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A BOUNDARY LAYER MODEL FOR THE KALUZA-KLEIN MONOPOLE*

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ABSTRACT

A simple model is presented for a five-dimensional Kaluza-Klein monopole with confined scalar field. The model consists of an interior region described by the Sorkin, Gross and Perry monopole solution, and an exterior region whose four-geometry is that of a Reissner-Nordstrom magnetic monopole, embedded in the Kaluza-Klein vacuum with constant internal radius. By postulating that the two regions are separated by a thin boundary layer, it is possible to use the Gauss-Codazzi formalism to place stringent constraints on the effective stress energy of the boundary layer.

Kaluza-Klein theories¹ have recently been the subject of much research.² A useful probe of the physical content of higher dimensional theories is provided by the analysis of exact solutions to the vacuum field equations. One particularly interesting solution of this type is the recently discovered Kaluza-Klein monopole^{3,4,5} which is an exact, non-singular solution to the five-dimensional vacuum Einstein equations. The reduction of this solution to four dimensions describes a magnetic monopole with magnetic charge $g = \frac{1}{2e}$ as well as a long range scalar field. A recent analysis⁶ has shown that the long range scalar field has striking consequences for test particle motion in the background of a Kaluza-Klein monopole. Since no symmetry requires the scalar field in Kaluza-Klein theory to stay massless, it is very likely that it does acquire a mass due to quantum corrections.⁷ A mass term in the effective Lagrangian would, however, alter the field equations so that one must ask what specific role classical solutions such as the Kaluza-Klein monopole play in the physical particle spectrum of the theory.

In the following, we adopt the attitude that the Kaluza-Klein monopole solution accurately describes the short range behavior of the geometry surrounding a physical monopole, while the long-range behavior must be modified in such a way as to "confine" the scalar field. Remarkably, this simple ansatz leads to an almost unique description of the global geometry surrounding a monopole. In particular, at large radii it looks like an ordinary 4-dimensional Reissner-Nordstrom monopole with constant scalar field, while at short radii, it is exactly described by the K.K. monopole solution. By making the further assumption that the intermediate region is thin compared to other length scales in the problem,* one can apply the Gauss-Codazzi formalism⁸ for the evolution of thin bubbles separating regions of known geometry. In the present case, the domain wall separating the two regions is necessarily static and the formalism provides remarkably stringent constraints on the stress energy of the bubble. Thus in some sense, this toy model allows us to probe the structure of the transition from an unconfined scalar field at short distances to a confined scalar field at large distances.

We consider a five-dimensional Kaluza-Klein theory with line element

* Even if this assumption is not strictly valid, the calculations nonetheless provide information about the stress energy in the intermediate region, averaged out in some suitable fashion.

parametrized as follows:**

$$d\bar{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + \lambda^2 (A_\mu dx^\mu + d\zeta)^2. \quad (1)$$

The Einstein-Hilbert action

$$\bar{I} = \frac{1}{16\pi\bar{G}} \int d^5\bar{x} \sqrt{-\bar{g}} (\bar{R} + \bar{L}_M) \quad (2)$$

yields the following field equation in the horizontal lift basis:⁹

$$\bar{G}_{\mu\nu} = G_{\mu\nu}(\theta) + \frac{1}{2}\lambda^2 \left(F_\mu{}^\rho F_{\rho\nu} + \frac{1}{4}g_{\mu\nu}F^2 \right) - \frac{\nabla_\mu \nabla_\nu \lambda}{\lambda} + g_{\mu\nu} \frac{\nabla^2 \lambda}{\lambda} = 8\pi \bar{G} \bar{T}_{\mu\nu} \quad (3a)$$

$$\bar{G}_{\mu 5} = \frac{\nabla_\mu (\lambda^3 F_\mu{}^5)}{2\lambda} = 8\pi \bar{G} \bar{T}_{\mu 5}, \quad (3b)$$

$$\bar{G}_{55} = \frac{3}{8}\lambda^4 F^2 - \frac{\lambda^2}{2}R(\theta) = 8\pi \bar{G} \bar{T}_{55}, \quad (3c)$$

where ∇_μ refers to covariant differentiation with respect to the four metric $g_{\mu\nu}$ and $F_{\mu\nu} \equiv A_{\mu,\nu} - A_{\nu,\mu}$.

The Kaluza-Klein monopole^{3,4,5} is a non-singular solution to the five-dimensional vacuum Einstein equations given by the following line element:³

$$d\bar{s}^2 = -dt^2 + \left(1 + \frac{m}{\sqrt{r^2 + m^2}}\right)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \left(\frac{4mr}{\sqrt{r^2 + m^2} + m}\right)^2 \left(\frac{1}{2}\cos\theta d\phi + d\zeta\right)^2. \quad (4)$$

The effective four-dimensional scalar and magnetic fields are, respectively:

$$\lambda(r) = \frac{4mr}{\sqrt{r^2 + m^2} + m} \quad (5)$$

and

$$F_3^2 = \frac{1}{2} \frac{\sin\theta}{r^2} \quad (6)$$

Note that as $r \rightarrow \infty$, $\lambda(r) \rightarrow 4m(1 - m/r)$. Thus, the invariant radius of the internal circle $\frac{1}{2\pi} \int \sqrt{g_{55}} dx^5 = \lambda(r)$ approaches a constant at $r = \infty$.

** In our notation, all five-dimensional tensors will have bars, and carry Latin indices $\{A, B, \dots = 0, 1, 2, 3, 5\}$, while four-dimensional tensors are unbarred and carry Greek indices $\{\mu, \nu, \dots = 0, 1, 2, 3\}$. The signature of the metric is $(-+++)$, $\{x^\mu\}$ refer to space-time coordinates while ζ is the internal coordinate which is periodic with range $[0, 2\pi]$.

We wish to construct a model for the Kaluza-Klein monopole in which the scalar field is short range. However, if $\bar{T}_{\mu\nu} = \bar{T}_{55} = 0$ then Eqs. (3a) and (3c) yield

$$\nabla^2 \lambda = \lambda^3 F^2. \quad (7)$$

Thus, if $g_{\mu\nu} \rightarrow \delta_{\mu\nu} + O(\frac{1}{r})$, $F^2 \rightarrow \theta^2/r^4 + O(\frac{1}{r^5})$ and $\lambda \rightarrow \text{constant} + O(\frac{1}{r^n})$, it can easily be shown that Eq. (7) requires $n \leq 2$. In the following, we impose the simplest ansatz which allows $\lambda(r)$ to fall off faster than $\frac{1}{r^2}$ at infinity. In particular, we require the space-time projections of the five dimensional stress energy to vanish, while the purely internal component, $\bar{T}_{55}$, is non-zero. In this case, the five dimensional Bianchi identities yield*

$$\frac{\lambda}{\lambda} \frac{\mu}{T_{55}} = 0. \quad (8)$$

Eq. (9) is in the form of a bifurcation equation, suggesting two possible phases for the "vacuum" surrounding the Kaluza-Klein monopole. In one phase $\bar{T}_{55} = 0$, so that the metric in Eq. (4), with its long-range scalar field, is valid**. In the other phase (i.e., $\bar{T}_{55} \neq 0$) the scalar field is constant, and the field equations (3a) and (3b) reduce to the four-dimensional Einstein-Maxwell equations for $g_{\mu\nu}$ and $F_{\mu\nu}$, whose spherically-symmetric monopole solution is the well-known Reissner-Nordstrom solution. The five-dimensional line element in the "confined phase" of the scalar field therefore takes the form

$$d\bar{S}^2 = -\left(1 - \frac{2M}{r} + \left(\frac{n\lambda_0}{4r}\right)^2\right) dt^2 + \left(1 - \frac{2M}{r} + \left(\frac{n\lambda_0}{4r}\right)^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \lambda_0^2 \left(\frac{n}{2} \cos\theta d\phi + d\zeta\right)^2. \quad (9)$$

The standard Kaluza-Klein ansatz⁶ requires that

$$\lambda_0 = \frac{\hbar \sqrt{16\pi G}}{e} \quad (10)$$

where G is the four dimensional Newton's constant. The dimensionless parameter n derives from an arbitrary constant of integration. The magnetic charge described by the four-dimensional reduction of the line element in Eq. (9) is:

$$g_n = \frac{n\lambda_0}{2\sqrt{16\pi G}} = \frac{n}{2} \hbar/e. \quad (11)$$

* We are grateful to Jack Gegenberg for pointing this out.

** We are assuming, of course, that the vacuum solution of Eq. (5) is the only (non-singular) solution with the right asymptotic behavior to describe a 4-dimensional monopole.

The remaining Eq. (3c) requires that

$$\bar{T}_{55} = \frac{3}{8} \frac{\lambda_0^2 F^2}{\pi \bar{G}} = \frac{3}{8} \frac{q^2 \lambda_0}{2\pi r^4}, \quad (12)$$

where we have used Eqs. (10) and (11), as well as the identification $G = \bar{G}/2\pi\lambda_0$.

The above analysis leads to the following model for a physical Kaluza-Klein monopole. At large energies or short distances, say $r \leq R$, for some as yet unknown R , the geometry surrounding the monopole is well-described by the line element given in Eq. (3). For large distances, we assume the scalar field to be short range, so that Eq. (9) defines the line element. By making the additional assumption that the boundary region separating these two phases is narrow, it is possible to apply the well-known Gauss-Codazzi formalism⁸ to analyze the dynamics of the boundary layer. Normally, this formalism is used to find the equations of motion for the layer. In the present case, we will use it to obtain constraints on the surface stress energy tensor of the bubble separating the confined and unconfined phases.

The generalization of the Gauss-Codazzi formalism to five dimensions is straightforward. We define an exterior region V^+ with coordinates $(t_+, r_+, \theta, \phi, \zeta)$ and line element given by Eq. (9). The interior region, V^- , has coordinates $(t_-, r_-, \theta, \phi, \zeta)$ and line element given by Eq. (4). The geometry of the boundary layer Σ is given in co-moving coordinates $\{\xi^i\} = \{r, \theta, \phi, \zeta\}$ by

$$(ds^2)_\Sigma = -dr^2 + R^2(\tau)(d\theta^2 + \sin^2\theta d\phi^2) + \lambda^2(\tau) \left(\frac{n'}{2} \cos\theta d\phi + d\zeta \right)^2. \quad (13)$$

The coordinates θ, ϕ , and ζ , which have been chosen to be continuous across the boundary, span a surface topologically equivalent to a three-sphere.³ The tangential line elements must be continuous across the boundary, so that

$$R(\tau) = r_- = r_+ \quad (14a)$$

$$\lambda(\tau) = \frac{4mR(\tau)}{\sqrt{R^2 + m^2} + m} = \lambda_0 \quad (14b)$$

$$n' = n = 1 \quad (14c)$$

Since λ_0 and m are constants, Eq. (14b) implies that only static matchings are allowed: $\dot{R}(\tau) = \dot{R}(\tau) = 0$. Eq. (14c) ensures that the Dirac quantization condition is satisfied by the magnetic charge.

We now define the extrinsic curvature $K_{ij}^\pm(K_{ij}^-)$ surface Σ embedded in regions $V^+(V^-)$, respectively:

$$K_{ij}^\pm = -\bar{g}_{AB} N^B \frac{D \partial X^A}{D \xi^i} \Big|^\pm, \quad (15)$$

where $\{i, j = 0, 1, 2, 3\}$ are indices referring to the intrinsic coordinates ξ^i on Σ and $M_\pm^B$ are the components of the unit normals to Σ embedded in $V_\pm$:

$$M_\pm^B = \left((\bar{g}_{11}^{-1/2})_\pm, (\bar{g}_{00})_\pm, 0, 0, 0 \right). \quad (16)$$

For a static bubble $\dot{R} = 0$ and $\dot{t}_\pm = (\bar{g}_{00})_\pm^{-1/2}$. In the above D refers to covariant differentiation with respect to the metric in V^+ or V^- , while a dot signifies differentiation with respect to τ .

The Gauss-Codazzi equations relevant to the present analysis are^{10,11}

$${}^{(5)}\bar{R}_{ij} = K_{i,j,n} + {}^{(4)}R_{ij} + 2K_i^k K_{kj} - K^k_k K_{ij} \quad (17)$$

where n refers to differentiation along the normal to the surface, and ${}^{(4)}R_{ij}$ is the intrinsic Ricci tensor. Using the five-dimensional Einstein equations in the left-hand side of Eq. (17), and integrating across the boundary region yields

$$\gamma_{ij} \equiv K_{ij}^+ - K_{ij}^- = -8\pi\bar{G} \left(S_{ij} - \frac{1}{3} S g_{ij} \right), \quad (18)$$

where

$$S_{ij} \equiv \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \bar{T}_{ij} dn, \quad (19)$$

and we have assumed that K_{ij} and ${}^{(4)}R_{ij}$ are bounded inside the layer. Eq. (18) is the five-dimensional analogue of the Lanczos equations.

The most general form of surface energy S_{ij} consistent with the symmetries of the line element (11) is in the horizontal lift basis

$$S_{ij} = \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p \sin^2 \theta & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad (20)$$

Note that since Σ is four dimensional S_{ij} has units of energy/(length)³.

Finally, using all of the above information, Eq. (18) yields the following non-trivial conditions on the five parameters R, M, σ, p and q :

$$(1 - 2Y + X^2)^{1/2} - (1 - X) = (\beta - \alpha - 1)\sigma' \quad (21a)$$

$$X = (2(\beta - \alpha) + 1)\sigma' \quad (21b)$$

$$\frac{Y - X^2}{(1 - 2Y + X^2)^{1/2}} = (2 + 2\alpha + \beta)\sigma' \quad (21c)$$

where $X \equiv \lambda_0/4R$, $Y \equiv M/R$, $\alpha \equiv p/\sigma$, and $\beta \equiv q/\sigma$ and $\sigma' \equiv \frac{8\pi\bar{G}R}{3}\sigma$. These equations can readily be solved for σ' and Y in terms of α and β :

$$\sigma' = \frac{\alpha}{\alpha(Z+2) - \frac{1}{2}(Z^2-1)}, \quad \alpha \neq 0, \quad \beta \neq 1 \quad (22)$$

$$Y = \sigma' \left(\frac{3}{2} \sigma' (Z^2 - 1) + Z + 2 \right), \quad (23)$$

where $Z \equiv \beta - \alpha$ and X is given by Eq. (21b) above.

Although in principle there exists a two parameter family of solutions to Eqs. (21a-c), it is necessary to impose the restrictions that M and R be positive. It is straightforward to show that these positivity conditions plus the condition that m be real, require the parameters α and β to lie in the shaded areas of Figure 1. Thus, the equation of state, ($p = \alpha\sigma$, $q = \beta\sigma$), for the bubble is severely restricted. For example, most of the regions for which the velocity of sound is not greater than the speed of light ($|\alpha| \leq 1$, $|\beta| \leq 1$) have been excluded. In particular, membrane-type solutions ($\alpha = -1$ and $\beta = -1$) are disallowed. For completeness we have included in Figure 1 those configurations with $\sigma < 0$ (dotted region). It is remarkable that only positive energy density configurations (cross-hatched regions) contribute significantly to the parameter space in the region of interest.

In Table I, we list the properties of some specific solutions: $Z = 1$, $Z = 0$ and $Z = -\alpha$. E_B is defined as the energy of the boundary in units of length

$$\begin{aligned} E_B &= G \int d^4x \sqrt{-g} \sigma \\ &= (4\pi R^3) (2\pi\lambda_0) \sigma G. \end{aligned} \quad (24)$$

In all cases, as $R \rightarrow \infty$, the effective gravitational mass M approaches $E_B + m/2$. As shown in reference 3, $m/2$ is the total energy of the vacuum Kaluza-Klein monopole. For $Z = 1$ and $Z = 0$, the boundary energy is independent of the radius, but as $R \rightarrow \infty$, the "predicted" masses differ by a factor of 2. Of particular interest is the second entry in Table I, in which $m \rightarrow \infty$. In this limit the interior region is flat, so that the physical monopole consists only of the static boundary layer and the exterior stress energy $\bar{T}_{55}$. Finally, we note that by choosing $Z = -\alpha$ and $(\alpha - 1/2)$ suitably small, it is possible to construct a monopole with arbitrary large mass, M , and radius R .

We have constructed a simple model for a Kaluza-Klein monopole with confined scalar field. By requiring the geometry to be well-defined across a thin boundary layer dividing the region with long range scalar field from the region with constant scalar field, we have derived constraints on the effective stress energy of the boundary layer. Further work must be done in order to discover whether such models are applicable to higher dimensional non-Abelian monopoles.¹⁰

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TABLE I

Description	α	β	σ	m	M	R	E_B
$Z = 1$ Solution	0	1	Undetermined $0 < \sigma < 1/(16\pi^2\lambda_0^2 G)$ $1/(16\pi^2\lambda_0^2 G)$	$\lambda_0[1 - (16\pi^2 G\lambda_0 \sigma)^{1/2}]^{1/2}$	$\lambda_0/4$	$1/(8\pi(G\sigma)^{1/2})$	$\lambda_0/8$
Limiting Case of Above: $X = \frac{1}{2}$	0	1		∞	$\lambda_0/4$	$\lambda_0/2$	$\lambda_0/8$
Limiting case of Above: $R \rightarrow \infty$	0	1	0	$\lambda_0/4$	$\lambda_0/4$	∞	$\lambda_0/8$
$Z = 0$ Solution Above: $\alpha < 0$	α	α	$\frac{3}{2}\alpha^2/(2\pi^2)\lambda_0^2(1+4\alpha)^2G$	$(1+4\alpha)^{\frac{1}{2}}\lambda_0/4$	$(4+10\alpha)\lambda_0/8(1+4\alpha)$	$(1+4\alpha)\lambda_0/8\alpha$	$3\lambda_0/8$
Limiting Case of Above: $\alpha \rightarrow 0$	0	0	0	$\lambda_0/4$	$\lambda_0/2$	∞	$3\lambda_0/8$
Dust type solution Above: Radiation Type Solution	$1/3$	$1/3$	$3/(196\pi^2 G\lambda_0^2)$	$(7/48)^{\frac{1}{2}}\lambda_0$	$11\lambda_0/28$	$7\lambda_0/8$	$3\lambda_0/8$
Specific case of Above: Ultrarelativistic	1	1	$3/(100\pi^2 G\lambda_0^2)$	$(5/16)^{\frac{1}{2}}\lambda_0$	$7\lambda_0/20$	$5\lambda_0/8$	$3\lambda_0/8$
$Z = -\alpha$	α	0	$\frac{3\alpha^2(1-2\alpha)}{(\pi^2 G\lambda_0^2)((1-\alpha^2)+2\alpha(2-\alpha))}$	$\frac{\lambda_0}{2} \frac{(1+5\alpha^2)}{(1+4\alpha-3\alpha^2)}$	$\frac{\lambda_0}{2} \left\{ \frac{3\alpha^2-5\alpha^2+2\alpha+1}{(1-2\alpha)(1+4\alpha-3\alpha^2)} \right\}$	$\frac{(1+4\alpha-3\alpha^2)\lambda_0}{8\alpha(1-2\alpha)}$	$3\lambda_0/8(1-2\alpha)$
Limiting Case of Above: $\alpha \rightarrow \frac{1}{2}$	$\frac{1}{2}$	0	0	$\lambda_0/4$	∞	∞	∞

Properties of several interesting solutions with different values of the parameter $\alpha = \sigma/p$ $\beta = \sigma/q$ are illustrated.

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Figure Caption

The allowed regions of the surface equation-of-state parameter space are shown. Diagonal lines denote regions of positive mass M , radius R and energy density σ . Solutions in the dotted regions yield positive M and R , but negative σ .