(2) \times U(1)^2$ and $E_7/SU(5) \times U(1)^3$ one can find complex structures whose $SU(5)$ content is different from (6). The $SU(5)$ anomaly remains, however, because the total number of 5, $\bar{5}$, 10 and $\bar{10}$ representations is odd. Therefore we do not list the complex structures for these Kähler manifolds.

The smallest Kähler manifold with E_8 symmetry and $SU(5)$ factor in H is $E_8/SU(5) \times SU(4) \times U(1)$ (Fig. 2b). From the $SU(5) \times SU(5)$ content of the adjoint representation of E_8 ¹⁴⁾,

*) See also C.L. Ong, 2nd Ref. in 3).

$$248 = (24, 1) + (1, 24) + (10, 5) + (\overline{10}, \overline{5}) + (5, \overline{10}) + (\overline{5}, 10) \quad , \quad (7)$$

one obtains with respect to $SU(5) \times SU(4) \times U(1)$

$$248 = [(24, 1)^0 + (1, 15)^0 + (1, 1)^0] + [(10, 4)^1 + (10, \overline{1})^4 + (\overline{5}, 6)^2 + (\overline{5}, 4)^{-3} + (1, 4)^5 + c.c.] \quad , \quad (8)$$

which yields the complex structure

$$\Omega = (\overline{5}, 6)^2 + (10, 4)^1 + (\overline{5}, 4)^3 + (\overline{10}, 1)^4 + (1, 4)^5 \quad . \quad (9)$$

The σ -model defined by (9) has again an $SU(5)$ anomaly which is also present for all complex structures on the Kähler manifolds $E_8/SU(5) \times SU(3) \times U(1)^2$, $E_8/SU(5) \times SU(2) \times U(1)^3$ and $E_8/SU(5) \times U(1)^4$.

If the "unbroken" subgroup H contains $SO(10)$ or E_6 as factor, G has to be E_8 in order to accommodate three families in G/H . The complex structure on the Kähler manifold $E_8/E_6 \times SU(2) \times U(1)$ (Fig. 2c) can be obtained from the $E_6 \times SU(3)$ content of the adjoint representation of E_8 ¹⁴⁾

$$248 = (78, 1) + (1, 8) + (27, 3) + (\overline{27}, \overline{3}) \quad , \quad (10)$$

which yields with respect to $E_6 \times SU(2) \times U(1)$:

$$248 = [(78, 1)^0 + (1, 3)^0 + (1, 1)^0] + [(27, 2)^1 + (27, 1)^{-2} + (1, 2)^3 + c.c.] \quad . \quad (11)$$

From (11) one reads off the complex structure

$$\Omega = (27, 2)^1 + (\overline{27}, 1)^2 + (1, 2)^3 \quad , \quad (12)$$

which corresponds to two E_6 families and one mirror family.

The complex structures for $E_8/E_6 \times U(1)^2$ (Fig. 2d) can be obtained from a further decomposition of (11). Although H contains now two $U(1)$ factors, there is a unique set of positive roots:

$$\Omega = \underline{27}^{1,1} + \underline{27}^{1,-1} + \underline{27}^{2,0} + \underline{1}^{3,1} + \underline{1}^{3,-1} + \underline{1}^{0,2} \quad , \quad (13)$$

which contains one E_6 singlet in addition to (12).

The Kähler manifolds with $SO(10)$ factors are $E_8/SO(10) \times SU(3) \times U(1)$, $E_8/SO(10) \times SU(2) \times U(1)^2$ and $E_8/SO(10) \times U(1)^3$ (Figs. 2 e-g). From the $SO(10) \times SU(4)$ decomposition ¹⁴⁾

$$248 = (45, 1) + (1, 15) + (10, 6) + (16, 4) + (\overline{16}, \overline{4}) \quad (14)$$

one has with respect to $SO(10) \times SU(3) \times U(1)$

$$248 = [(45, 1)^0 + (1, 8)^0 + (1, 1)^0] + [(16, 3)^1 + (16, 1)^{-3} + (10, 3)^{-2} + (1, 3)^4 + \text{c.c.}] \quad (15)$$

This yields for $E_8/SO(10) \times SU(3) \times U(1)$ the complex structure ^{7), 8)}

$$\Omega = (16, 3)^1 + (\overline{16}, 1)^3 + (10, \overline{3})^2 + (1, 3)^4, \quad (16)$$

containing three $SO(10)$ families and one mirror family.

For the Kähler manifold $E_8/SO(10) \times SU(2) \times U(1)^2$ there exist three inequivalent complex structures. Two of them contain three 16 and one $\overline{16}$ representations, and one has two 16 and two $\overline{16}$ representations:

$$\Omega_{\underline{I}} = (16, 2)^{1,1} + (16, 1)^{1,-2} + (\overline{16}, 1)^{3,0} + (10, 2)^{2,-1} + (10, 1)^{2,2} + (1, 2)^{4,1} + (1, 1)^{4,-2} + (1, 2)^{0,-3}, \quad (17a)$$

$$\Omega_{\underline{II}} = (16, 2)^{1,1} + (16, 1)^{1,-2} + (\overline{16}, 1)^{3,0} + (10, 2)^{2,-1} + (10, 1)^{2,2} + (1, 2)^{4,1} + (1, 1)^{4,-2} + (1, 2)^{0,3}, \quad (17b)$$

$$\Omega_{\underline{III}} = (16, 2)^{1,1} + (\overline{16}, 1)^{-1,2} + (\overline{16}, 1)^{3,0} + (10, 2)^{2,-1} + (10, 1)^{2,2} + (1, 2)^{4,1} + (1, 1)^{4,-2} + (1, 2)^{0,3}. \quad (17c)$$

The complex structure Ω_{II} has previously been obtained by Ong ⁷⁾.

On the coset space $E_8/SO(10) \times U(1)^3$ even more complex structures exist. Two examples are:

$$\begin{aligned} \Omega_I = & \underline{16}^{1,1,1} + \underline{16}^{1,1,-1} + \underline{16}^{1,-2,0} + \underline{16}^{3,0,0} + \underline{10}^{2,-1,1} + \underline{10}^{2,-1,-1} + \underline{10}^{2,2,0} \\ & + \underline{1}^{4,1,1} + \underline{1}^{4,1,-1} + \underline{1}^{4,-2,0} + \underline{1}^{0,-3,1} + \underline{1}^{0,-3,-1} + \underline{1}^{0,0,2}, \end{aligned} \quad (18a)$$

$$\begin{aligned} \Omega_{II} = & \underline{16}^{1,1,1} + \underline{16}^{1,1,-1} + \underline{16}^{-1,2,0} + \underline{16}^{3,0,0} + \underline{10}^{2,-1,1} + \underline{10}^{2,-1,-1} + \underline{10}^{2,2,0} \\ & + \underline{1}^{4,1,1} + \underline{1}^{4,1,-1} + \underline{1}^{4,-2,0} + \underline{1}^{0,3,1} + \underline{1}^{0,3,-1} + \underline{1}^{0,0,2}. \end{aligned} \quad (18b)$$

There exists no complex structure with four families and no mirror family.

In Eqs. (6), (9), (12), (13), (16), (17) and (18) we have given the complex structures for coset spaces G/H with $G = E_7, E_8$ and $H = H' \times K$, $H' = SU(5), SO(10), E_6$. The listed representations are the representations of the chiral superfields of the supersymmetric σ -models for the corresponding coset spaces. $SU(5)$ models always have an $SU(5)$ anomaly and therefore, unless further fields are added, they are not suited as basis of a grand unified theory. All $SO(10)$ and E_6 models are anomaly free. E_6 models contain two families and one mirror family. $SO(10)$ models can have two families and two mirror families as well as three families and one mirror family. We conclude that the three family $SO(10)$ models first considered by Ong, Irié and Yasui, are the most interesting candidates for a grand unified theory. Their inevitable prediction is the existence of one mirror family with masses not heavier than the weak interaction scale $G_F^{-1/2}$ in addition to three ordinary families.

We have restricted ourselves to a discussion of homogeneous Kähler manifolds. It is known ^{16), 17)} that the corresponding σ -models can not be the effective low energy Lagrangians of supersymmetric theories with spontaneous symmetry breaking. The effective Lagrangians of Goldstone superfields always involve a partial "doubling" and contain some scalar fields, "quasi-Goldstone bosons", in addition to the Goldstone fields. In some cases, however, this "doubling" affects only singlets, i.e., Goldstone superfields which are trivial with respect to the non-Abelian part of the unbroken subgroup H (cf., e.g., Ref. 18)). Therefore, except for the singlets, our analysis of complex structures applies to such theories as well. Another possibility is that the partial "doubling" involves also fields which transform non-trivially with respect to the non-Abelian part of H . In this way one can avoid the anomaly problem of the $SU(5)$ models. An interesting coset space of this type has been suggested by Ibanez ¹⁹⁾.

The main open questions with respect to grand unified supersymmetric σ -models are the breaking of the GUT gauge group and the generation of masses. We hope that the further investigation of the nonlinear interactions of the σ -models and the effects of gauge interactions and soft supersymmetry breaking will provide an answer to these questions.

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FIGURE CAPTIONS

Fig. 1 : The coset space $SU(n+2)/SU(n)\times U(1)^2$. a) $SU(n+2)$ Dynkin diagram with two dots crossed out, b) the three regions for the complex structures Ω_I , Ω_{II} , Ω_{III} , c) the two inequivalent complex structures Ω_I and Ω_{II} .

Fig. 2 : Dynkin diagrams for Kähler manifolds with E_7 and E_8 symmetry:

- a) $E_7/SU(5)\times SU(3)\times U(1)$,
- b) $E_8/SU(5)\times SU(4)\times U(1)$,
- c) $E_8/E_6\times SU(2)\times U(1)$,
- d) $E_8/E_6\times U(1)^2$,
- e) $E_8/SO(10)\times SU(3)\times U(1)$,
- f) $E_8/SO(10)\times SU(2)\times U(1)^2$,
- g) $E_8/SO(10)\times U(1)^3$.

Exceptional groups	Dynkin diagrams	Coset spaces $E_{n+1}/E_n \times U(1)$
$E_3 = SU(3) \times SU(2)$		$(3, 2) + c.c.$
$E_4 = SU(5)$		$10 + c.c.$
$E_5 = SO(10)$		$16(=\overline{5}+10+1) + c.c.$
E_6		$27(=16+10+1) + c.c.$
E_7		$56(=27+\overline{27}+1+1) + 1 + c.c.$
E_8

TABLE: The "exceptional sequence". The coset spaces $E_{n+1}/E_n \times U(1)$ of the exceptional groups contain the E_n representations which are used for quark-lepton multiplets in E_n gauge theories.

a)

b)

c)

Fig. 1

a)

b)

c)

d)

e)

f)

g)

Fig. 2