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INSTANTON CALCULUS WITH MAJORANA FERMIONS

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A B S T R A C T

We show that the treatment of Majorana fermions in Euclidean space used in the constructive approach can be extended to instanton calculations. We briefly outline how this method can be used for supersymmetric gauge theories.

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1. The possibility of developing a four-dimensional Euclidean field theory with fermions has been investigated several years ago by Osterwalder et al.¹⁾ An important conclusion reached by these authors was that the construction of the Schwinger functions requires a doubling of the fermionic degrees of freedom. They also showed that the hermiticity of the action is replaced by an unitary involution (the so-called Osterwalder-Schrader positivity). In particular, it is clear from covariance considerations that one cannot introduce Euclidean Majorana fermions and preserve hermiticity at the same time. Thus, there is no Hermitean (N=1) supersymmetry in Euclidean space²⁾. However, Nicolai³⁾ has shown that it was possible to apply the methods described in Ref. 1) to the Wess-Zumino model in order to construct (perturbatively) Schwinger's functions satisfying the supersymmetric Ward identities at imaginary time. This is just what we need⁴⁾ and the fact that in Euclidean space, supersymmetry is non-unitary does not cause any problem.

Another method has been proposed⁵⁾ to functionally integrate Majorana fermions, namely to take the square root of the determinant for a corresponding Dirac fermion. However, this is not always convenient for practical calculations. The aim of this letter is to show that the Nicolai procedure³⁾ can be used for instanton calculations. Various aspects of instanton calculus in supersymmetric gauge theories have been treated elsewhere⁶⁾⁻⁸⁾. Here we will concentrate mainly on the problem of integration of Majorana fermions.

In section 2, we derive the consistency conditions to be imposed on the Euclidean action. Section 3 is devoted to the diagonalization of $iD_\mu \gamma_\mu$. Finally, in section 4 we evaluate in this basis the action of a Majorana fermion, we check the consistency of the procedure and we show how it can be used in the case of supersymmetric QCD.

2. Let us consider an SU(2) gauge theory with a Majorana fermion ψ in an arbitrary representation R. The procedure of Ref. 3) can be repeated when gauge fields are introduced and the fermionic part of the action reads:

$$A_F = \int d^4x \, \psi^T \not{D} \psi \quad (1)$$

with the convention

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ \bar{e}_\mu & 0 \end{pmatrix} ; \quad e_\mu = (i\vec{\sigma}, 1) ; \quad \bar{e}_\mu = e_\mu^\dagger \quad (2)$$

$$D_\mu = \partial_\mu - ig A_\mu^a T_R^a$$

The rotation group in Euclidean space is $SO(4)$ and the two $SU(2)$ factors are no longer related by complex conjugation. Consequently, the four components have to be independent while two of these are the complex conjugate of the two others in Minkowski space. Although the fermionic degrees of freedom are thus doubled, the complex conjugate field $\phi^\dagger$ is an irrelevant object which plays no further role.

In Eq. (1), C is the direct product of two matrices C and K acting respectively on spinor and gauge indices. The conditions to be imposed on C are more involved in presence of gauge fields and we shall discuss them in detail. Requiring the action (1) to be a) $SO(4)$ invariant, b) gauge invariant, c) consistent with the anti-commuting properties of ψ , gives respectively the following conditions⁹⁾

$$C \gamma^\mu C^{-1} = \pm \gamma^\mu \tau \quad (3a)$$

$$K T_R^a K^{-1} = -T_R^a \tau \quad (3b)$$

and

$$K^T (C \gamma_\mu)^T = K C \gamma_\mu \quad (3c)$$

Condition (3a) can be satisfied with the (unitary) matrices

$$\begin{aligned} C_+ &= \gamma_1 \gamma_3 \\ C_- &= \gamma_2 \gamma_4 \end{aligned} \quad (4)$$

where the subscript $(\pm)$ indicates the sign appearing in (3a). We note that both C_+ and C_- are antisymmetric. Condition (3b) means that the representation R is real or pseudoreal in which case K is respectively symmetric or antisymmetric. For simplicity, we have considered an $SU(2)$ gauge group where all representations are real or pseudoreal but the analysis can be easily extended to any real or pseudoreal representation of a larger group. On the other hand, it is clear that the treatment of chiral fermions would require more care¹⁰⁾. Finally, from the

previous remarks and condition (3c) we conclude that $C = KC_- (KC_+)$ for real (pseudoreal) representations.

3. In order to integrate the fermionic fluctuations in the background of a BPST instanton¹¹⁾ we shall first expand the field ψ in terms of the eigenfunctions of the background covariant derivative $iD_\mu \gamma_\mu$. Namely

$$\psi = \sum_n a_n \psi_n \quad (5)$$

with

$$i D_\mu \gamma_\mu \psi_n = \lambda_n \psi_n \quad (6)$$

and

$$\{a_n, a_m\} = 0$$

Since the operator $iD_\mu \gamma_\mu$ is Hermitean for the scalar product

$$\langle \chi, \psi \rangle = \int d^4x \chi^\dagger \psi \quad (7)$$

its eigenvalues are real. The diagonalization of $iD_\mu \gamma_\mu$ can be performed using the methods developed by 't Hooft¹²⁾. First, it is clear that Eq. (6) implies

$$(i D_\mu \gamma_\mu)^2 \psi_n = \lambda_n^2 \psi_n \quad (8)$$

Due to the self-duality of the background this equation reduces to¹²⁾

$$\begin{pmatrix} -D^2 + \frac{i}{2} g e_\mu \bar{e}_\nu F_{\mu\nu} & 0 \\ 0 & -D^2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda_n^2 \begin{pmatrix} v \\ w \end{pmatrix} \quad (9)$$

where ψ has been expressed in terms of the two component objects v and w . Setting

$$L_{(2)}^a = -\frac{i}{2} \eta_{a\mu\nu} x_\mu \partial_\nu ; S_{(2)}^a = -\frac{i}{8} \eta_{a\mu\nu} \gamma_\mu \gamma_\nu \quad (10)$$

We first recall¹²⁾ that D^2 commutes with $\vec{L}_1 + \vec{T}_R$ and $\vec{L}_2$. Consequently, the eigenfunctions of

$$-D^2 A_n = \lambda_n^2 A_n \quad (11)$$

are also eigenfunctions of $(\vec{L}_1 + \vec{T}_R)^2$, $L_1^3 + T_R^3$, $(L_2^+)^2$, L_2^3 with eigenvalues noted J_1, J_1^3, ℓ, m respectively. These eigenfunctions will be written $A_n^{J_1, J_1^3, \ell, m}$. Also, a straightforward computation yields

$$[i D_\mu \gamma_\mu, \vec{J}_1] = [i D_\mu \gamma_\mu, \vec{J}_2] = 0 \quad (12)$$

with

$$\vec{J}_1 = \vec{L}_1 + \vec{T} + \vec{S}_1 ; \quad \vec{J}_2 = \vec{L}_2 + \vec{S}_2 \quad (13)$$

On the other hand, spinors of given chirality are not eigenstates of $i D_\mu \gamma_\mu$ (γ_5 and $i D_\mu \gamma_\mu$ do not commute). Rather, the general form of the eigenfunctions for $\lambda_n \neq 0$ is

$$\Psi_{\pm n} = \begin{pmatrix} \frac{i D_\mu e_\mu}{\pm i \lambda_n} w_n \\ w_n \end{pmatrix} \quad (14)$$

where the two components of w_n satisfy Eq. (11).

From Eq. (12) and Eq. (14) we can now see that ϕ_n is eigenfunction of $(J_1)^2$, $J_1^3, (\vec{J}_2)^2$, J_2^3 with respective eigenvalues $J_1, J_1^3, J_2 = 1 \pm \frac{1}{2}, J_2^3 = m \pm \frac{1}{2}$ provided that w_n has the form

$$w_n = \begin{pmatrix} \alpha_m^\pm A_n^{J_1, J_1^3, \ell, m} \\ \alpha_{m+1}^\pm A_n^{J_1, J_1^3, \ell, m+1} \end{pmatrix} \quad (15)$$

where $\alpha_m^\pm$ are the Clebsch-Gordan coefficients of the reduction $\ell \times \frac{1}{2} = \ell + \frac{1}{2} + \ell - \frac{1}{2}$. The $\pm$ in (14) and (15) are unrelated. The eigenfunctions $\phi_{\pm n}^{J_1, J_1^3, J_2, J_2^3}$ defined by Eq. (14) and (15) are orthogonal for the scalar product (7) and $i D_\mu \gamma_\mu$ is now diagonalized.

4. In the basis constructed in the previous section, the action (1) takes the form

$$A_F = \sum a_{\pm n}^{J_1, J_1^3, J_2, J_2^3} a_{(\pm n)'}^{J_1', J_1'^3, J_2', J_2'^3} \int d^4x \left[\psi_{\pm n}^{J_1, J_1^3, J_2, J_2^3} \right. \\ \left. \mathcal{C} D_\mu \gamma_\mu \psi_{(\pm n)'}^{J_1', J_1'^3, J_2', J_2'^3} \right] \quad (16)$$

where the summation runs over

$$\begin{aligned} \ell - t \leq J_1 \leq \ell + t & \quad ; J_1 \geq 0 \quad ; \dim R = 2t + 1 \\ J_2 = \ell \pm 1/2 & \quad ; J_2 \geq 0 \quad ; \ell = 0, 1/2, 1, \dots \\ -J_1 \leq J_1^3 \leq J_1 \\ -J_2 \leq J_2^3 \leq J_2 \end{aligned} \quad (17)$$

and $|\lambda_n|$ depends only on J_1 and J_2 . In order to compute the integrals appearing in Eq. (16) we first remark that

$$\begin{aligned} i D_\mu \gamma_\mu \mathcal{C}^{-1} \psi^* &= \mathcal{C}^{-1} (i D_\mu \gamma_\mu \psi)^* && \text{if } R \text{ is real} \\ &= -\mathcal{C}^{-1} (i D_\mu \gamma_\mu \psi)^* && \text{if } R \text{ is pseudoreal} \end{aligned} \quad (18)$$

and

$$\begin{aligned} \vec{J}_{(2)} \mathcal{C}^{-1} \psi^* &= \mathcal{C}^{-1} (-\vec{J}_{(2)} \psi)^* \\ (\vec{J}_{(2)}^1 + i \vec{J}_{(2)}^2) \mathcal{C}^{-1} \psi^* &= \mathcal{C}^{-1} (-(\vec{J}_{(2)}^1 - i \vec{J}_{(2)}^2) \psi)^* \end{aligned} \quad (19)$$

Using, (18) and (19) together with the properties of $\mathcal{C}$ described in section 2, we can write

$$\begin{aligned} \left(\psi_{\pm n}^{J_1, J_1^3, J_2, J_2^3} \right)^T &= (-1)^{J_1 - J_1^3 + J_2 - J_2^3 + 1} \left(\psi_{\pm n}^{J_1, J_1^3, J_2, J_2^3} \right)^+ \mathcal{C}^{-1} && \text{if } R \text{ is real} \\ &= (-1)^{J_1 - J_1^3 + J_2 - J_2^3} \left(\psi_{\mp n}^{J_1, J_1^3, J_2, J_2^3} \right)^+ \mathcal{C}^{-1} && \text{if } R \text{ is pseudoreal.} \end{aligned} \quad (20)$$

From this and the orthogonality properties of the ψ_n it follows that the only non-vanishing integrals in (16) are

$$\int d^4x (\psi_{\pm n}^{J_1, J_1^3, J_2, J_2^3})^T \not{D}_\mu \gamma_\mu \psi_{\pm n}^{J_1, J_1^3, J_2, J_2^3} = \pm i (-1)^{J_1 - J_1^3 + J_2 - J_2^3} |\lambda_n|$$

if R is real

or

$$\int d^4x (\psi_{\pm n}^{J_1, J_1^3, J_2, J_2^3})^T \not{D}_\mu \gamma_\mu \psi_{\mp n}^{J_1, -J_1^3, J_2, -J_2^3} = \pm i (-1)^{J_1 - J_1^3 + J_2 - J_2^3} |\lambda_n| \quad (21)$$

if R is pseudoreal.

We now turn to the properties of the integrals²¹⁾ under the interchange of the two eigenfunctions. First, it is clear from Eq. (17) and the fact that t is integer (half-integer)₃ for real (pseudoreal) representations, that when R is real only one of the J_1, J_2 is integral while the other is half-integral, whereas they are both integral or both half-integral when R is pseudoreal. In particular, this implies that when R is real, eigenfunctions cannot be related to themselves by the conjugation relation (20) (for R pseudoreal this property is obvious as far as $\lambda_n \neq 0$). If we now exchange the two eigenfunctions in Eq. (21)₃ (remembering they are commuting quantities) we get a factor $(-1)^{2(J_1 + J_2)}$ $[(-1)^{2(J_1 + J_2) + 1}]$ for a real (pseudoreal) representation. It thus follows that this factor is (-1) in both cases consistently with the anticommutativity of the expansion coefficients a_n . Consequently, the action (16) picks systematically one half of the eigenvalues of $iD_\mu \gamma_\mu$ according to (21). If no higher order interactions are taken into account, the integration of the non-zero modes gives the expected result.

$$\int [da_m]_{\lambda_n \neq 0} e^{-A_F} = \sqrt{\det_{\lambda_n \neq 0} (iD_\mu \gamma_\mu)} \quad (22)$$

We note that for a single pseudoreal representation, the regularization of (22) is problematic. This is due to the fact that C is symmetric and it is impossible to introduce a gauge-invariant Pauli-Villars regulator. A theory which is safe from this point of view and which allows non-trivial applications of our method is supersymmetric QCD.

This theory contains gauge bosons, a Majorana fermion λ in the adjoint representation (i.e. real), a Dirac fermion which is represented in Euclidean space by two independent four component spinors $\psi^{(1)}$ and $\psi^{(2)}$ and two scalar fields A and $\tilde{A}$. The Yukawa coupling reads

$$i\sqrt{2}g \left[A^\dagger T^a \lambda^a C \left(\frac{1+\gamma_5}{2} \right) \psi^{(1)} - \psi^{(2)} \left(\frac{1-\gamma_5}{2} \right) \lambda^a T^a A \right. \\ \left. - \psi^{(2)} \left(\frac{1+\gamma_5}{2} \right) \lambda^a T^a \tilde{A}^* + \tilde{A}^\dagger \lambda^a T^a C \left(\frac{1-\gamma_5}{2} \right) \psi^{(1)} \right] \quad (23)$$

Here, A and $\tilde{A}$ are quantum fields, and the most we can do analytically is to use a perturbation expansion in g . In that case, the square root procedure seems unsuitable whereas the one proposed here can be easily applied. Examples of such an expansion are given in Ref. (8).

In conclusion, we have shown that the treatment of Majorana fermions in Euclidean space used in the constructive approach can also be applied for instanton calculations. This method has interesting applications in the case of supersymmetric gauge theories.

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