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HIGGS AND CONFINEMENT PHASES IN THE FUNDAMENTAL
SU(2) HIGGS MODEL: MEAN FIELD ANALYSIS

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ABSTRACT

The phase diagram of the four-dimensional SU(2) gauge-Higgs model with Higgs field in the fundamental representation is derived by mean field techniques. When the Higgs field is allowed to fluctuate in magnitude, the analytic connection between Higgs and confinement phases breaks down for sufficiently small values of the quark Higgs coupling, indicating that the Higgs and confinement phases for these couplings are strictly distinct phases.

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A proper identification of the "Higgs" and "confinement" phases of a gauge theory coupled to one or more scalar fields meets some intriguing obstacles. It is known¹⁾ that no gauge-invariant local order parameter exists which can distinguish the two phases. [The well-known order parameter $\langle \phi(x) \rangle$, $\phi(x)$ being the scalar field with non-trivial tree-level effective potential, used perturbatively in the standard model of weak and electromagnetic interactions, relies on the choice of a unitary gauge, and hence is not gauge-invariant.] This statement goes far beyond Elitzur's theorem²⁾. It has been argued that the non-existence of a local gauge-invariant order parameter, as well as the Fradkin-Shenker analyticity argument³⁾, is related to the ability of expressing the physical states of the theory (e.g., the W's and the Z of the Weinberg-Salam model) in terms of local, gauge-invariant, composite operators⁴⁾. This, however, has never been rigorously established; in fact, several of the problems associated with a gauge-invariant description of Higgs and confinement phases remain, in our opinion, yet to be understood in depth^{*}).

In this letter, we show how the approximate phase diagram of the latticized SU(2) fundamental Higgs model can be obtained by mean field techniques. In so doing, we hope to shed some light on the connection between Susskind's "complementarity principle"⁴⁾ and the Fradkin-Shenker phenomenon, as well as putting some recent Monte Carlo simulations^{7),9)} into perspective.

Our starting point is the lattice action

$$S = S_g - \lambda \sum_x (\phi(x) \phi^\dagger(x) - 1)^2 - \sum_{x,y,\mu} \phi^\dagger(x) Q(x,y) \phi(y) \quad (1)$$

with

$$S_g = \frac{1}{g^2} \sum_P \text{Tr} [U_P + U_P^\dagger] \quad (2)$$

*) Several confusing statements concerning this problem exist in the literature. In Ref. 5), for example, an order parameter $\langle \text{Tr} [\phi(0) U \phi(x) U^\dagger] \rangle$ (with U a string operator) was proposed. In the axial gauge $U = 1$ this goes towards $\langle \phi(0) \rangle^2$ as $x \rightarrow \infty$, and may hence appear as a gauge-invariant order parameter. Unfortunately, however, this object has been shown to vanish in the axial gauge as $x \rightarrow \infty$ ¹⁾, and it therefore cannot possibly serve as an order parameter. For a possible non-local order parameter, see Ref. 6).

(p indicating plaquettes), and

$$Q(x,y) = \delta_{x,y} - \kappa [\delta_{x,y-\mu} U_\mu(x) + \delta_{x,y+\mu} U_\mu^\dagger(x)] \quad (3)$$

in the notation of Ref. 7).

The measure of integration is

$$d\phi d\phi^\dagger dU = \rho^3 d\rho d\Omega dU \quad (4)$$

with ρ being the (real, positive) radial part of the Higgs field, and $\Omega, U \in SU(2)$ are integrated over the Haar measure. In terms of these new variables,

$$S = S_g - \lambda \sum_x (\rho(x)^2 - 1)^2 - \sum_x \rho(x)^2 + \kappa \sum_{x,\mu} \rho(x) \rho(x+\mu) \text{Tr}[\Omega^\dagger(x) U_\mu(x) \Omega(x+\mu)] \quad (5)$$

First, let us perform a gauge transformation $U_\mu(x) \rightarrow \Omega(x) U_\mu(x) \Omega^\dagger(x+\mu)$, and do the (trivial) integration over $d\Omega$. We are then left with an action

$$S = S_g - \lambda \sum_x (\rho(x)^2 - 1)^2 - \sum_x \rho(x)^2 + \kappa \sum_{x,\mu} \rho(x) \rho(x+\mu) \text{Tr}[U_\mu(x)] \quad (6)$$

which can be analyzed by standard mean field techniques. We will do this in two steps: first we compute the mean field solutions for $U_\mu(x)$, next we substitute this solution into the action to derive the effective potential $V_{\text{eff}}(\rho)$ for $\rho(x)$. Later on in this letter, we shall return to an alternative method.

Our mean field theory notation follows that of Ref. 9). [For a comprehensive review of the method, with references to the original literature, we refer the reader to Ref. 10).] First parametrize $U \in SU(2)$ as $U = u_0 \mathbb{1} + i \vec{u} \cdot \vec{\sigma}$, so that

$$S_g = \frac{1}{g^2} \sum_P \text{Tr}[U_P + U_P^\dagger] = \frac{2}{g^2} \text{Re} \sum_P \text{Tr} U_P = \frac{4}{g^2} \sum_P u_0^P \equiv \beta \sum_P u_0^P \quad (7)$$

In order to introduce unconstrained variables, we Fourier transform

$$Z(\beta) = \int d[U] e^{S(\beta, u_\ell^i)}$$

$$= \int d[U] \int_{-\infty}^{\infty} d[v_\ell^i] \int_{-i\infty}^{+i\infty} \frac{d[\alpha_\ell^i]}{(2\pi i)^4} e^{S(\beta, u_\ell^i) + \sum_\ell \alpha_\ell^i (u_\ell^i - v_\ell^i)} \quad (8)$$

and define

$$e^{w(\alpha_\ell)} \equiv \int dU e^{\alpha_\ell^i u_\ell^i} \quad (9)$$

Identifying $u_\ell = v_\ell$ in S , we can perform the $d[U]$ integration in (8) to get

$$Z(\beta) = \int_{-\infty}^{\infty} \prod_\ell dv_\ell^i \int_{-i\infty}^{+i\infty} \prod_\ell \frac{d\alpha_\ell^i}{(2\pi i)^4} e^{S(\beta, v_\ell^i) + \sum_\ell [w(\alpha_\ell^i) - \alpha_\ell^i v_\ell^i]} \quad (10)$$

which can be treated by saddle point methods^{*)}.

Solving the saddle point equations

$$v_\ell^i = \frac{\partial w(\alpha_\ell)}{\partial \alpha_\ell^i} \quad ; \quad \alpha_\ell^i = \frac{\partial S(\beta, v_\ell)}{\partial v_\ell^i} \quad (11)$$

by a mean field ansatz $v_\ell = (V, \vec{0})$, $\alpha_\ell = (\alpha, \vec{0})$ and $\rho(x) = \rho$, independent of x and ℓ , we obtain the coupled equations

$$V = I_2(\alpha) / I_1(\alpha) \quad (12)$$

*) Note the apparent ambiguity in substituting $u_\ell = v_\ell$ in all of S , or just in S_g (11). Actually, when one solves the mean field equations, one finds that the answer is completely independent of the fraction of $(S-S_g)$ which identifies $u_\ell = v_\ell$. This is a consequence of $(S-S_g)$ being linear in $U_\mu^g(x)$. In this case, the mean free energies derived from Jensen's inequality and the saddle point method also agree.

$$\alpha = 2\beta(d-1)v^3 + 2\kappa g^2 \quad (13)$$

in d dimensions. The effective potential for ρ is then

$$\begin{aligned} V_{\text{eff}} &= -S_{\text{MF}}(g) - \frac{3}{2} \ln[g^2] - d[w(\alpha(g)) - \alpha(g)v(g)] \\ &= \frac{3}{2} \beta d(d-1)v(g)^4 + \lambda(g^2-1)^2 + g^2 - \frac{3}{2} \ln[g^2] \\ &\quad - d \ln \left[\frac{I_1(2\beta(d-1)v(g)^3 + 2\kappa g^2)}{2\beta(d-1)v(g)^3 + 2\kappa g^2} \right] \end{aligned} \quad (14)$$

with $v(\rho)$ determined by the mean field equation (12).

It is interesting to note that in the case $\beta = 0$ the effective potential can be found explicitly; now $\alpha = 2\kappa\rho^2$, and hence $V(\rho) = I_2(2\kappa\rho^2)/I_1(2\kappa\rho^2)$. $V_{\text{eff}}(\rho)$ is then

$$V_{\text{eff}}(g) = \lambda(g^2-1)^2 + g^2 - \frac{3}{2} \ln[g^2] - d \ln \left[\frac{I_1(2\kappa g^2)}{2\kappa g^2} \right] \quad (15)$$

This agrees exactly with the $\beta = 0$ effective potential obtained from Eq. (6) by exact integration of $d[U]$. Hence, in the $\beta = 0$ limit, this mean field solution becomes exact.

The effective potential V_{eff} of Eq. (15) can be used to locate phase transition points by following how the absolute minimum of the potential changes as a function of λ and κ . To illustrate this, we show in Fig. 1 the effective potential $V_{\text{eff}}(\rho)$ for $\lambda = 0.005$ and $\kappa = 0.1820, 0.1825$ and 0.1830 , respectively. At $\kappa = 0.1825$, the absolute minimum jumps discontinuously from $\rho_{\text{min}}^{(1)} = 1.35$ to $\rho_{\text{min}}^{(2)} = 5.80$, indicating a first-order phase transition point at these values of the coupling constants. Mapping out the line of first-order phase transition points in this coupling constant plane leads to the graph shown in Fig. 2. The line of first-order phase transition points ends at $(\lambda^*, \kappa^*) = (0.035, 0.2450)$ in what is presumably a second-order critical point. The line starts at $\lambda = 0$ with $\kappa_c = 0.125$, a critical value of κ which is not difficult to understand: using the asymptotic expansion for the Bessel function, we see from Eq. (15) that the action becomes unbounded from below when $2d\kappa > 1$, i.e., $\kappa_c = 1/8$ in four dimensions, in agreement with the value found above.

The fact that for $\lambda = 0$ the theory ceases to exist for $\kappa > \kappa_c = 1/8$ is in disagreement with claims of Ref. 7) based on Monte Carlo (MC) data. We interpret this disagreement as an artifact of the MC procedure, which probes the region around the local unstable minimum at small values of $\rho(x)^2$. From our effective potential we see that to pass over the potential barrier requires $\rho(x)^2 \approx 200^*$ and an alignment of $\Omega^+(x)U_\mu(x)\Omega(x+\mu)$ to ≈ 1 , a very unlikely situation in a local MC sequence. We have verified this in a simple MC simulation on a 4^4 lattice by performing runs from different initial configurations. Starting with sufficiently large $\rho(x)^2$, the instability of the theory was clearly seen.

Let us also comment on the order of the phase transition. Lowest order mean field theory of pure gauge theories in four dimensions is known to predict (incorrectly) first-order phase transitions for all $SU(N)$ gauge groups, including $SU(2)$. For $\beta = 0$ our mean field solution is not of this type. Instead, it corresponds to integrating out exactly the gauge field degrees of freedom, and only using the effective potential ansatz $\rho(x) = \rho$ (translational invariance) for one scalar variable. We therefore believe that the prediction of a first-order phase transition at $\beta = 0$ is firm; it is very difficult to imagine how higher order corrections around the solution $\rho(x) = \rho$ could change the order of the phase transition.

For $\beta \neq 0$ we have used a similar method to locate the phase transitions: determine which values of the parameters (λ, κ, β) lead to a discontinuous jump in the absolute minimum of $V_{\text{eff}}(\rho)$. This method is efficient, but of course becomes increasingly more difficult as one approaches an end-point of a first-order phase transition line.

The phase diagram obtained in this way is shown by the fully drawn lines to the left in Fig. 3. For reasons of comparison with recent MC data⁷⁾, we have chosen to restrict ourselves to $\lambda = 0.001, 0.1$ and 1.0 respectively. For $\lambda > 0.035$ (as exemplified by $\lambda = 0.1$ and $\lambda = 1.0$), lines of first-order phase transitions are found to end before reaching $\beta = 0$. Again, their end-points are presumably second-order critical points.

For $0 < \beta < \beta^* \approx 1.57$, the phase transition is associated exclusively with a discontinuous jump in ρ , since for this range of β the mean field equations allow

*) Depending on the value of κ ($> 1/8$).

for only one solution for v , at fixed ρ . For $\beta > \beta^*$ the solutions become far more complex, allowing up to five different saddle points in v . $V_{\text{eff}}(\rho)$ is now obtained by always choosing the solution v^* for which $V_{\text{eff}}(\rho(v^*))$ is minimized. $V_{\text{eff}}(\rho)$ obtained in this fashion turns out to be a continuous function of ρ^* , but it will typically have discontinuities in the first derivative with respect to ρ .

Interestingly, for $\beta > \beta^*$, the phase transition is triggered by a switch in mean field solutions for v . Thus, although the phase transition line joins smoothly across β^* , the mechanism behind the phase transition is entirely different. In fact, the Higgs system treated in this fashion for $\beta > \beta^*$ "remembers" the spurious pure SU(2) gauge transition at $\beta_c \approx 2.82^{10)$, and sends all phase transition lines down to the point $(\kappa = 0, \beta_c)$ for all λ . Clearly, for some intermediate values of $\beta > \beta^*$ this lowest order mean field treatment is inadequate.

The fact that this first mean field treatment of the SU(2) Higgs model must fail for sufficiently large β can be understood from a completely different point of view. As $\beta \rightarrow \infty$, the gauge field configurations freeze into pure gauges, $U_\mu(x) = \Omega(x)\Omega^\dagger(x+\mu)$, and the model reduces to a pure ϕ^4 system with global internal $SU(2) \times SU(2) \approx O(4)$ symmetry. This behaviour cannot be recovered from the mean field equations (12) and (13) which can be viewed as having been obtained in a unitary gauge.

The correct large- β behaviour can be determined from a second mean field analysis of the system given in Eq. (5) by introducing unconstrained variables, not only for the link fields $U_\mu(x)$, but also for the angular part of the Higgs field, $\Omega(x)$. Denoting the analogue of α_ℓ by r_x and the analogue of v_ℓ by s_x , and making a translationally-invariant mean fields ansatz, the mean field equations now become

$$V = I_2(\alpha) / I_1(\alpha) \quad (15a)$$

$$S = I_2(r) / I_1(r) \quad (15b)$$

*) One can conceive examples where this may not necessarily be the case.

$$r = 4d\kappa g^2 s v \quad (15c)$$

$$\alpha = 2\beta(d-1)v^3 + 2\kappa g^2 s \quad (15d)$$

The effective potential for ρ is now

$$\begin{aligned} V_{\text{eff}} = & \frac{3}{2}\beta d(d-1)v(g)^4 + \lambda(g^2-1)^2 + g^2 - \frac{3}{2}\ln[g^2] \\ & - d\ln\left[\frac{I_1(2\beta(d-1)v(g)^3 + 2\kappa g^2 s(g)^2)}{\beta(d-1)v(g)^3 + \kappa g^2 s(g)^2}\right] \\ & + 4d\kappa g^2 s(g)^2 v(g) - \ln\left[\frac{I_1(4d\kappa g^2 s(g)^2 v(g))}{d\kappa g^2 s(g)^2 v(g)}\right] \end{aligned} \quad (16)$$

Solving Eqs. (15a) and (15d) for fixed s and r as $\beta \rightarrow \infty$, using the asymptotic form of the Bessel functions, and inserting into (15c), this becomes

$$r = 4d\kappa g^2 s \left(1 - \frac{3}{4(d-1)\beta} + \mathcal{O}(1/\beta^2)\right) \quad (17)$$

which, together with (15b), is recognized as the mean field equation of the pure Higgs system in the limit $\beta \rightarrow \infty$. Equation (17) also shows that the leading contributions from the gauge fields at weak coupling are renormalizations of κ . The pure Higgs system has a second-order phase transition at

$$\kappa_\infty = \frac{1}{3d} \left[(1-2\lambda) + \sqrt{(1-2\lambda)^2 + 12\lambda} \right] \quad (18)$$

for $\lambda > 0.5$. This is a transition in the mean value s of the angular part of the Higgs field, and has previously been found in the limit $\lambda \rightarrow \infty$, where the pure Higgs system reduces to the $SU(2) \times SU(2)$ four-dimensional chiral model¹²⁾. For smaller values of λ , the effective potential $V_{\text{eff}}(\rho)$ develops a second minimum for κ -values smaller than κ_∞ , which induces a first-order transition at a κ somewhat below κ_∞ . This feature persists in the solution of the mean field equations (15) of the coupled gauge-Higgs system for β values down to $\beta_c \approx 2.82$, i.e., in the whole coupling range where the pure gauge system is in its weak coupling phase.

It would be interesting to investigate whether the switch from second-order to first-order transitions at weak coupling is an artifact of the mean field ansatz, or a genuine property of the full gauge-Higgs system. This is particularly important in view of the claimed irrelevance of the value of the ϕ^4 coupling λ in the continuum limit.

We have plotted the phase transition lines computed from this second mean field method in Fig. 3 (fully-drawn lines continuing to $\beta = \infty$). Although this second mean field method yields the correct $\beta \rightarrow \infty$ behaviour, it clearly does not reproduce correctly the small β region. The two methods therefore complement each other nicely, with a small strip in β in which both methods are unreliable.

Let us end by pointing out a rather intriguing phenomenon. For $\lambda \lesssim 0.035$, we saw that the phase transition lines continued all the way down to $\beta = 0$. This indicates that for sufficiently small λ , the Higgs and confinement phases are in fact two distinct phases which cannot be continuously transformed into one another. [Recall that the Fradkin-Shenker analyticity argument³⁾ is valid only for fixed-length Higgs fields, i.e., $\lambda \rightarrow \infty$.] This is not just an artifact of the mean field approximation; recent MC results [by Jersak et al., Ref. 8)] indicate that for $\lambda = 0.01$ the phase transitions even go into the region $\beta < 0$, and in fact continue all the way down to $\beta = -\infty$. Since even in this case the theory can still be expressed completely in terms of gauge-invariant fields⁴⁾, it demonstrates that caution is required when "complementarity" ideas are used to argue for an analytic connection between the Higgs and confinement phases of such systems.

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FIGURE CAPTIONS

- Fig. 1 : Example of how one can locate a first-order phase transition by solving for the effective potential $V_{\text{eff}}(\rho)$, and subsequently vary the parameters until a global minimum of the effective potential becomes just local. Here exemplified for $\beta = 0$, $\lambda = 0.005$. The phase transition occurs at $\kappa = 0.1825$.
- Fig. 2 : The line of first-order phase transition points at $\beta = 0$. The end-point at $(\lambda^*, \kappa^*) = (0.035, 0.2450)$ is presumably a second-order critical point.
- Fig. 3 : The phase diagram in the κ - β plane for different values of λ . The fully-drawn lines for $\beta < \beta_c \approx 2.82$ correspond to the first mean field method, and the fully-drawn lines at $\beta > \beta_c$ correspond to the second method, as discussed in the text. MC data are taken from Jersak et al., Ref. 8).

Fig. 1

Fig. 2

Fig. 3