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OF KALUZA-KLEIN THEORIES I: $D = 1 + 4$

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THE INFINITE-DIMENSIONAL GAUGE STRUCTURE
OF KALUZA-KLEIN THEORIES I: $D = 1 + 4$ *

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ABSTRACT

We show that the untruncated four-dimensional theory obtained by reducing the ($D = 5$) Einstein action, is a gauge theory of the Virasoro and the Kac-Moody extended Lorentz algebras $(\mathfrak{g}_0 \oplus \mathfrak{so}(1,3) \times \mathfrak{sl}(\theta))$. We write the effective four-dimensional action in closed form and show how its four-dimensional symmetries guarantee its consistency as a system of interacting spin-2 particles which are massless for the unbroken theory and massive for the spontaneously broken one. Finally, we discuss how the Kaluza-Klein prescription circumvents the problems encountered in previous attempts to obtain consistent spin-2-gravity couplings.

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Dimensional reduction in Kaluza-Klein theories (KKTs) is normally restricted to extracting only the zero mass modes or, at most, to calculating the associated mass spectra [1]. For purposes of deciding their relevance to low-energy physics, this is generally sufficient. However, when the issues are those of over-all consistency, of identifying reliable truncation procedures, of calculating quantum effects or of studying the excitation of massive modes, most notably in the early universe, we must necessarily come to grips with the complete structure of the effective theory.

In recent papers [2,3] it was shown that the mass spectrum of small fluctuations in the effective four dimensional theory is patterned by an infinite dimensional Higgs effect. These results strongly suggested that the full effective theory was an infinite dimensional gauge theory.

In this paper we show that this is in fact true: i.e., viewed from a four-dimensional standpoint, KKTs are simply gauge theories of an infinite-dimensional algebra, which is a natural extension of the gauge and Poincaré algebras operative at low energies and the zero-mass modes are part of infinite dimensional multiplets forming representations of this algebra [2,3]. In view of this, the complete structure, far from being intractable, is really no more difficult to understand than the zero-mass sector. We use this insight to put down the dimensionally reduced Lagrangian in its entirety, for the five-dimensional theory which we treat first, since it provides a clear and simple paradigm for the general case [4]. We relegate to separate publications the technically more complicated higher dimensional and supersymmetric cases [4] here proceeding to demonstrate that the Lagrangian reduced to four dimensions indeed provides a consistent prescription for coupling spin-2 particles to gravity -- a prescription that has long been sought for [5]. We note that this prescription does for spin-2 spin-2 couplings what supergravity had earlier achieved for the spin 3/2-spin 2 case [6].

The Kaluza-Klein programme begins with Einstein gravity coupled to matter in $d=4+D$ dimensions and proceeds by assuming that the coordinates for the extra dimensions parameterize a compact space which is too small to permit direct observation. Thus, for a "four dimensional" observer the compact coordinates $\{\theta_i\}$ take on the character of internal rather than space time labels. Furthermore all quantities must thenceforth be periodic in the $\{\theta_i\}$ due to the basic assumption of compactness. It is conceptually important to recognize that the postulate of compact extra dimensions corresponds to a more basic level of interpretation than that at which contact with field theories

of small fluctuations about a classical solution of the field equations is made. For, as is well known, the field equations -- being local -- provide no information on the global structure of space time which is therefore necessarily an ad hoc input of the nature of a boundary condition. Thus the analysis in harmonics of the corresponding compact space. This of course in no way detracts from the necessity to ensure that the choice of compact space for this purpose is consistent with the background solution and the corresponding field theory of small fluctuations. A mismatch between the two will in general lead to non-diagonality of the mass-spectrum of small fluctuations with respect to the labels of the harmonic expansion. For a more detailed description of these issues see ref. [4].

We shall proceed by first setting forth the operations of the symmetries, the structure of the action and consistency of the field equations in terms of the fields $\Phi_i(x, \theta) \equiv \{e_\mu^\alpha(x, \theta), A_\mu(x, \theta), \varphi(x, \theta)\}$ without integrating explicitly over the compact label θ . By utilizing some simple properties of the Fourier expansion on S^1 we will then be able to carry over all our results in terms of infinite multiplets of four-dimensional fields $\Phi_i^n(x)$ ($n=0,1,\dots,\infty$). This is essentially because, as noted above, the θ -dependence of $\Phi_i(x, \theta)$ functions as an alternative (albeit continuous) label for the infinite dimensional symmetries operative at the four-dimensional level and because the four-dimensional space time derivatives commute with the operation of taking Fourier components.

The Einstein action in five dimensions*

$$S = \int d^4x d\theta ER(x, \theta) \quad (1)$$

$$R(x, \theta) = E^{MA} E^{NB} (\partial_M B_{N[A\theta]} - \partial_N B_{M[A\theta]} + B_{M[A\theta]} B_{N\theta B]} - B_{N\theta[A\theta]} B_{M\theta B]}) \quad (2)$$

$$B_{M[A\theta]} = \frac{1}{2} E_M^C (\Omega_{CAB\theta]C} - \Omega_{[BA\theta]B} + \Omega_{[BC]A}) \quad (3)$$

$$\Omega_{[AB]C} = E_A^M E_B^N \partial_M E_{NC} \quad (4)$$

* Our conventions are as follows. Upper case latin indices take the values 0,1,2,3,5 while the lower case greek ones range over 0,1,2,3. Early alphabet letters denote frame labels, while the mid-alphabet is used for world indices.

is invariant under both the five-dimensional general coordinate transformations $Z^M \rightarrow Z'^M = Z^M + \xi^M(x)$ and the local Lorentz rotations $\omega^A_{\ B}(x, \theta)$ of the 5-bein E_M^A :

$$\delta E_M^A = \xi^L \partial_L E_M^A + \partial_M \xi^L E_L^A + \omega^A_{\ B} E_M^B \quad (5)$$

We parametrize E_M^A as

$$E_M^A(x, \theta) = \begin{pmatrix} e_\mu^\alpha(x, \theta) & \sqrt{\varphi} A_\mu(x, \theta) \\ 0 & \sqrt{\varphi(x, \theta)} \end{pmatrix} \quad (6)$$

where we have used the off-diagonal rotation ω_5^α in (5) to set $E_5^\alpha = 0$.

This leaves us with the local Lorentz rotations corresponding to $\omega_\beta^\alpha(x, \theta)$ and the five coordinate transformations parametrized by the $\xi^\mu(x, \theta)$. In four dimensions, the latter split into a scalar parameter $\xi^\theta(x, \theta)$ and a vector $\xi^\mu(x, \theta)$.

The action of these symmetry transformations is effected on the appropriate object (Lorentz vector or world coordinate) through the operators $\omega_{\alpha\beta}(x, \theta)$ $L_{\alpha\beta} \equiv$ Lorentz generators $i \xi^\theta(x, \theta) \partial_\theta$ and $i \xi^\mu(x, \theta) \partial_\mu$ respectively, the first two of which, on being Fourier analysed, become

$$\omega_{\alpha\beta}(x, \theta) L_{\alpha\beta} = \omega_{\alpha\beta}^n(x) e^{in\theta} L_{\alpha\beta} = \omega_{\alpha\beta}^n L_{\alpha\beta} \quad (\omega_{\alpha\beta}^n = \omega_{\alpha\beta}^{-n*}) \quad (7a)$$

$$\xi^\theta(x, \theta) i \partial_\theta = \xi^{\theta n}(x) i e^{in\theta} \partial_\theta = \xi^{\theta n}(x) Q_n \quad (\xi^{\theta n} = \xi^{\theta(-n)*}) \quad (7b)$$

Thus in four dimensions $\omega_{\alpha\beta}(x, \theta)$ produces an infinite number of generalized local Lorentz rotations with generators $L_{\alpha\beta}^n$ satisfying

$$[L_{\alpha\beta}^n, L_{\gamma\delta}^m] = \eta_{\gamma\beta} L_{\alpha\delta}^{m+n} - \eta_{\alpha\gamma} L_{\beta\delta}^{m+n} + \eta_{\beta\delta} L_{\gamma\alpha}^{m+n} - \eta_{\alpha\delta} L_{\gamma\beta}^{m+n} \quad (8)$$

while $\xi^\theta(x, \theta)$ leads to an infinite number of local internal symmetries with parameters $\xi^{\theta n}(x)$ and generators Q_n obeying the Virasoro loop algebra [2]

$$[Q_m, Q_n] = (m-n) Q_{m+n} \quad (9)$$

The action of these transformations on the fields of the theory follows from eqs. (5) and (6). We see that the Lorentz rotations ω_β^α are linearly

realized on the 4-bein e_μ^α ($\xi^\mu = 0, \omega^\alpha \neq 0$)

$$\delta e_\mu^\alpha = \omega^\alpha{}_\beta e_\mu^\beta \quad (10a)$$

and furthermore respect the gauge condition $E_5^\alpha = 0$:

$$\delta E_5^\alpha = \omega^\alpha{}_\beta E_5^\beta = 0 \quad (10b)$$

Similarly, under algebra (9), φ, A_μ and e_μ^α transform as ($\omega^\alpha = \xi^\alpha = 0, \xi^\theta \neq 0$):

$$\delta \varphi = 2(\partial_\theta \xi^\theta) \varphi + \xi^\theta \partial_\theta \varphi \quad (11a)$$

$$\delta A_\mu = \partial_\mu \xi^\theta + \xi^\theta \partial_\theta A_\mu - (\partial_\theta \xi^\theta) A_\mu \quad (11b)$$

$$\delta e_\mu^\alpha = \xi^\theta \partial_\theta e_\mu^\alpha \quad (11c)$$

while

$$\delta E_5^\alpha = \xi^\theta \partial_\theta E_5^\alpha + E_5^\alpha \partial_\theta \xi^\theta = 0 \quad (11d)$$

once again.

In terms of Fourier components eq. (10) read:

$$\delta \varphi^n = i(n+l) \xi^\theta e^l \varphi^{n-l} \quad (12a)$$

$$\delta A_\mu^n = \partial_\mu \xi^\theta e^{in} + i(n-2e) \xi^\theta A_\mu^{n-e} = \partial_\mu \xi^\theta e^{in} - i f_{lmn} e^l \xi^\theta A_\mu^{n-l} \quad (12b)$$

$$\delta e_\mu^{\alpha n} = i(n-l) \xi^\theta e^l e_\mu^{\alpha n} \quad (12c)$$

Eq. (12b) allows us to identify A_μ^n unambiguously as the gauge field corresponding to the generator Q_n . Furthermore, since,

$$(Q_e(\omega))_{mn} = -[m + (n-l)e] \delta_{n,m-l} \quad (13)$$

provides a representation of algebra (9) for any value of α , it is clear from eqs. (12a) and (12c) that φ^n and $e_\mu^{\alpha n}$ transform as matter fields corresponding to the $\alpha = 2$ and $\alpha = 0$ representations respectively of the gauge algebra (9).

In contrast to these two symmetries, those corresponding to $\xi^\mu(x)$ do not respect the condition $E_5^\alpha = 0$. Indeed

$$\delta E_5^\alpha = (\partial_5 \xi^\lambda) e_\lambda^\alpha + \omega^\alpha{}_\beta E_5^\beta \quad (14a)$$

Our gauge condition can nevertheless be maintained by performing a compensating ω_5^α rotation. This fixes ω_5^α to be

$$\omega_5^\alpha = -\frac{1}{4\varphi} (\partial_5 \xi^\lambda) e_\lambda^\alpha \quad (14b)$$

Inserting this into eq. (5) and keeping only ξ^μ non-zero we get the following set of vector transformations which are now symmetries of the effective four-dimensional action written in terms of the fields in eq. (6)

$$\delta \varphi = \xi^\lambda \partial_\lambda \varphi + 2(\partial_\theta \xi^\lambda) A_\lambda \varphi \quad (15a)$$

$$\delta A_\mu = \xi^\lambda \partial_\lambda A_\mu + (\partial_\mu - A_\mu \partial_\theta) \xi^\lambda A_\lambda + \frac{1}{\varphi} g_{\mu\lambda} \partial_\theta \xi^\lambda \quad (15b)$$

$$\delta e_\mu^\alpha = \xi^\lambda \partial_\lambda e_\mu^\alpha + (\partial_\mu - A_\mu \partial_\theta) \xi^\lambda e_\lambda^\alpha \quad (e_\lambda^\alpha e_{\mu\alpha} = g_{\lambda\mu}) \quad (15c)$$

In spite of their non-linearity these variations will be extremely useful to us in proving the consistency of the effective four-dimensional action.

Since both linear symmetries are local it must be possible to write the total reduced Lagrangian with all space-time derivatives covariantized with respect to either symmetry. Explicit evaluation indicates that this covariantization organises the action neatly into three terms

$$S = \int d^4x d\theta e \sqrt{\varphi} \left[R_4^{\text{cov}}(e) - \frac{\varphi}{4} F_{\alpha\beta}^2 - \frac{1}{4\varphi} (e_\mu^\alpha \partial_\theta e_{\mu\beta})^2 + \frac{1}{\varphi} (e_\mu^\alpha \partial_\theta e_{\mu\alpha})^2 \right] \quad (16)$$

where $R_4^{\text{cov}}(e)$ is the usual Ricci scalar (cf eq. (2)) but with each $\partial_\mu e_{\nu\alpha}$ replaced by

$$D_\mu^{\text{cov}} e_{\nu\alpha} = \partial_\mu e_{\nu\alpha} - \frac{1}{2} e_\mu^\beta \partial_\theta e_{\beta\alpha} A_\mu e_\nu^\beta \quad (17)$$

$$D_\mu^\theta = \partial_\mu - A_\mu \partial_\theta \quad (18)$$

$$F_{\alpha\beta} = e_\mu^\mu e_\nu^\nu D_{[\mu}^\theta D_{\nu]}^\theta A_{\mu\nu} \quad (19)$$

The D_μ^{cov} and D_μ^θ so defined are covariant with respect to the local $SO(1,3)$ and θ -rotations respectively

$$D_\mu^L(D_\nu^{\text{cov}} e_\alpha^\mu) = \omega_\beta^\alpha (D_\mu^{\text{cov}} D_\nu^{\text{cov}} e_\beta^\mu) \quad (20a)$$

$$\delta^\theta (D_\mu^\theta e_\nu^\alpha) = \xi^\theta \partial_\theta (D_\mu^\theta e_\nu^\alpha) \quad (20b)$$

Using this fact, it is easy to check that each term in S is separately invariant under the two linear symmetries.

Expressed, alternatively, in terms of the metric $g_{\mu\nu} = e_\mu^\alpha e_\nu^\alpha$, the action takes the form

$$S = \int d^4x d\theta \sqrt{g} \sqrt{\varphi} \left[R_4^{\text{cov}}(g) - \frac{\varphi}{4} F_{\mu\nu}^2 - \frac{1}{4\varphi} \{ (g^{\mu\nu} \partial_\theta g_{\mu\nu})^2 - (g^{\mu\nu} \partial_\theta g_{\lambda\mu})^2 \} \right] \quad (21)$$

where $R_4^{\text{cov}}(g)$ is now the Ricci scalar in terms of the Christoffel symbols but

with all derivatives, ∂_μ , including those in the connections replaced by their covariant counterparts $\mathcal{D}_\mu^\theta = \partial_\mu - A_\mu \partial_\theta$.

We note that in either formulation, the action has a particularly simple form. In eq. (16) e.g. it consists of an R-type curvature term for the $SO(1,3) \times \mathbb{C}^1$ loop algebra (8), an F^2 -term for the algebra (9) of the $\mathcal{Q}_n$ and a third piece which on linearization gives rise to a Pauli-Fierz mass term for the spin-2 particles.

This form also makes clear that the Kaluza-Klein 'miracle' produces a gauged Lagrangian not only at the zero-mass level but right through the massive sector as well. The symmetries of the low energy sector can be studied in isolation consistently because they form a subgroup of the full symmetry group, as can be seen by setting $m = n = 0$ in (8) and (9).

It is, furthermore, worth remarking that the absence of a kinetic term for the φ -field in the intricately coupled action of eq. (16) or eq. (21), should not lead one to conclude that φ does not propagate. Indeed, if we perform a conformal rescaling of the 5-bein $E_{\mu\gamma}^A \rightarrow \tilde{\varphi}^{\frac{1}{2}} E_{\mu\gamma}^A$,

$$R_+^{\text{cov}} \rightarrow \varphi^{\frac{1}{2}} R_+^{\text{cov}} - \frac{1}{6} \left(\frac{D_\mu \varphi}{\varphi} \right)^2 \quad (22)$$

apart from a total divergence, where the modified covariant derivatives are

$$D_\mu e_{\alpha}^{\text{cov}} = \{ \partial_\mu - A_\mu \partial_\theta - \frac{1}{2} \partial_\theta A_\mu \} e_{\alpha}^{\text{cov}} + \frac{1}{2} e_{\alpha}^{\text{cov}} \partial_\theta e_{\beta}^{\text{cov}} A_\mu e_{\beta}^{\text{cov}} \quad (23)$$

$$D_\mu^{\text{cov}} \varphi = \{ \partial_\mu - A_\mu \partial_\theta - 3 \partial_\theta A_\mu \} \varphi \quad (24)$$

and a kinetic term appears explicitly.

Alternatively if we simply expand the Lagrangian (21) to bilinear order in the fluctuations about a classical $M^4 \times S^1$ background: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\varphi = 1 + \tilde{\varphi}$ and diagonalise this bilinear $\mathcal{L}^{(2)}$ in the light-cone gauge $(h_{\mu\alpha} = A_\alpha = 0)$ we obtain the standard expression

$$\int d\theta \mathcal{L}^{(2)} = \int d\theta \left[\frac{1}{4} h_{\alpha\beta} (\partial^2 + \partial_\theta^2) h_{\alpha\beta} + \frac{1}{2} A_\alpha (\partial^2 + \partial_\theta^2) A_\alpha + \frac{1}{4} \tilde{\varphi} (\partial^2 + \partial_\theta^2) \tilde{\varphi} \right] \quad (25)$$

wherein $\tilde{\varphi}$ is once again seen to propagate.

By Fourier analysing each field in (25) and integrating over θ we obtain the physical spectrum for the broken theory viz. a massless graviton $(h_{\mu\nu}^{\text{tr}(\omega)})$,

a photon $A_\mu^{(\omega)}$, a pseudo-Goldstone boson, $\varphi^{(\omega)}$, and a tower of massive spin-2 particles $(h_{\mu\nu}^{\text{tr}(\ell)}, A_\mu^\ell, \varphi^\ell; \ell=1, 2, \dots, \infty)$ with five helicity states on each floor of the tower. As shown previously [2] all masses arise from the spontaneous breaking of the infinite dimensional gauge invariances by $\langle g_{\mu\nu}^0 \rangle = \eta_{\mu\nu}$ and $\langle \varphi^0 \rangle = 1$.

Finally, we point out that the contribution of the last terms in (21)

to $\int d\theta \mathcal{L}^{(2)}$ is precisely of the Pauli-Fierz form

$$-\sum_\ell \frac{\ell^2}{4} (h_{\mu\nu}^\ell h^{\mu\nu\ell} - h_{\mu}^{\mu(\ell)} h_{\nu}^{\nu(\ell)}) \quad (26)$$

as promised.

With the complete dimensionally reduced action in hand, we next ask whether the corresponding field eqs. are consistent or not. The problem which normally arises is this. If we act with ∂_ν on a typical field eq.

$$\partial_\mu F^{\mu\nu} = j^\nu \quad (27)$$

the l.h.s. vanishes by the antisymmetry of $F^{\mu\nu}$ while the r.h.s. viz. $\partial_\nu j^\nu$ is zero iff j^ν is the current corresponding to some symmetry of the action. If it is not, we get an inconsistency.

To carry out the relevant analysis for our Lagrangian, it is convenient to eliminate through integration by parts all derivatives acting on the spin-connections. The action density then reads

$$\mathcal{L} \rightarrow \mathcal{L}'(\chi, \theta) = E [B_{A[BC]} B_{C[AB]} + B_{D[AB]}^2] \quad (28a)$$

$$= e \sqrt{\varphi} [B_{\alpha[\beta\gamma]} B_{\gamma[\alpha\beta]} + B_{\beta\alpha\gamma}^2] - (D_\alpha^{\text{cov}} \varphi) B_{\beta\alpha\gamma}^2 + \frac{1}{2} e^{\text{cov}} \partial_\theta e_{\mu\beta} F^{\alpha\beta} - \frac{\varphi}{4} F_{\alpha\beta}^2 - \frac{1}{4} \varphi (e_{\alpha}^{\text{cov}} \partial_\theta e_{\mu\beta})^2 + \frac{1}{\varphi} (e_{\alpha}^{\text{cov}} \partial_\theta e_{\mu\alpha})^2] \quad (28b)$$

where

$$D_\mu^{\text{cov}} \varphi = D_\mu^\theta \varphi - 2(\partial_\theta A_\mu) \varphi \quad (29)$$

and

$$B_{\alpha[\beta\gamma]}^\theta = e_{\alpha}^{\text{cov}} B_{\mu[\beta\gamma]} (\partial_\mu e_{\nu\delta} \rightarrow D_\mu^\theta e_{\nu\delta}) \quad (30)$$

The simplifying feature of $\mathcal{L}'$ from our viewpoint is that it depends on $\partial_\mu e_{\nu\alpha}$ only through $B_{\beta\gamma\delta}$, or, in turn, through $\Omega_{[\beta\gamma\delta]}$ and on $\partial_\mu A_\nu$ only

through $F_{\mu\nu}$. Since

$$\frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \partial_\nu} = e_\mu^\sigma e_\nu^\tau \delta_{\sigma\tau} \quad (31a)$$

while

$$\frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu A_\nu} = \delta_{\mu\nu}^\sigma \delta_{\sigma\tau} \quad (31b)$$

both $\delta \mathcal{L}' / \delta \partial_\mu A_\nu$ and $\delta \mathcal{L}' / \delta \partial_\mu A_\nu$ are antisymmetric under $\sigma \leftrightarrow \tau$. The consistency test for the potentially problematic A_μ and e_μ^σ eqs. then reduces to verifying that

$$\partial_\mu \frac{\delta \mathcal{L}_{eff}}{\delta A_\mu} = 0 \quad (32a)$$

and

$$\partial_\mu \frac{\delta \mathcal{L}_{eff}}{\delta e_\mu^\sigma} = 0 \quad (32b)$$

identically. (Here $\mathcal{L}_{eff}$ is the Lagrangian, $\mathcal{L}'$, integrated over θ and $\delta \mathcal{L}_{eff}(x) / \delta A_\mu$ e.g. is, accordingly, short-hand for

$$\frac{\delta \mathcal{L}_{eff}}{\delta A_\mu} = \int d\theta \frac{\delta \mathcal{L}'(x, \theta)}{\delta A_\mu(x, \theta)} \quad (33)$$

As we now show, the Virasoro symmetries (11) guarantee the former while a combination of the Virasoro and vector symmetries (15) ensure the latter.

The symmetry transformations (11) lead in an altogether straightforward

manner to the infinite set of conserved currents

$$j^\mu(\theta) = \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu e_\nu^\sigma} \partial_\nu e_\mu^\sigma(\theta) + \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu A_\nu} \partial_\nu A_\mu(\theta) + \partial_\mu \left(\frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu A_\nu} A_\nu \right) + \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \varphi} \partial_\mu \varphi - 2 \partial_\mu \left(\frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \varphi} \varphi \right) \quad (34)$$

Eq. (37a) can now be trivially established by checking that $-\delta \mathcal{L}_{eff} / \delta A_\mu(\theta)$ coincides exactly with this current.

The argument for eq. (32b) is similar.

We first check that under the vector symmetries, $\mathcal{L}_{eff}$ is invariant up to a pure divergence:

$$\delta \mathcal{L}_{eff}(x) = \int d\theta \partial_\mu \left(\xi^\mu \mathcal{L}' + 2(\partial_\theta \xi^\mu) e_\lambda^\sigma \partial_\theta(e e^\mu_\sigma) \right) \quad (35)$$

where the second term results from the off-diagonal rotation $\omega^\alpha_\gamma = -\frac{1}{\theta}(\partial_\theta \xi^\alpha) e_\gamma^\lambda$ incorporated into the vector variations.

Furthermore

$$\delta \mathcal{L}_{eff}(x) = \partial_\mu \int d\theta \left\{ \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu e_{\nu\rho}(\theta)} \delta e_{\nu\rho}(\theta) + \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu A_\nu} \delta A_\nu + \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \varphi} \delta \varphi \right\} \quad (36)$$

Substituting into this the relevant variations from eq. (15), using eq. (34) in rewriting the r.h.s. in terms of covariant quantities and combining the result with eq. (35) we are led to conclude that

$$\int d\theta \xi^\alpha(\theta) \partial_\mu \left\{ e_\mu^\nu \mathcal{L}' + 2e e^\alpha_\nu \partial_\nu e_\mu^\sigma e_\sigma^\mu \partial_\theta e_{\nu\rho} \partial_\theta e_{\lambda\sigma} - 2 \partial_\theta(e e^\mu_\sigma) \frac{e_{\nu\mu} \delta \mathcal{L}_{eff}}{\partial \partial_\mu A_\nu} + A_\nu \frac{\delta \mathcal{L}_{eff}}{\partial \partial_\mu A_\nu} \right\} \\ - e^\lambda_\alpha \left[\frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu e_{\nu\rho}(\theta)} \partial_\mu e_{\nu\rho} + \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu A_\nu} (F_{\lambda\nu}^{\text{cov}} - e_\nu^\sigma \partial_\theta e_{\lambda\sigma}) + \frac{\delta \mathcal{L}_{eff}}{\partial \partial_\mu \varphi} D_\lambda \varphi \right] \quad (37)$$

where we have chosen $\mathcal{L}'(\theta) = \mathcal{L}'(x, \theta) e_\mu^\sigma(x, \theta)$ to be independent of x but dependent on θ . The quantity in brackets thus represents yet another infinite set of conserved currents -- one for each value of θ . Explicit evaluation now easily shows that this current precisely equals $\delta \mathcal{L}_{eff} / \delta e_\mu^\sigma(\theta)$ and that eq. (32b) is therefore truly an identity.

All these eqs. can be equally well written out in terms of the discrete labels 'n'--- by recalling a few simple properties of Fourier sums.

If a_ℓ are the Fourier components of

$$a_\ell = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\ell\theta} a(\theta) \quad (38)$$

then

$$(a_\ell)^\ell = a^{\ell-m} b^{-m} \quad (39)$$

By repeated use of this formula we can get at the Fourier components of an arbitrary product. Each factor will produce an additional index, making for very unwieldy expressions if we write all the indices in. We shall accordingly often use the single-indexed l.h.s. of eq. (39) to mean the more complete expression on the r.h.s.

Specialised to the case where $b=a^{-1}$ eq. (39) leads to an implicit definition of the components of a^{-1} . Thus,

$$(\varphi^{-1})^{\ell-m} \varphi^m = (\varphi \varphi^{-1})^\ell = \delta_0^\ell \quad (40)$$

$$g^{\mu\lambda} a^{-m} g_{\lambda\nu}^m = (g^{\mu\lambda} g_{\lambda\nu})^\ell = \delta_\nu^\mu \delta_0^\ell \quad (41)$$

Fractional powers are defined similarly. E.g.

$$\sqrt{\varphi}^{\ell-m} \sqrt{\varphi}^m = \varphi^\ell \quad (42)$$

Using these formulae we can easily write out the effective four-dimensional action (21) in Fourier components:

$$S = \int d^4x \sqrt{g} \left\{ \sqrt{-g} \left[\frac{1}{4} R_4^{(n)} - \frac{(\sqrt{-g})^{L-m}}{4} F_{\mu\nu}^{mn} F^{\mu\nu} + \frac{(\sqrt{-g})^{L-m}}{4} [(\sqrt{-g})^2 g_{\mu\nu}]^2 (\sqrt{-g})^2 g_{\mu\nu} \right] \right\} \quad (43)$$

where typically

$$g^{\mu\nu} = \varepsilon^{\mu_1\mu_2\mu_3\mu_4} g_{\mu_1\mu_2}^{L-m} g_{\mu_3\mu_4}^{m-n} g_{\mu_1\mu_2}^{n-p} g_{\mu_3\mu_4}^{p-q} \quad (44)$$

$$[g^{\mu\nu} \partial_\mu g_{\nu\lambda}]^L = g^{\mu\nu\lambda\sigma} (\partial_\mu g_{\nu\sigma})_{L-m} g_{\mu\nu}^m \quad (45)$$

All covariant derivatives now have the form

$$D_{\mu}^{cov} \Phi = (\partial_{\mu} + i A_{\mu}^n Q_n(\alpha)) \Phi \quad (46)$$

where a (cf eq. (13)) is determined by the representation to which Φ belongs. ($\alpha = 0$ for $g_{\mu\nu}$, $\sqrt{-g}$ and R ; $\alpha = -1$ for A_{μ} , $F_{\mu\nu}$ and Q_n for $(\sqrt{-g})^2$.)

Note that $\sum_i \alpha_i = 1$ for each term of the Lagrangian in (43). In fact $\sum_i \alpha_i = 1$ is a necessary and sufficient condition for obtaining singlets of the Virasoro algebra. Indeed, if

$$\delta \Phi^L(\alpha) = -i \xi^{\sigma m} (Q_m(\alpha))_{\ell n} \bar{\Phi}^n(\alpha) \quad (47)$$

then it is easily checked that

$$\delta [\bar{\Phi}^L(-\alpha) \Phi^L(\alpha+1)] = 0 \quad (48)$$

Moving next to the consistency proof for the A_{μ} eq. we find that it is even more transparent in the Fourier analysed version than in terms of θ . This is because all variations under the Virasoro algebra have the form (47) which follows closely the pattern for finite-dimensional gauge theories. The corresponding current

$$j^{\mu\ell} = -i \left\{ \frac{\delta \mathcal{L}_{eff}}{\delta \partial_{\mu} e_{\nu\sigma}} (Q_{\ell}(\alpha=0))_{mn} e_{\nu\sigma}^n + \frac{\delta \mathcal{L}_{eff}}{\delta \partial_{\mu} A_{\nu}} (Q_{\ell}(\alpha=-1))_{mn} A_{\nu}^n + \frac{\delta \mathcal{L}_{eff}}{\delta \partial_{\mu} \Phi} (Q_{\ell}(\alpha=2))_{mn} \bar{\Phi}^n \right\} \quad (49)$$

is, in complete analogy, equal to $-\delta \mathcal{L}_{eff} / \delta A_{\mu}^{\ell}$.

Finally the consistency proof of the spin-2 equation can be trivially translated from the θ - to the α - language.

$$\partial_{\mu} \frac{\delta \mathcal{L}_{eff}}{\delta e_{\mu\alpha}} = \partial_{\mu} \int d^4x \frac{\delta \mathcal{L}_{eff}}{\delta e_{\mu\sigma}(\alpha, \theta)} \frac{\delta e_{\mu\sigma}(\alpha, \theta)}{\delta e_{\mu\sigma}} = \int d^4x \partial_{\mu} \frac{\delta \mathcal{L}_{eff}}{\delta e_{\mu\sigma}(\alpha, \theta)} = 0 \quad (50)$$

Before concluding, we discuss the action (43) in the light of previous attempts to obtain consistent spin 2-gravity couplings. The problem has always been one of extending the general coordinate invariance of the one graviton case to an entire set of such invariances, with the simultaneous introduction of compatible symmetries to handle any additional non-scalar fields which may have been required by this extension. As is clear from the Kaluza-Klein approach succeeds in doing this. That is, it generates a general coordinate invariance to the infinite set of vector symmetries (15) and concurrently introduces a local Virasoro symmetry to ensure consistency for the A_{μ} fields required by the vector symmetries (15).

Although implementation of the vector symmetries required simultaneous consideration of all three terms of the action, that of the Virasoro symmetries can be understood in steps and explains the detailed structure of the first two terms from a four dimensional standpoint:

The graviton is promoted to being part of an infinite-dimensional multiplet transforming as the $Q_n(\alpha=0)$ -representation (cf eq. (13)) of the Virasoro loop algebra (9). The standard Einstein-Hilbert action $\int \sqrt{-g} R$ -- the correct one graviton limit -- is generalized by gauging the loop algebra (i.e. $\partial_{\mu} \rightarrow D_{\mu} = \partial_{\mu} + i A_{\mu}^n Q_n(\alpha=0)$). The gauge field multiplet (transforming as the $Q_n(\alpha=-1)$, adjoint representation) is then furnished with a kinetic term, $-1/4 F^2$.

All objects thus progressively introduced transform as definite representations of the loop algebra. $(\sqrt{-g})^L, R^{(L)}$ as the $\alpha=0$, and F^2 as the $\alpha=-2$ representations, respectively). However, to form Q_n -singlets we require products whose (1) Fourier indices add up to zero and (2) Q_n -values sum to 1. Neither $\sqrt{-g}^{(L)} R^{(L)}$ nor $\sqrt{-g}^{(L)} F^2$ forms a singlet as it stands. This forces us to introduce yet another multiplet $\sqrt{-g}^L$, which for $L=0$ must be a general coordinate scalar since $\sqrt{-g}^{(0)} R^{(0)}$ and $\sqrt{-g}^{(0)} F^2(0)$ are so, and whose r -th power $(\sqrt{-g}^L)^r$ transforms as the representation $\alpha=r$. The conditions for the formation of Q_n -singlets then dictate that $\sqrt{-g} R$ is to be multiplied by $\sqrt{-g}$ and $\sqrt{-g} F^2$ by $(\sqrt{-g})^3$.

In conclusion, then, we have demonstrated that the five-dimensional Kaluza-Klein theory corresponds in four-dimensions to a gauge theory of the Virasoro algebra. We have used this understanding to write the effective four-dimensional action in closed form and to thus produce a theory which in its unbroken version contains an infinite number of massless, and in its spontaneously broken phase, one massless and an infinite number of massive spin-2 particles interacting consistently with themselves and with the other spin-1 and spin-0 particles in the theory. We have further shown how the symmetries of this action ensure the consistency of its field eqs.

We should mention that the program suggested by this paper for higher dimensions has already been partially implemented. The infinite dimensional algebras for which we trade in the extra dimensions have been identified and put into a convenient form [3]. These and other matters will be reported elsewhere [4].

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