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SPONTANEOUS COMPACTIFICATION AND THE HIGHER
DIMENSIONAL VACUUM EINSTEIN EQUATIONS*

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ABSTRACT

The vacuum Einstein equations in $(4 + m + n)$ dimensions are displayed for a class of metrics obtained by the Kaluza-Klein geometrization of the $(4 + m)$ dimensional Einstein-Maxwell equations. Consistency problems are solved by requiring that the metric be one of two types. In both cases the field equations split into space-time and internal field equations, and the full metric includes components which can be interpreted as internal, mutually orthogonal electromagnetic fields. We show that spontaneous compactification in vacuum occurs only if observable space-time has a huge cosmological constant, or if the internal spaces have no global isometries, or if the number of internal dimensions is very large.

INTRODUCTION

There has been considerable effort recently to construct a self-consistent field theory which unifies the four fundamental interactions and correctly predicts the spectrum of known elementary particles. The thread which links together the most promising candidates - *e.g.*, superstrings in ten dimensions¹, supergravity in eleven dimensions², quasi-Riemannian "Kaluza-Klein" theory³, etc. is their common setting in a spacetime M^{4+N} with dimension $(4 + N)$, $N \geq 1$. In order to account for the non-observability, at least in our epoch, of the extra N dimensions, it is assumed that the structure of M^{4+N} is locally, a product of ordinary four dimensional space-time M^4 and a compact (or at least finite-volume)⁴ "internal space" B^N , whose mean radius of curvature is approximately the Planck length.

The simplest class of such theories endows M^{4+N} with a Lorentzian signature pseudo-Riemannian metric $\hat{g}$, whose dynamics are governed by the Einstein-Hilbert functional,

$$I[\hat{g}, \hat{\phi}] = \int_{M^{4+N}} d^{4+N}x \sqrt{-\hat{g}} [\kappa^{-1} (\hat{R}(\hat{g}) + 2\Lambda) + \hat{L}(\hat{\phi})] \quad (1.1)$$

where $\hat{g}$ is the determinant of $\hat{g}$, $\hat{R}(\hat{g})$ is the curvature scalar of $\hat{g}$, $\hat{L}(\hat{\phi})$ is the matter Lagrangian density constructed from field(s) $\hat{\phi}$ which carry a representation of $O(3 + N, 1)$ and which are minimally coupled to $\hat{g}$. Λ is the $(4 + N)$ -dimensional cosmological constant, and κ is the $(4 + N)$ -dimensional gravitational constant. The classical equations of motion are thus:

$$\hat{R}_{AB} - \frac{1}{2} \hat{R} \hat{g}_{AB} - \Lambda \hat{g}_{AB} = -\kappa \frac{\delta}{\delta \hat{g}_{AB}} (\sqrt{-\hat{g}} \hat{L}(\hat{\phi})) \quad (1.2)$$

$$\frac{\delta}{\delta \hat{\phi}} \hat{L}(\hat{\phi}) = 0 \quad (1.3)$$

where $A, B, \dots, \epsilon(1, 2, \dots, 4 + N)$ denote tensor indices on M^{4+N} . $\hat{R}_{AB}$ are the components of the Ricci tensor of $\hat{g}$ and $\frac{\delta}{\delta \hat{\phi}}$ denote the usual functional derivatives.

The viability of such theories at the classical level requires the following, usually dubbed spontaneous compactification:

There must exist solutions $^{(0)}\hat{g}$, $^{(0)}\hat{\phi}$ of equations (1.2) and (1.3) such that $^{(0)}\hat{g}_i, i, j, \dots \in \{5, 6, \dots, 4 + N\}$ are, at least up to an M^4 -dependent scale factor, the components of a Riemannian metric on a compact manifold B^N . Furthermore, the isometry group G of $(B^N, ^{(0)}\hat{g}_i)$ must contain $SU(3) \times SU(2) \times U(1)$ as a subgroup.

If such solutions $\{^{(0)}\hat{g}, ^{(0)}\hat{\phi}\}$ existed, and satisfied the further requirements of stability, an acceptably small prediction for the value of the cosmological constant and agreement with the phenomenology of elementary fermions, then they would be ideal candidates for the ground state of a successful unified field theory. Such solutions have been found – e.g., for the case² of $N = 7$ and for $N = 2$, but with a gauge group too small to achieve “grand unification”.⁵ However, these solutions violate the spirit of the original Kaluza-Klein program, in that the fields $\hat{\phi}$ are themselves external gauge fields, and not of geometrical origin. Compactification has also been achieved for the case where $\hat{L}(\hat{\phi})$ is the Lagrangian density of a non-linear sigma model.⁶ In this case the fields $\hat{\phi}$ are not gauge fields, but their physical significance is obscure.

In fact, it is quite commonly held that vacuum spontaneous compactification satisfying all the above requirements does not occur in pure Kaluza-Klein theory, and that one must of necessity include non-trivial, non-geometric gauge, or other matter field configurations in the ground state.⁷ As a result, little effort has been expended in examining the vacuum Einstein equations (1.2) (with $\hat{A}_B \equiv 0$), except for the non-physical case of $N = 1$ (i.e., the original Kaluza-Klein theory).

The purpose of this paper is to begin the process of filling in this gap in the understanding of higher dimensional gravity theory by investigating solutions of the vacuum $(4 + N)$ -dimensional Einstein equations with $N \geq 2$. We are motivated in particular by the solutions of Randjbar-Daemi, *et al.*⁸ in which spontaneous compactification to $M^4 \times S^2$ is achieved in six dimensional Einstein-Maxwell theory by allowing a non-zero magnetic monopole field on the internal two sphere. In this solution, the six-dimensional cosmological constant is fine-tuned so that it precisely cancels the non-zero trace of the Maxwell stress-energy tensor (which is only traceless in four dimensions). Consequently, the effective four-dimensional cosmological constant vanishes.

In the following we will begin a systematic analysis of the vacuum Einstein equations on $M^4 \times B^N$, where $B^N = B^n \times T^m$, in order to discover whether solutions analogous to those of Randjbar-Daemi *et al.* exist, but with the internal magnetic field generated by the standard Kaluza-Klein mechanism from the metric components associated with the extra torus $T^m = S^1 \times S^1 \times \dots$. The possible existence of such solutions was also noted in a slightly different context by Salam and Szegin.⁹ Unfortunately, our conclusions will be somewhat negative. As we shall see, cancellation of the effective cosmological constant using “geometrical” magnetic fields does not seem possible, no matter what the dimensionality of the torus, T^m , unless ridiculously large values of m are allowed.

The paper is organized as follows: In Section 2 we derive the vacuum Einstein equations for a dimensionally reduced metric $\hat{G}_{AB}$ on $M^4 \times B^n \times T^m$, whose pro-

jected components describe ordinary gravity and a set of scalar fields on M^4 as well as a set of magnetic fields F_{ij} on B^n . We also show that consistency conditions of the type discussed by Duff *et al.*⁹ reduce the field equations to two possible forms. The resulting equations split cleanly into two sets of equations: one on M^4 , and the other on B^n , the two sets being linked only by some constants related to the curvature of B^n . In Section 3 we discuss general properties of the field equations and present some particular solutions. We show that if the internal dimensions are compactified to Planck scales everywhere on M^4 (except possibly on some finite domains), then the effective cosmological constant on M^4 can only be made small by the somewhat unphysical expedient of introducing a huge number of internal dimensions. The final section closes with some conclusions and prospects for future research.

2. THE FIELD EQUATIONS

In this section, we shall display the field equations which result when the metric $\hat{g}$ is restricted to the following form:

$$\hat{g} = g_{\mu\nu} dx^\mu dx^\nu + \lambda^2 \gamma_{ij} dy^i dy^j + \sum_{a=1}^m w_a^2 [A_a^i dy^i + dx^a]^2, \quad (2.1)$$

The indices $\mu, \nu, \dots \in \{1, 2, 3, 4\}$, $i, j, \dots \in \{5, 6, \dots, 4 + n\}$, and $a, b, \dots \in \{5 + n, 6 + n, \dots, 4 + n + m\}$ with $n + m = N$. The coordinates $\hat{x}^A$ on a chart of M^{4+N} are identified by $\hat{x}^\mu = x^\mu$, $\hat{x}^i = y^i$, $\hat{x}^a = x^a$. We also impose the following coordinate dependence on the various fields that appear in Eq. (2.1):

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}(x), \\ \lambda &= \lambda(x), \\ \gamma_{ij} &= \gamma_{ij}(y), \\ w_a &= w_a(x, y), \\ A_a^i &= A_a^i(y). \end{aligned} \quad (2.2)$$

Our motivation for this ansatz is provided by the solutions discovered in reference⁵ of the six-dimensional Einstein-Maxwell equations. These solutions contain a six-dimensional metric, which in our notation is:

$$^{(5)}\hat{g} = g_{\mu\nu}(x) dx^\mu dx^\nu + \gamma_{ij}(y) dy^i dy^j,$$

with $g_{\mu\nu}$ a metric with constant curvature, and γ_{ij} the standard metric on S^2 . As well, there is a magnetic monopole field $A_i(y)$ over S^2 . It seems natural⁸ to “geometrize” the magnetic-monopole field by incorporating it into a metric $^{(7)}\hat{g}$ on M^7 :

$$^{(7)}\hat{g} = ^{(5)}\hat{g} + (A_i dy^i + dx)^2.$$

This metric is a solution to the seven-dimensional vacuum Einstein equations, as long as observable spacetime $(M^4, g_{\mu\nu})$ has a positive cosmological constant, and

appropriate relations among various constants are specified. It seems possible, and it will be born out in the sequel, that the generalizations (2.1) may include other spontaneously compactified vacuum solutions.

We compute the Ricci tensor $\hat{R}_{AB}$ of the metric (2.1) in the horizontal lift basis $\hat{\theta}^A$: (underlined indices denote components in the horizontal lift basis):

$$\begin{aligned}\hat{\theta}^\mu &= dx^\mu, \\ \hat{\theta}^i &= dy^i, \\ \hat{\theta}^a &= A_i^a dy^i + dx^a.\end{aligned}\quad (2.3)$$

In the Appendix, the dual basis, the rotation coefficients and our conventions on the Riemann and Ricci tensors, etc., are displayed. The result is:

$$\hat{R}_{\mu\nu} = R_{\mu\nu} - n\lambda^{-1} \lambda_{[\mu\nu]} - \sum_{a=1}^m w_a^{-1} w_a |_{\mu\nu}, \quad (2.4a)$$

$$\hat{R}_{\mu i} = \sum_{a=1}^m [-w_a^{-1} (w_a |_{\mu i} - \lambda^{-1} \lambda_{[\mu} w_a |_{i})], \quad (2.4b)$$

$$\hat{R}_{\mu a} = 0, \quad (2.4c)$$

$$\begin{aligned}\hat{R}_{ij} &= \hat{R}_{ij} - \gamma_{ij} [\lambda \nabla^2 \lambda + (n-1) |\nabla \lambda|^2] \\ &\quad + \sum_{a=1}^m [-w_a^{-1} (w_a |_{ij} + \lambda \lambda^{[\mu} w_a |_{\mu} \gamma_{ij}) \\ &\quad + \frac{1}{2} \lambda^{-2} w_a^2 F_a^a{}_{i k} F_a^a{}_{k j}],\end{aligned}\quad (2.4d)$$

$$\hat{R}_{ia} = \frac{1}{2} \lambda^{-2} [(w_a^2 F_a^a{}_{i j})_{|j} + \left(\sum_{b=1}^m \frac{w_b |_{j}}{w_b} \right) w_a^2 F_a^a{}_{i j}], \quad (2.4e)$$

$$\begin{aligned}\hat{R}_{ab} &= \delta_{ab} \{ -w_a [\nabla^2 w_a + \lambda^{-2} \tilde{\nabla}^2 w_a \\ &\quad + (n\lambda^{-1} \nabla \lambda + \sum_{c=1}^m w_c^{-1} \nabla w_c) \cdot \nabla w_a \\ &\quad + \sum_{c=1}^m w_c^{-1} \lambda^{-2} \tilde{\nabla} w_c \cdot \tilde{\nabla} w_a] + |\nabla w_a|^2 \\ &\quad + \lambda^{-2} |\tilde{\nabla} w_a|^2 \} + \frac{1}{4} \lambda^{-4} w_a^2 w_b^2 F_a^a{}_{i j} F_b^b{}_{i j}.\end{aligned}\quad (2.4f)$$

Note that the summation convention is suspended for the indices a, b, c , and that covariant derivatives and raising and lowering of indices on the right-hand-sides are with respect to the coordinate bases dx^μ and dy^i on M^4 and B^n , respectively.

We now impose the vacuum field equations with cosmological constant:

$$\hat{R}_{AB} = \Lambda \hat{\theta}_{AB}. \quad (2.5)$$

At this stage consistency problems become apparent.⁹ If we use (2.4e), the field equations $\hat{R}_{ia} = \Lambda \hat{\theta}_{ia} = 0$ can be written in the form (since $w_a \neq 0$):

$$F_a^a{}_{i j} (y) Z_{a,j}(x, y) + Q^a{}_i(y) = 0, \quad (2.6)$$

where

$$Z_a(x, y) := \log |w_a^2(x, y) \prod_{b \neq a} w_b(x, y)|,$$

$$Q^a{}_i(y) := F_a^a{}_{i j}(y).$$

If we differentiate (2.6) with respect to x^μ , then for each μ and each a there are n homogeneous linear equations $F_a^a{}_{i j} Z_{a|\mu|j} = 0$ for the n $Z_{a|\mu|j}$'s. Hence, except for the special case where $\det(F_a^a{}_{i j}) = 0$, we must have $Z_{a|\mu|j} = 0$. It follows, after some algebra, that $w_a(x, y)$ must be of the form:

$$w_a(x, y) = \lambda_a(x) \beta_a(y), \quad (2.7)$$

with the λ_a and β_a being as yet arbitrary functions which are sufficiently smooth.

The equation $\hat{R}_{\mu i} = \Lambda \hat{\theta}_{\mu i} = 0$, however, yields:

$$\sum_{a=1}^m [\lambda_{a|\mu} \beta_{a|i} - \lambda_{a,\mu} \beta_{a|i}] = 0, \quad (2.8)$$

which implies for each a that, if $\beta_{a|i} \neq 0$, then

$$\lambda_a(x) = \lambda(x) + \text{constant}. \quad (2.9)$$

Finally, the remaining equations are consistent only if this constant is equal to zero.

We are left with only two possible forms for $w_a(x, y)$:

$$\text{A. } w_a(x, y) = \lambda(x) \beta_a(y), \quad (2.10)$$

$$\text{B. } w_a(x, y) = \bar{w}_a(x). \quad (2.11)$$

With restriction (A) in place, the field equations (2.5) reduce to

$$R_{\mu\nu} - (n+m)\lambda^{-1} \lambda_{[\mu\nu]} - \Lambda g_{\mu\nu} = 0. \quad (2.12a)$$

$$\hat{R}_{ij} + \sum_{a=1}^m \left[-\beta_a^{-1} \beta_{a|ij} + \frac{1}{2} \beta_a^2 F_a^a{}_{i k} F_a^a{}_{k j} \right] - M \gamma_{ij} = 0, \quad (2.12b)$$

$$\lambda \nabla^2 \lambda + (n+m-1) |\nabla \lambda|^2 + \Lambda \lambda^2 - M = 0, \quad (2.12c)$$

$$(\beta_a^2 F_a^a{}_{i j})_{|j} + \left(\sum_{b=1}^m \beta_b^{-1} \beta_{b|j} \right) \beta_a^2 F_a^a{}_{i j} = 0, \quad (2.12d)$$

$$\beta_a \tilde{\nabla}^2 \beta_a + \sum_{b \neq a} \beta_a \beta_b^{-1} \tilde{\nabla} \beta_a \cdot \tilde{\nabla} \beta_b + M \beta_a^2 - \frac{1}{4} \beta_a^4 F_a^a{}_{i j} F_a^a{}_{i j} = 0, \quad (2.12e)$$

$$F_a^a{}_{i j} F_b^b{}_{i j} = 0 \quad \text{if } a \neq b. \quad (2.12f)$$

The constant M appears because our restriction (A) amounts to a separation of variables in the components (2.4d) and (2.4f) of R_{AB} , and the corresponding field equations are then satisfied by Eqs. (2.12b,c,e,f).

The field equations (2.12) can be re-written in terms of tensors on the full internal space B^{n+m} with metric:

$$\bar{\gamma} = \bar{\gamma}_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = \gamma_{ij} dy^i dy^j + \sum_{a=1}^m \beta_a^2 (A_i^a dy^i + dx^a)^2, \quad (2.13)$$

where $\alpha, \beta \in \{5, 6, \dots, 4+n, 5+n, \dots, 4+n+m\}$, and the coordinates $[y^i, x^a]$. The components with respect to the horizontal lift basis of $\bar{\gamma}$, $\bar{R}_{\alpha\beta}$ are then:

$$\begin{aligned} \bar{R}_{ij} &= \bar{R}_{ij} + \sum_a [-\beta_a^{-1} \beta_{ij} + \frac{1}{2} \beta_a^2 F^a{}_i F^a{}_j], \\ \bar{R}_{ia} &= \frac{1}{2} \left[(\beta_a^2 F^a{}_i)_{,j} + \left(\sum_{b=1}^m \beta_b^{-1} \beta_{bij} \right) \beta_a^2 F^a{}_j \right], \\ \bar{R}_{ab} &= \delta_{ab} \left[-\beta_a \tilde{\nabla}^2 \beta_a - \sum_{c \neq a} \beta_a \beta_c^{-1} \tilde{\nabla} \beta_c - \tilde{\nabla} \beta_a \right] \\ &\quad + \frac{1}{4} \beta_a^2 \beta_b^2 F^a{}_{ij} F^b{}_{ij}. \end{aligned} \quad (2.14)$$

Hence, Eqs. (2.12) are equivalent to:

$$\begin{aligned} R_{\mu\nu} - (n+m)\lambda^{-1} \lambda_{[\mu\nu]} - \Lambda g_{\mu\nu} &= 0 & (2.14a) \\ \lambda \nabla^2 \lambda + (n+m-1) |\nabla \lambda|^2 + \Lambda \lambda^2 - \epsilon(n+m-1) &= 0, & (2.14b) \\ \bar{R}_{\alpha\beta} = \epsilon(n+m-1) \bar{\gamma}_{\alpha\beta}, & & (2.14c) \end{aligned}$$

where $\epsilon = 0, +1, -1$. The constant M has been normalized so that (2.14c) is identically satisfied if $\bar{\gamma}_{\alpha\beta}$ is the metric on the unit $(n+m)$ -sphere. That M can be so normalized without loss of generality follows from the fact that λ and $\bar{\gamma}_{\alpha\beta}$ appear in the full metric $\hat{g}$ only in the combination $\lambda^2 \bar{\gamma}_{\alpha\beta}$. Restriction (B), on the other hand, yields:

$$\begin{aligned} R_{\mu\nu} - n\lambda^{-1} \lambda_{[\mu\nu]} - \sum_{a=1}^m \bar{w}_a^{-1} \bar{w}_{a[\mu\nu]} - \Lambda g_{\mu\nu} &= 0, & (2.15a) \\ F^a{}_i F^a{}_{kj} &= -2\epsilon^{\delta} \gamma_{ij}, & (2.15b) \\ \bar{R}_{ij} &= \epsilon(n-1) \gamma_{ij}, & (2.15c) \\ \lambda \nabla^2 \lambda + (n-1) |\nabla \lambda|^2 + \Lambda \lambda^2 + \sum_{a=1}^m [\lambda \bar{w}_a^{-1} \nabla \lambda \cdot \nabla \bar{w}_a \\ &\quad + \epsilon^{\delta} \lambda^{-2} w_a^2] - \epsilon(n-1) = 0, & (2.15d) \end{aligned}$$

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$$\begin{aligned} F^a{}_{ij} &= 0, & (2.15e) \\ F^a{}_{ij} F^b{}_{ij} &= 0 \text{ if } a \neq b, & (2.15f) \\ \bar{w}_a \nabla^2 \bar{w}_a + n\lambda^{-1} \bar{w}_a \nabla \lambda \cdot \nabla \bar{w}_a + \sum_{a \neq b} \bar{w}_b^{-1} \bar{w}_a \nabla \bar{w}_a \cdot \nabla \bar{w}_b \\ &\quad - \frac{n\epsilon^{\delta}}{2} \lambda^{-4} \bar{w}_a^4 + \Lambda \bar{w}_a^2 = 0. & (2.15g) \end{aligned}$$

In the above, the constant ϵ^{δ} arose from the separation of variables in Eqs. (2.4d,f).

3. DISCUSSION OF SOLUTIONS

First note that both forms of the field equations, (2.12) and (2.15) contain the strong restriction:

$$F^a{}_{ij} F^b{}_{ij} = 0, \quad a \neq b. \quad (3.1)$$

If we consider the special case $n = 2$, with γ_{ij} the metric for a unit 2-sphere S^2 and the m fields $F^a{}_{ij}$ being those of magnetic monopoles on S^2 , then equations (3.1) imply that only one magnetic charge can be non-zero.¹⁰ That is, if $\gamma = d\theta^2 + \sin^2 \theta d\phi^2$ and $A_i^a dy^i = k^a (\cos \theta \pm 1) d\phi$, then it follows from (3.1) that:

$$0 = F^a{}_{ij} F^b{}_{ij} = 2k^a k^b, \quad a \neq b. \quad (3.2)$$

Somewhat more generally, let γ_{ij} be the metric on S^n expressed in euler-angle type coordinates θ^i :

$$\begin{aligned} \gamma &= \gamma_{ij} d\theta^i d\theta^j \\ &= (d\theta^1)^2 + \sin^2 \theta^1 [(d\theta^2)^2 + \sin^2 \theta^2 [(d\theta^3)^2 + \dots]], \end{aligned} \quad (3.3)$$

and let the vector potential A_i^a be given by:

$$A_i^a d\theta^i = k^a (\cos \theta^1 \pm 1) d\theta^n. \quad (3.4)$$

Then,

$$0 = F^a{}_{ij} F^b{}_{ij} = \frac{2k^a k^b \sin^2(\theta^1)}{\prod_{j=1}^{n-1} [\sin^2(\theta^j)]^{n-j}}, \quad (3.5)$$

so that again all but one magnetic charge k^a must be zero.

We now discuss the two sets of equations (2.12) (or, equivalently, (2.14)) and (2.15), separately, starting with the former:

1. First, note that there are solutions of (2.14) in which four-dimensional space-time (M^4 , $g_{\mu\nu}$) and the internal space (B^{n+m} , $\bar{\gamma}_{\alpha\beta}$) have maximal symmetry. These solutions are characterized by:

$$\begin{aligned} \lambda &= \text{constant}, & (3.6a) \\ \Lambda \lambda^2 &= n+m-1, & (3.6b) \\ (M^4, g_{\mu\nu}) &= (\text{anti-}) \text{ deSitter space-time}, \\ (S^{n+m}, \bar{\gamma}_{\alpha\beta}) &= \text{standard } (n+m)\text{-sphere, with unit radius}. \end{aligned}$$

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The physical size of the internal space S^{n+m} is given by λ , and hence $\Lambda\lambda^2 = n + m - 1$ results in a huge cosmological constant for $(M^4, g_{\mu\nu})$ under the usual assumption that $\lambda \sim$ Planck length.

The field equations (2.14) with positive internal cosmological constant ($\epsilon = +1$) do not allow solutions in which $\lambda \rightarrow$ constant in a region of M^4 in which $g_{\mu\nu} \rightarrow$ flat metric. To see this, first note that in a region of M^4 where $R_{\mu\nu} \rightarrow 0$, equations (2.14a) imply that $(\lambda_{\mu\nu} - N^{-1}\lambda g_{\mu\nu}) \rightarrow 0$ which can only be satisfied if $\Lambda = 0$. Then, (2.14b) forces $|\nabla\lambda|^2 \rightarrow n + m - 1 > 0$ in this region. So λ cannot settle down to a constant in the flatter parts of $(M^4, g_{\mu\nu})$.

By similar reasoning, it follows that $g_{\mu\nu}$ can only be flat if $\Lambda = \epsilon = 0$ and $\lambda =$ constant. In this case, $\bar{\gamma}_{\alpha\beta}$ is Ricci-flat. If the internal space B^{n+m} has dimension $N = n + m \leq 3$, then it must be flat, and hence no non-trivial solutions exist for $N \leq 3$ with $g_{\mu\nu}$ flat. However, with $N = 4$, one may take B^4 to be the so-called $K3$ manifold.¹¹ The $K3$ manifold is Ricci-flat and compact without boundary, but to date an explicit metric for $K3$ has not been discovered. Although $K3$ has no global isometries, it does admit local isometries.

2. We now discuss some important facts about the solutions of (2.15). The metric on the internal space B^{n+m} cannot, in general, have its x^μ -dependence factored out. Rather, one may split the space into a smaller space B^n with metric γ_{ij} , whose size is determined by $\lambda(x)$, and m circles $S^1_{(a)}$, whose sizes are determined by the $\bar{w}_a(x)$. Effectively, γ_{ij} satisfies the (Riemannian) cosmological Einstein equations (2.15), and it admits M mutually orthogonal internal electromagnetic fields F^a_{ij} which satisfy the interesting relation:

$$\tilde{T}^a_{ij} := F^a_k F^a_{ij} + \frac{1}{n} \gamma_{ij} F^a_{kl} F^a_{kl} = 0, \quad (3.7)$$

on account of (2.15b). Only for $n = 4$ is $\tilde{T}^a_{ij}$ a stress-energy-momentum tensor, and hence only in that case can we say that the F^a_{ij} are zero-energy fields.

The field equations (2.15), in general, admit solutions in which the complete metric $\hat{g}$ does not split cleanly into a metric $g_{\mu\nu}(x)$ on observable spacetime M^4 and a metric $\bar{\gamma}_{\alpha\beta}(y)$ on the internal space B^N . Instead, there is the possibility that the internal space B^N changes shape as well as overall size over M^4 . This is because the internal metric $\bar{\gamma}$ given by

$$\bar{\gamma} = \lambda^2(x) \gamma_{ij}(y) dy^i dy^j + \sum_{a=1}^m \bar{w}_a^2(x) [A_a^c(y) dy^c + dx^a]^2, \quad (3.8)$$

contains the $(m+1)$ distinct M^4 scalar fields $\lambda(x)$ and $\bar{w}_a(x)$. The metric $\gamma_{ij}(y)$ satisfied the n -dimensional cosmological Einstein equations (2.15c), while the electromagnetic fields F^a_{ij} satisfy (2.15b), (2.15e) and (2.15f) and in the background of γ_{ij} . The behavior of the M^4 metric $g_{\mu\nu}(x)$ and the scalar fields λ , $\bar{w}_a$ is determined by (2.15a), (2.15d) and (2.15g). It is easy to see that if λ and the $\bar{w}_a$ are constant, then the M^4 -cosmological constant Λ must satisfy:

$$\Lambda = \frac{\epsilon n(n-1)}{\lambda^2(n+2m)}, \quad (3.9)$$

and hence Λ is large if $\lambda \sim$ Planck, unless m is huge!

A solution $\{\gamma_{ij}, A_i^a, C^a\}$ of (2.15b), (2.15c), (2.15e) and (2.15f), in which $n = 4$ and $m = 2$ is the following:

$$\gamma = \ell^{-1} [(dy^5)^2 + \sin^2 y^5 (dy^6)^2 + (dy^7)^2 + \sin^2 y^7 (dy^8)^2] \quad (3.10a)$$

$$A_i^9 dy^i = k^9 (\cos y^5 dy^6 - \cos y^7 dy^8), \quad (3.10b)$$

$$A_i^{10} dy^i = k^{10} (\cos y^5 dy^6 + \cos y^7 dy^8), \quad (3.10c)$$

$$C^a = \frac{1}{2} \ell^2 (k^a)^2; \quad a = 9, 10, \quad (3.10d)$$

where the k^a and ℓ are constants. (We could normalize ℓ to one by absorbing it into λ .) The metric γ is the standard metric on $S^2 \times S^2$.¹² The electromagnetic fields, given by the potentials A_i^a , are mutually orthogonal on $S^2 \times S^2$, and may be interpreted as magnetic monopole fields centered at the origins of the two spheres. Furthermore, F^9_{ij} is anti-self-dual, while F^{10}_{ij} is self-dual, and the stress-energy-momentum tensors:

$$T^a_{ij} = F^a_k F^a_{ij} + \frac{1}{4} \gamma_{ij} F^a_{kl} F^a_{kl}$$

are both zero. The properties of this solution will be discussed in more detail elsewhere.¹³

4. CONCLUSIONS

We have examined in some detail the properties of $(4+N)$ -dimensional metrics which decompose, locally, into product metrics on $M^4 \times B^n \times T^m$, with B^n compact, and which are exact solutions of the $(4+n+m)$ -dimensional vacuum Einstein equations, with, in general, a non-zero cosmological constant. The set of such metrics bifurcates into two classes. In one class, the internal space $B^N = B^n \times T^m$ maintains its shape as it is moved around M^4 , while for the other class $B^n \times T^m$ changes shape since B^n and each of the circles comprising T^m change their relative size as the internal space is moved over M^4 .

Associated with each circle $S^1_{(a)}$ in T^m is an internal electromagnetic field $F^a_{ij}(y)$. If the components F^a_{ij} all vanish for a given a , then $S^1_{(a)}$ is a trivial bundle over B^n . For the case where F^a_{ij} are magnetic monopole fields, then of necessity all the $S^1_{(a)}$'s but one are trivial bundles over S^n .

To achieve spontaneous compactification in classical Kaluza-Klein theories, it seems that one must introduce undesirable features. If we demand that the internal space $(B^N, \bar{\gamma}_{\alpha\beta})$ and observable space-time $(M^4, g_{\mu\nu})$ have maximal symmetry, then the latter acquires an unacceptably large cosmological constant. On the other hand, if one forces $(M^4, g_{\mu\nu})$ to be Ricci-flat, then $(B^N, \bar{\gamma}_{\alpha\beta})$ is not symmetrical enough to account for the non-gravitational interactions via the isometries of internal space.¹⁴

Our analysis assumed the existence of charts on M^{4+N} in which the components $\hat{g}_{ij}(x, y)$ of the full metric were products of an M^4 -dependence scale factor $\lambda^2(x)$ and an internal metric $\gamma_{ij}(y)$. We conjecture that dropping this assumption would not affect the conclusions reached here for the less general case. It remains to test this conjecture.

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APPENDIX

The horizontal lift basis with respect to the metric (2.1) is $\hat{\theta}^A$, with

$$\begin{aligned}\hat{\theta}^u &= dx^u, \\ \hat{\theta}^i &= dy^i, \\ \hat{\theta}^a &= A_i^a dy^i + dz^a,\end{aligned}\tag{A.1}$$

and the corresponding dual basis $\hat{w}_A$ is:

$$\begin{aligned}\hat{w}_u &= \partial_u, \\ \hat{w}_i &= \partial_i - A_i^a \partial_a, \\ \hat{w}_a &= \partial_a,\end{aligned}\tag{A.2}$$

with commutators $[\hat{w}_A, \hat{w}_B] = C_{AB}^D \hat{w}_D$ given by

$$\begin{aligned}C_{ij}^a &= -C_{ij}^a = -F_{ij}^a, \\ C_{AB}^D &= 0, \text{ otherwise,}\end{aligned}\tag{A.3}$$

where

$$F_{ij}^a = A_{j,i}^a - A_{i,j}^a.$$

The Ricci rotation coefficients w_{AB}^D with respect to θ^A are defined by:

$$w_{AB}^D = \frac{1}{2} \theta^D \left[\hat{w}_A \hat{\theta}_B + \hat{w}_B \hat{\theta}_A - \hat{w}_E \hat{\theta}_A \hat{\theta}_B - C_{EAB}^D - C_{BAE}^D - C_{ABE}^D \right],\tag{A.4}$$

where the $\hat{\theta}_A$ are the (anholonomic) components of $\hat{\theta}$ with respect to $\hat{\theta}^A$.

$$[\hat{\theta}_A \hat{\theta}_B] = \begin{bmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & \lambda^2 \gamma_{ij} & 0 \\ 0 & 0 & w_a^2 \delta_{ab} \end{bmatrix},$$

and $C_{EAB} = \hat{\theta}_E C_{AB}^D$. The non-vanishing components of w_{AB}^D are:

$$\begin{aligned}w_{\mu\nu}^{\tau} &= \Gamma_{\mu\nu}^{\tau} \equiv \frac{1}{2} \theta^{\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}), \\ w_{ij}^{\tau} &= -\lambda \lambda^{\tau} \gamma_{ij}, \\ w_{ab}^{\tau} &= -w_b w_a^{\tau} \delta_{ab}, \\ w_{j\mu}^i &= w_{j\mu}^i = \delta_j^i \lambda^{-1} \lambda_{,\mu}, \\ w_{jk}^i &= \tilde{\Gamma}^i_{jk} \equiv \frac{1}{2} \gamma^{i\ell} (\gamma_{j\ell,k} + \gamma_{k\ell,j} - \gamma_{jk,\ell}), \\ w_{ij}^k &= w_{ij}^k = \frac{1}{2} \lambda^{-2} w_b^2 F^{b\,i}_j, \\ w_{ab}^i &= w_{ba}^i = -\lambda^{-1} w_b w_a^i \delta_{ab}, \\ w_{\mu b}^a &= w_{b\mu}^a = w_b w_a^{-2} w_{b,\mu} \delta_{ab}, \\ w_{ij}^a &= -w_{ji}^a = \frac{1}{2} F^a_{ij}, \\ w_{ba}^i &= w_{ab}^i = w_b w_a^{-2} w_{b,i} \delta_{ab},\end{aligned}\tag{A.5}$$

where on the right hand sides, indices are raised and lowered with respect to $g_{\mu\nu}$ or γ_{ij} , as appropriate. Note that the summation convention is suspended for the "torus" indices $a, b, \dots, h$. The components $\hat{R}_{AB}$ of the Ricci tensor in the $\hat{\theta}^A$ basis are computed from:

$$\begin{aligned} \hat{R}_{AB} = & \hat{w}_D w^D_{AB} - \hat{w}_B w^D_{AD} + W^D_{CD} w^C_{AB} \\ & - w^D_{CB} w^C_{AD} - w^D_{AE} C^E_{DB} . \end{aligned} \quad (\text{A.6})$$

The expressions (2.4) follow. In those equations, $|\mu| (|i|)$ denote the covariant derivative with respect to the Christoffel symbols $\Gamma^\pi_{\mu\nu} (\tilde{\Gamma}^k_{ij})$. Similarly, $R_{\mu\nu} (\tilde{R}_{ij})$ is the Ricci tensor with respect to $g_{\mu\nu} (\gamma_{ij})$. Finally, we use the notation:

$$\begin{aligned} \nabla^2 \lambda &:= g^{\mu\nu} \lambda_{|\mu\nu} , \\ \tilde{\nabla}^2 w_a &:= \gamma^{ij} w_{a|ij} , \\ |\nabla \lambda|^2 &:= g^{\mu\nu} \lambda_{,\mu} \lambda_{,\nu} , \\ \nabla \lambda \cdot \nabla w_a &:= g^{\mu\nu} \lambda_{,\mu} w_{a|\nu} , \\ &\text{etc.} \end{aligned}$$