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STRONG COUPLING ANALYSIS OF THE CHIRAL PHASE TRANSITION
AT FINITE TEMPERATURE

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ABSTRACT

The finite temperature chiral symmetry restoration phase transition in lattice gauge theories is investigated analytically in the strong coupling region, and prediction are made regarding the order of the phase transition. We discuss the temperature dependence of m_π and f_π near the phase transition point. When the effect of baryons is ignored, we show explicitly how the phase transition is insensitive to the introduction of a chemical potential for quark densities. Finally, we derive the effective action for $\sigma \sim \langle \bar{\psi}\psi \rangle$ with different numbers of flavours.

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1. INTRODUCTION

Within the last couple of years we have witnessed a surge of renewed interest in the phenomena which are associated with finite temperature (or finite density) effects in gauge theories. No doubt, a large amount of this interest has been sparked by the possibility of achieving conditions in very high energy heavy ion collisions which could lead to the formation of the celebrated, but so far unobserved, quark-gluon plasma. Another part of the interest has, in some sense, been aroused by purely theoretical considerations. With the realization that gauge theories coupled to either scalar or fermionic matter fields generally will exhibit quite a rich phase structure, it becomes important to understand the nature and the location of the various phases. Finite temperature, then, provides a physical and tunable external parameter with which one can explore some of the different phases, and through which one can obtain information about the critical properties of the theories.

At least two phase transitions at finite temperature are of relevance to QCD: one is the deconfinement phase transition^{1),2)} in which the theory enters the liberated quark-gluon plasma phase; the other is the chiral symmetry restoration phase transition where the physical vacuum shifts from a domain of spontaneously broken chiral invariance to the chirally symmetric phase³⁾.

These two phase transitions stand on quite different grounds from a theoretical point of view. In the case of a pure $SU(N)$ invariant gauge theory at finite temperature, a well-defined order parameter (the gauge invariant Wilson-Polyakov line variable in the periodic temporal direction) exists, by means of which the deconfinement phase transition can be analyzed directly. For such a pure $SU(N)$ gauge theory a vanishing/non-vanishing Wilson-Polyakov line variable can be traced back to the spontaneous symmetry breakdown of a global $Z(N)$ invariance in the gauge sector^{1),2),4)}. Unless this $Z(N)$ symmetry is broken spontaneously, the system will remain in the confined phase. One may then hope to use universality arguments to predict the critical behaviour of gauge theories in $(d+1)$ dimensions from known properties of $Z(N)$ spin systems in d dimensions⁵⁾, or one can attempt to pin down this equivalence directly at strong coupling⁶⁾ in lattice gauge theories.

The addition of fermions to this picture makes the phase structure far more complex. If the fermions are chiral, a new phase transition - the chiral symmetry restoration phase transition - is introduced. Furthermore, light fermions are believed to have significant consequences for the deconfinement

phase transition itself: second order phase transitions should effectively disappear, and previous first order phase transitions should go away if the fermion masses are sufficiently small⁷⁾⁻¹⁰⁾. This scenario is, however, still a matter of some controversy^{11),12)}.

The chiral symmetry restoration phase transition should in principle be easier to understand than the deconfinement phase transition in the presence of light fermions. In contrast to the latter case, where the Wilson-Polyakov line variable ceases to be a good order parameter and no other viable order parameter is known, the chiral phase transition can be analyzed by a unique order parameter: the quark-antiquark vacuum condensate $\langle \bar{\psi}\psi \rangle$.

In practice, however, the situation is not nearly as simple. Using a lattice regularization, which presently is the only controllable and yet truly non-perturbative regularization scheme available, the introduction of fermions with a chiral symmetry is notoriously difficult¹³⁾, and several unwanted features (e.g., non-locality, absence of a chiral charge, etc.) are unavoidably introduced. In Monte Carlo computations^{3),14)} the introduction of light (ideally massless) dynamical quarks is a tremendously difficult task.

Most analytic discussions of chiral symmetry breaking on the lattice have so far mostly been restricted to the zero-temperature regime^{15),18)}, although there has recently appeared three independent analyses of chiral symmetry breaking (and its restoration) in the Hamiltonian formalism^{7),19),20)}. We shall return briefly to these analyses in the continuous-time Hamiltonian formalism later on in this paper.

Our aim with the present paper is to develop some understanding of the finite temperature chiral symmetry restoration phase transition directly in the Lagrangian formulation of lattice gauge theories. By going to the strong coupling region, all gauge degrees of freedom can explicitly be integrated out of the functional integral. The resulting effective action is then a functional of only the fermionic variable ψ and $\bar{\psi}$. From a standard effective Lagrangian analysis, a composite field $\phi \sim \langle \bar{\psi}\psi \rangle$ is introduced, whereupon the Grassmann integrations are performed. The result is an effective action for $\langle \bar{\psi}\psi \rangle$. This work, which is an extended and detailed description of results reported briefly elsewhere²¹⁾, can therefore be viewed as the finite temperature generalization of the effective potential analyses of Refs. 17) and 18).

The paper is organized as follows. In Section 2 we introduce the strong coupling lattice action and define what we mean by "chiral symmetry" on the lattice. Section 3 is devoted to a step-by-step derivation of the effective action for the chiral condensate, here temporarily restricted to the case $N_f = 1$. The temperature dependence of this effective action is analyzed in Section 4, and a second order phase transition is shown to occur for all finite values of N . (As $N \rightarrow 0$ this phase transition disappears.) At the point of the phase transition, chiral symmetry becomes exact and a suitably defined "pion" field becomes massless. Near the phase transition one can derive the temperature dependence of the pion mass, as well as the variation on T of the pion decay constant f_π . In Section 5 we introduce a chemical potential μ coupled to the quark density $\psi^\dagger \psi$. Restricting ourselves to the gauge group $U(N)$, where colour singlet baryons are absent, we show explicitly how the chiral phase transition is insensitive to the chemical potential. (As expected, since the effective action is written explicitly in terms of mesonic operators which carry zero baryon number.) The dependence on N_f , the number of "strong coupling flavours" is analyzed in Section 6. Finally, Section 7 contains a discussion of our results.

2. THE LATTICE ACTION

We begin this section with a short description of our choice of fermionic action. As we are going to discuss the chiral properties of lattice gauge theories, a natural starting point for the fermionic part of the action is the "naive" fermion formulation:

$$S = - \sum_x \bar{\psi}(x) M \psi(x) + \frac{1}{2} \sum_x \sum_\mu \{ \bar{\psi}(x) \gamma_\mu U_\mu(x) \psi(x+\mu) - \bar{\psi}(x+\mu) \gamma_\mu U_\mu^\dagger(x) \psi(x) \} \quad (2.1)$$

Here, we choose the link variables $U_\mu(x)$ to be elements of the colour gauge group $U(N)$. Working in Euclidean space, the Hermitian γ -matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. From now on, let the fermions always belong to the fundamental representation of the gauge group.

When $M = 0$ the above action contains a hidden global $U(N_f) \times U(N_f)$ symmetry, with N_f being the number of independent flavours. This continuous symmetry can be made explicit by spin diagonalization¹⁷⁾:

$$\Psi(x) = T(x) \chi(x), \quad \bar{\Psi}(x) = \bar{\chi}(x) T^\dagger(x) \quad (2.2)$$

where

$$T(x) = (\gamma_0)^{x_0} (\gamma_1)^{x_1} (\gamma_2)^{x_2} \dots (\gamma_d)^{x_d} \quad (2.3)$$

in $(d+1)$ dimensions.

In terms of the χ -fields the action (2.1) now reads

$$S = - \sum_x \bar{\chi}(x) M \chi(x) + \frac{1}{2} \sum_x \sum_\mu \gamma_\mu(x) \{ \bar{\chi}(x+\mu) U_\mu(x) \chi(x) - \bar{\chi}(x) U_\mu^\dagger(x) \chi(x+\mu) \} \quad (2.4)$$

where

$$\gamma_\mu(x) = T^\dagger(x) \gamma_\mu T(x+\mu) = T^\dagger(x+\mu) \gamma_\mu T(x) \quad (2.5)$$

or, explicitly,

$$\gamma_0(x) = 1, \quad \gamma_1(x) = (-1)^{x_0}, \quad \gamma_2(x) = (-1)^{x_0+x_1}, \quad \dots \quad (2.6)$$

Note that by the unitary transformation (2.2) the action has become diagonal in the spinor indices. Without any loss of generality, we can therefore restrict ourselves to just one spinor component.

Furthermore, note that in the limit $M = 0$ the action (2.4) is invariant under the transformations

$$\chi(x) \rightarrow \begin{cases} V \chi(x), & \epsilon(x) = +1 \\ W \chi(x), & \epsilon(x) = -1 \end{cases} \quad (2.7)$$

$$\bar{\chi}(x) \rightarrow \begin{cases} \bar{\chi}(x) W^\dagger, & \epsilon(x) = +1 \\ \bar{\chi}(x) V^\dagger, & \epsilon(x) = -1 \end{cases} \quad (2.8)$$

where $\varepsilon(x) = (-1)^{x_0+x_1+x_2+x_3+\dots}$ and $V, W \in U(N_f)$. This "odd-even site" symmetry is precisely the global $U(N_f) \times U(N_f)$ invariance adhered to above. The mass term in the action (2.4) explicitly breaks this symmetry: $U(N_f) \times U(N_f) \rightarrow U(N_f)$. The action is now only invariant under the transformations (2.7), (2.8) if $V = W$ globally. We identify this $U(N_f) \times U(N_f)$ symmetry as "chiral symmetry" on the lattice.

Even in the absence of a mass term the $U(N_f) \times U(N_f)$ chiral symmetry is spontaneously broken at zero temperature: $U(N_f) \times U(N_f) \rightarrow U(N_f)$ ^{17),18)}. This pattern of chiral symmetry breaking agrees precisely with what was expected²²⁾ from a large- N analysis in the continuum. The purpose of this paper is to analyze how the symmetry can be recovered at finite temperature.

Before proceeding to this point, let us try to clarify the relationship between the lattice "chiral symmetry" (as we have defined it above) and the standard global chiral-flavour symmetry of continuum QCD. Ignoring the pure gauge piece, the continuum QCD Lagrangian reads

$$\mathcal{L}_{\text{QCD}}^{\text{cont.}} = \bar{\Psi}_\alpha^a (i \not{\partial} \delta_{ab} - \not{A}_{ab}) \Psi_\alpha^b = \bar{\Psi}_L (i \not{\partial} - \not{A}) \Psi_L + \bar{\Psi}_R (i \not{\partial} - \not{A}) \Psi_R \quad (2.9)$$

with

$$\begin{aligned} \Psi_L(x) &= \frac{1-\gamma_5}{2} \Psi(x), & \bar{\Psi}_L(x) &= \bar{\Psi}(x) \frac{1+\gamma_5}{2} \\ \Psi_R(x) &= \frac{1+\gamma_5}{2} \Psi(x), & \bar{\Psi}_R(x) &= \bar{\Psi}(x) \frac{1-\gamma_5}{2} \end{aligned} \quad (2.10)$$

Here a and b are colour indices, and $\alpha = 1, 2, \dots, N_f$ is the flavour index.

$\mathcal{L}_{\text{QCD}}^{\text{cont.}}$ is invariant under the following transformations:

$$SU_R(N_f): \Psi_R' = e^{i\theta_a \tau^a \gamma_5} \Psi_R, \quad \bar{\Psi}_R' = \bar{\Psi}_R e^{i\theta_a \tau^a \gamma_5} \quad (2.11)$$

and

$$SU_L(N_f): \Psi_L' = e^{i\tilde{\theta}_a \tau^a \gamma_5} \Psi_L, \quad \bar{\Psi}_L' = \bar{\Psi}_L e^{i\tilde{\theta}_a \tau^a \gamma_5} \quad (2.12)$$

where τ^a is an $SU(N_f)$ generator.

In addition, $\mathcal{L}_{\text{QCD}}^{\text{cont.}}$ has two further symmetries. The first is quark number conservation:

$$U_B(1): \quad \psi' = e^{i\theta} \psi, \quad \bar{\psi}' = \bar{\psi} e^{-i\theta} \quad (2.13)$$

and the second is the axial flavour singlet invariance:

$$U_A(1): \quad \psi' = e^{i\theta\gamma_5} \psi, \quad \bar{\psi}' = \bar{\psi} e^{i\theta\gamma_5} \quad (2.14)$$

This last symmetry is broken by the axial anomaly (or by instantons), so the only residual symmetry of $\mathcal{L}_{\text{QCD}}^{\text{cont.}}$ is $SU_L(N_f) \times SU_R(N_f) \times U_B(1)$. The $SU_L(N_f) \times SU_R(N_f)$ part of this symmetry is finally believed to be broken spontaneously to $SU(N_f)$ by the formation of a (flavour independent) chiral condensate $\langle \bar{\psi}\psi \rangle$.

In our lattice formulation $\epsilon(x)$ [defined after Eq. (2.8)] plays the rôle of γ_5 , and "left" and "right" symmetry is replaced by "odd" and "even". Here the "odd" and "even" characters are originated by the number $\sum_{\mu=0}^d x_\mu$ or, equivalently, by the sign of $\epsilon(x)$. To be explicit, we can define

$$\begin{aligned} \chi_L(x) &= \frac{1-\epsilon(x)}{2} \chi(x), & \bar{\chi}_L(x) &= \bar{\chi}(x) \frac{1+\epsilon(x)}{2} \\ \chi_R(x) &= \frac{1+\epsilon(x)}{2} \chi(x), & \bar{\chi}_R(x) &= \bar{\chi}(x) \frac{1-\epsilon(x)}{2} \end{aligned} \quad (2.15)$$

in analogy with Eq. (2.10)

Then,

$$\chi(x) = \frac{1-\epsilon(x)}{2} \chi(x) + \frac{1+\epsilon(x)}{2} \chi(x) = \chi_L(x) + \chi_R(x) \quad (2.16)$$

and similarly $\bar{\chi}(x) = \bar{\chi}_L(x) + \bar{\chi}_R(x)$. Since

$$\bar{\chi}_R(x) \chi_L(x+\mu) = \bar{\chi}(x) \frac{1-\epsilon(x)}{2} \cdot \frac{1-\epsilon(x+\mu)}{2} \chi(x+\mu) = 0 \quad (2.17)$$

we also have

$$\bar{\chi}(x)\chi(x+\mu) = \bar{\chi}_L(x)\chi_L(x+\mu) + \bar{\chi}_R(x)\chi_R(x+\mu) \quad (2.18)$$

Our original lattice action (2.4) then takes the form

$$\begin{aligned} S = & \frac{1}{2} \sum_{x,\mu} \gamma_\mu(x) \{ \bar{\chi}_L(x) U_\mu(x) \chi_L(x+\mu) - \bar{\chi}_L(x+\mu) U_\mu^\dagger(x) \chi_L(x) \} \\ & + \frac{1}{2} \sum_{x,\mu} \gamma_\mu(x) \{ \bar{\chi}_R(x) U_\mu(x) \chi_R(x+\mu) - \bar{\chi}_R(x+\mu) U_\mu^\dagger(x) \chi_R(x) \} \end{aligned} \quad (2.19)$$

This action is invariant under

$$\begin{aligned} \bar{\chi}'_L(x) &= \bar{\chi}_L(x) W^\dagger, \quad \bar{\chi}_R(x) = 0 \\ \chi_R(x+\mu) &= W \chi_L(x+\mu), \quad \chi_R(x+\mu) = 0 \end{aligned} \quad (2.20)$$

and

$$\begin{aligned} \bar{\chi}'_R(x) &= \bar{\chi}_R(x) V^\dagger, \quad \bar{\chi}_L(x) = 0 \\ \chi'_R(x+\mu) &= V \chi_R(x+\mu), \quad \chi_L(x+\mu) = 0 \end{aligned} \quad (2.21)$$

This shows that we can interpret our lattice symmetry as a $U_L(N_f) \times U_R(N_f)$ "left-right" symmetry.

It should be emphasized that the above chiral-flavour symmetry is not equivalent to the flavour symmetries of the Susskind fermion formalism, but rather has a more direct connection to flavours in continuum QCD. Let us clarify this point by noting that (in four dimensions) we can introduce four times as many flavours as N_f by identifying the 16 "thinned out" species doublers with new flavours. The two flavours sitting on even sites [corresponding to $\varepsilon(x) = +1$] have positive chirality, while the two flavours sitting on odd sites [with $\varepsilon(x) = -1$] have negative chirality. The full chiral symmetry of the lattice action can then be identified as

$$(U_L(N_f) \times e^{i\theta\gamma_5} \mathbb{1})_+ \times (U_R(N_f) \times e^{-i\theta\gamma_5} \mathbb{1})_-$$

for $4N_f$ flavours. ($\mathbb{1}$ is a 2×2 matrix which acts only on the same chiral fermion.) Here $()_+$ and $()_-$ act on even odd sites, respectively.

To illustrate this, consider as an example a two-dimensional free fermion theory,

$$S = \frac{1}{2} \sum_{x,\mu} \gamma_\mu(x) \{ \bar{\chi}(x) \chi(x+\mu) - \bar{\chi}(x+\mu) \chi(x) \} \quad (2.22)$$

Now define, with the notation of Fig. 1, where $x_1(E,E)$ is defined on the site; $(x_1, x_2) = (\text{Even}, \text{Even})$ and so on:

$$u \equiv \begin{pmatrix} \chi_2 \\ \chi_3 \end{pmatrix} \quad d \equiv \begin{pmatrix} \chi_1 \\ \chi_4 \end{pmatrix}$$

$$\bar{u} \equiv (\bar{\chi}_1, \bar{\chi}_4) \quad \bar{d} \equiv (\bar{\chi}_2, \bar{\chi}_3)$$

We can identify u and d as the up and down quarks, respectively. S is then invariant under

$$\chi_3 \rightarrow \chi_2, \chi_1 \rightarrow \chi_4, \chi_2 \rightarrow -\chi_3 \text{ and } \chi_4 \rightarrow -\chi_1$$

which translates into invariance under the transformation

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = e^{i\theta\gamma_5\tau_3} \begin{pmatrix} u \\ d \end{pmatrix} \quad (2.23)$$

with

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Equivalently,

$$u' = e^{i\theta\gamma_5} u \quad (\epsilon(x) = +1)$$

$$d' = e^{-i\theta\gamma_5} d \quad (\epsilon(x) = -1) \quad (2.24)$$

This shows that the u quark has positive chirality, while the d quark has negative chirality. The d -dimensionally case works equivalently, and will hence not be constructed explicitly here.

3. THE EFFECTIVE LAGRANGIAN AT FINITE TEMPERATURE

Finite temperature on the lattice is introduced in the standard way by making the link variables $U_\mu(x)$ periodic in the Euclidean time direction with period β : $U_\mu(\tau+\beta, x) = U_\mu(\tau, x)$. (From now on the variable x will be used to indicate spatial co-ordinates only.) Similarly, the fermion fields are constrained to be antiperiodic: $\chi(\tau+\beta, x) = -\chi(\tau, x)$. In the strong coupling limit, our finite temperature action reads

$$S = \sum_{\tau} \sum_x \left[-\bar{\chi}(\tau, x) M \chi(\tau, x) + \frac{1}{2} a_s^{-1} \sum_j \eta_j(\tau, x) \left\{ \bar{\chi}(\tau, x) U_j(\tau, x) \chi(\tau, x+j) - \bar{\chi}(\tau, x+j) U_j^\dagger(\tau, x) \chi(\tau, x) + \frac{1}{2} a_\tau^{-1} \left\{ \bar{\chi}(\tau, x) U_0(\tau, x) \chi(\tau+\delta, x) - \bar{\chi}(\tau+\delta, x) U_0^\dagger(\tau, x) \chi(\tau, x) \right\} \right] \right. \quad (3.1)$$

and the physical temperature equals $T = (a_\tau \beta)^{-1}$. Here the τ -sum runs from $\tau = 1$ to β and the j -sum runs from $j = 1$ to d . In the following, let us for notational convenience set $a_\tau = a_s = 1$, which corresponds to a symmetric lattice. Appropriate factors of a_τ and a_s will be reinstated in our final results.

The periodicity constraint on $U_0(\tau, x)$, together with the fact that the Wilson-Polyakov line $\text{Tr}[\prod_\tau U_0(\tau, x)]$ now is a gauge invariant quantity, severely restricts our freedom in choosing a gauge in the Euclidean time direction. Clearly, we are allowed to choose $U_0(\tau+n\delta, x) = 1$ for $n = 0, \dots, \beta-2$. The remaining last link $U_0(\tau+(\beta-1)\delta, x)$ can be chosen to be in diagonal form

$$U_0(\tau+(\beta-1)\delta, x) = \text{diag.}(e^{i\theta_1(x)}, e^{i\theta_2(x)}, \dots, e^{i\theta_N(x)})$$

By a gauge transformation, this can be put in a more symmetric form, in which all timelike links are equal^{8), 21)}:

$$U_0(\tau+n\delta, x) = D(x) = \text{diag.}(e^{i\theta_1(x)/\beta}, e^{i\theta_2(x)/\beta}, \dots, e^{i\theta_N(x)/\beta}) \quad (3.2)$$

for $n = 0, \dots, \beta-1$. As U_0 is τ -independent, this corresponds to a "static" gauge choice.

The measure corresponding to Eq. (3.2) is readily found²³⁾:

$$\int dU_0 = \int_{-\pi}^{\pi} \prod_{l=1}^N \left(\frac{d\theta_l}{2\pi} \right) \Delta[D] \quad (3.3)$$

with the Jacobian

$$\Delta[U_0] = \Delta[D] = \prod_{k < l}^N |e^{i\theta_k(x)} - e^{i\theta_l(x)}|^2 \quad (3.4)$$

It is worthwhile emphasizing at this point that as we shall later integrate out exactly dU_0 (and $d\chi d\bar{\chi}$), our final answer will be completely gauge invariant. The gauge choice (3.2) is simply a convenient computational device. Our action now reads

$$\begin{aligned} S = \sum_{\tau} \sum_x & \left[-\bar{\chi}(\tau, x) M \chi(\tau, x) + \frac{1}{2} \sum_j \gamma_j(\tau, x) \{ \bar{\chi}(\tau, x) U_j(\tau, x) \chi(\tau, x+j) \right. \\ & - \bar{\chi}(\tau, x+j) U_j^\dagger(\tau, x) \chi(\tau, x) \} + \frac{1}{2} \{ \bar{\chi}(\tau, x) D(x) \chi(\tau+\delta, x) \\ & \left. - \bar{\chi}(\tau+\delta, x) D^*(x) \chi(\tau, x) \} \right] \end{aligned} \quad (3.5)$$

where the sum on j runs from 1 to d .

From now on consider the case of what in this strong coupling region corresponds to just one flavour, i.e., $N_f = 1$.

Note that if we define (a, b being colour indices)

$$J_j^\dagger(\tau, x)_a^b = \frac{1}{2} \bar{\chi}^b(\tau, x) \chi_a(\tau, x+j) \gamma_j(\tau, x) \quad (3.6)$$

and

$$J_j(\tau, x)_a^b = -\frac{1}{2} \bar{\chi}^b(\tau, x+j) \chi_a(\tau, x) \gamma_j(\tau, x) \quad (3.7)$$

then all U_j -integrations factor into products of one-link integrals of the form

$$\int dU_j e^{\text{Tr} \{ J_j^\dagger U_j + U_j^\dagger J_j \}} = \exp(W[J, J^\dagger]) \quad (3.8)$$

Such one-link integrals are well-known from the literature²⁴⁾⁻²⁶⁾. [The

fact that J and $J^\dagger$ in our case are composed of Grassmann parameters does not lead to any complications, see Refs. 17) and 18).] For the gauge group $U(N)$ the function $W[J, J^\dagger]$ of Eq. (3.8) has an expansion²⁶⁾:

$$W[J, J^\dagger] = \text{Tr}[J^\dagger J] / N - \text{Tr}[J^\dagger J]^2 / 2N(N^2 - 1) + \dots \quad (3.9)$$

We are therefore now able to explicitly perform the U_j -integrations. From the analysis of the zero-temperature case^{17), 18)}, it is known that retaining only the leading order with respect to the composite field $J_j(\tau, x)$ is a very good approximation compared with the full effective action. We therefore keep only the leading order term in our analysis. This corresponds to dropping higher order terms in a systematic $1/d$ expansion¹⁸⁾, and hence becomes exact in the limit of high dimensionality. Explicitly displaying the colour index dependence in the four-fermion term, our action (3.5) then becomes

$$S = \sum_{\tau} \sum_x \left[-\bar{\chi}(\tau, x) M \chi(\tau, x) - \frac{1}{4N} \sum_j \bar{\chi}^a(\tau, x) \chi_b(\tau, x+j) \bar{\chi}^b(\tau, x+j) \chi_a(\tau, x) \right. \\ \left. + \frac{1}{2} \left\{ \bar{\chi}(\tau, x) D(x) \chi(\tau+\delta, x) - \bar{\chi}(\tau+\delta, x) D^*(x) \chi(\tau, x) \right\} \right] \quad (3.10)$$

We now come to an important point in the derivation of our effective action. Roughly speaking, we wish to trade the fermionic integrations in our partition function with an integration over a single scalar field $\sigma(\tau, x)$ which will serve as an elementary field of the composite operator $\bar{\chi}(\tau, x) \chi(\tau, x)$. [The definition of an effective action for such composite operators is a problem beset with ambiguities, some of which are discussed in Ref. 27).] Fortunately, as long as the action does not contain fermionic terms higher than four-fermion interactions, there is one well-defined way of introducing an elementary field $\sigma(\tau, x)$ for a composite fermionic operator. This is based on the identity

$$\begin{aligned}
 & e^{\frac{1}{4N} \sum_{x,j} \chi_a(\tau, x) \bar{\chi}^a(\tau, x) \chi_b(\tau, x+j) \bar{\chi}^b(\tau, x+j)} \\
 & = \int d[\sigma] \exp \left\{ \frac{1}{4N} \sum_{x,j} (N\sigma(\tau, x) - \chi_a(\tau, x) \bar{\chi}^a(\tau, x)) (N\sigma(\tau, x+j) - \chi_a(\tau, x+j) \bar{\chi}^a(\tau, x+j)) \right. \\
 & \quad \left. + \frac{1}{4N} \sum_{x,j} \chi_a(\tau, x) \bar{\chi}^a(\tau, x) \chi_b(\tau, x+j) \bar{\chi}^b(\tau, x+j) \right\} \quad (3.11) \\
 & = \int d[\sigma] \exp \left\{ -\frac{N}{4} \sigma(\tau, x) \sigma(\tau, x+j) + \sigma(\tau, x) \chi_a(\tau, x+j) \bar{\chi}^a(\tau, x+j) + \sigma(\tau, x+j) \chi_a(\tau, x) \bar{\chi}^a(\tau, x) \right\}
 \end{aligned}$$

This way of introducing a scalar field $\sigma(\tau, x)$ provides us with an effective linearization of the four-fermion term. Furthermore, as can be seen from Eq. (3.11), it also gives us the correspondence $\sigma(\tau, x) \sim N^{-1} \bar{\chi}_a(\tau, x) \chi^a(\tau, x)$, an identification which is made precise in vacuum expectation values.

Note that the effective Lagrangian obtained in this way has kept the $U(N_f) \times U(N_f)$ symmetry discussed in the previous section. In addition to the transformations (2.7) and (2.8), we see that $\sigma(\tau, x)$ transforms as

$$\sigma(\tau, x) \rightarrow \begin{cases} W \sigma(\tau, x) V^\dagger, & \epsilon(x) = -1 \\ V \sigma(\tau, x) W^\dagger, & \epsilon(x) = +1 \end{cases}$$

Under these combined transformations the action is still invariant.

Imposing translational invariance (in the discrete sense of the lattice), we expand $\sigma(\tau, x)$ around its constant background value σ and ignore translationally non-invariant vacuum fluctuations. In this way we automatically ensure a constant $\langle \bar{\chi} \chi \rangle$ vacuum condensate. Computationally, this effectively reduces the fermion dynamics to a tractable one-dimensional problem. With the help of the identity (3.11) our effective action becomes

$$\begin{aligned}
 S = \sum_{\tau} \sum_x & \left[-\bar{\chi}(\tau, x) M \chi(\tau, x) - \frac{Nd}{4} \sigma^2 - \frac{d}{2} \bar{\chi}(\tau, x) \sigma \chi(\tau, x) \right. \\
 & \left. + \frac{1}{2} \{ \bar{\chi}(\tau, x) D(x) \chi(\tau+\delta, x) - \bar{\chi}(\tau+\delta, x) D^*(x) \chi(\tau, x) \} \right] \quad (3.12)
 \end{aligned}$$

The final step in deriving the effective action for σ now consists in integrating out the fermion fields and in performing the U_0 -integration. We consider each in turn.

As the action (3.12) is bilinear in the fermion fields, the only complication in connection with the Grassmann integrations is the incorporation of the antiperiodicity constraint in the temporal direction. Introducing the partial Fourier transform

$$\chi(\tau, x) = \frac{1}{\sqrt{\beta}} \sum_{n=1}^{\beta} e^{2\pi i (n+1/2)\tau/\beta} \chi_n(x) \quad (3.13)$$

[and similarly for the $\bar{\chi}(\tau, x)$ field] we automatically take care of the antiperiodicity:

$$\chi(\tau+\beta, x) = -\chi(\tau, x) \quad (3.14)$$

The functional integral over the fermion fields can now be performed directly over the unconstrained Fourier components $\chi_m(x)$, $\chi_n(x)$ in a standard way.

After performing the $\sum_{\tau=1}^{\beta}$ summation in Eq. (3.12) the action S , upon substitution of Eq. (3.13), reads

$$S = \sum_x \left[-\frac{Nd}{4} \beta \sigma^2 - \sum_b \sum_n \bar{\chi}_n^b(x) \left\{ (M + \frac{1}{2} \sigma d) - i \sin([2\pi(n+1/2) + \theta_0]/\beta) \right\} \chi_b^n(x) \right] \quad (3.15)$$

which after integrating out the fermion fields gives rise to the following partition function:

$$Z = \int d[\sigma] d[U_0] e^{-\frac{Nd}{4} \beta \sum_x \sigma^2} \prod_x \prod_b \prod_n \left\{ \frac{1}{2} d\sigma - i \sin([2\pi(n+1/2) + \theta_0]/\beta) \right\} \quad (3.16)$$

In Eq. (3.16) we have dropped the quark mass term. The effect of such a mass term can easily be taken into account by a substitution $\sigma \rightarrow \sigma + 2M/d$ in the second term of Eq. (3.16)

Observe that in this "staggered" fermion formalism [e.g., Eq. (2.4)], the periodicity parameter β is an even integer. The fact that β is even can be used to derive the identity

$$\begin{aligned} & \prod_{n=1}^{\beta} [\sinh(s) - i \cos(2\pi n/\beta + \theta)] \\ &= \frac{1}{2^{\beta-1}} [\cosh(\beta s) - i^{\beta} \cos(\beta \theta)] \end{aligned} \quad (3.17)$$

Introducing

$$s \equiv \sinh^{-1}(\frac{1}{2} \sigma d) \quad (3.18)$$

the identity (2.25) can now be used to eliminate the $\prod_{n=1}^{\beta}$ product in the fermion determinant appearing in the partition function (3.16). With a suitable normalization the partition function then takes the form

$$Z = \int d[\sigma] d[U_0] e^{-\frac{Nd}{4} \beta \sum_x \sigma^2} \prod_x \prod_b [2 \cosh(\beta s) + 2 \cos(\theta_b)] \quad (3.19)$$

where s is as defined in Eq. (3.18).

Remarkably, the U_0 -integration can now be performed exactly for all N . This is noted by recognizing that in the measure $d[U_0]$ of Eq. (3.3) the Jacobian $\Delta[U_0]$ equals

$$\Delta[D] = \prod_{k < l}^N |e^{i\theta_k(x)} - e^{i\theta_l(x)}|^2 = |V(e^{i\theta_1(x)}, \dots, e^{i\theta_N(x)})|^2 \quad (3.20)$$

where $V(x_1, \dots, x_N)$ is the Vandermonde determinant of order N . If we define

$$I_n \equiv \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} W(\theta) e^{in\theta} \quad (3.21)$$

with $W(\theta)$ chosen in such a way that $I_n = I_{-n}$, then an orthogonal polynomial method²⁸⁾ provides us with the following formula:

$$\int_{-\pi}^{\pi} \prod_b \left\{ \frac{d\theta_b}{2\pi} W(\theta_b) \right\} |V(e^{i\theta_1}, \dots, e^{i\theta_N})|^2 = N! \det \begin{vmatrix} I_0 & I_1 & \dots & I_{N-1} \\ I_1 & I_0 & & \\ & \ddots & \ddots & \\ I_{N-1} & & & I_0 \end{vmatrix} \quad (3.22)$$

We shall present the solution to the general case where $I_n = I_{-n}$ later on. Here it suffices to note that with

$$W(\theta) = 2 \cosh(\beta s) + 2 \cos(\theta) \quad (3.23)$$

we in fact do have $I_n = I_{-n}$. Explicitly, in our case:

$$I_0 = 2 \cosh(\beta s), \quad I_1 = 1, \quad I_n = 0 \text{ for } n \geq 2 \quad (3.24)$$

which leaves us with the evaluation of the determinant

$$C_N = \det \begin{vmatrix} I_0 & I_1 & & & 0 \\ I_1 & I_0 & I_1 & & \\ & I_1 & I_0 & \ddots & \\ 0 & & \ddots & I_1 & I_0 \end{vmatrix} \quad (3.25)$$

Using the recursion relation

$$C_N = I_0 C_{N-1} - I_1^2 C_{N-2} \quad (3.26)$$

this determinant is readily evaluated for all N . First the associated quadratic equation $t^2 - I_0 t + I_1^2 = 0$ is solved, and the two solutions $t_{\pm}$ are then used to construct $C_N = a_1 (t_+)^N + a_2 (t_-)^N$. Solving for the normalization constants a_1 and a_2 by means of the fixed values $C_0 = 1$, $C_1 = I_0$ we find the solutions

$$C_N = \begin{cases} I_1^N \frac{\sinh[(N+1)\cosh^{-1}(I_0/2I_1)]}{\sinh[\cosh^{-1}(I_0/2I_1)]} & , I_0 \geq 2I_1 \\ I_1^N \frac{\sin[(N+1)\cos^{-1}(I_0/2I_1)]}{\sin[\cos^{-1}(I_0/2I_1)]} & , I_0 < 2I_1 \end{cases} \quad (3.27)$$

Substituting Eqs. (3.18) and (3.24) this gives us the solution

$$C_N = \sinh[(N+1)\beta s] / \sinh[\beta s] \quad (3.28)$$

and we are then finally left with the partition function

$$Z = \int d[\sigma] \exp\left\{ \sum_x \left[-\frac{Nd}{4} \beta \sigma^2 + \ln[C_N] \right] \right\} \quad (3.29)$$

corresponding to the effective potential V_{eff} :

$$V_{\text{eff}} = \frac{Nd}{4} \beta \sigma^2 - \ln \left\{ \frac{\sinh[(N+1)\beta s]}{\sinh[\beta s]} \right\} \quad (3.30)$$

and s as given by $s = \sinh^{-1}(\frac{1}{2}\sigma d)$.

In particular, note that the effective potential takes a particularly simple form in the large N limit:

$$V_{\text{eff}} = N\beta[\sigma^2 d/4 - \sinh^{-1}(\frac{1}{2}\sigma d)] \quad \text{as } N \rightarrow \infty \quad (3.31)$$

(here assuming $\sigma > 0$; for $\sigma < 0$ the formula is similar, with only the sign of the second term changed. The effective potential is always even in σ).

This completes our derivation of the effective potential at finite temperature. Having integrated out all gauge degrees of freedom from the partition function, and having substituted an integration over $\sigma \sim N^{-1} \bar{\chi}\chi$ for the fermion integrations, we are left with an effective action $V_{\text{eff}}(\sigma)$ from which chiral symmetry breaking and chiral symmetry restoration can be analyzed directly. This will be the subject of the next section.

4. THE PHASE TRANSITION

The effective potential V_{eff} which we derived in the previous section has a characteristic form as shown in Fig. 2 where we have displayed the potential for the case $N = 3$ in $(3+1)$ dimensions. Clearly, for sufficiently large β the vacuum will not be invariant under chiral transformations: chiral symmetry is spontaneously broken. However, as the temperature increases (and β becomes smaller), the value of σ at which the effective potential V_{eff} acquires its minimum moves continuously towards zero. For sufficiently small σ we can expand the potential:

$$V_{\text{eff}} = N\beta d \left(1 - \frac{(N+2)\beta d}{6} \right) \sigma^2/4 + \mathcal{O}(\sigma^4) \quad (4.1)$$

We note that we have

$$\left(\partial^2 V_{\text{eff}} / \partial \sigma^2 \right)_{\sigma=0} = 0 \quad (4.2)$$

at the point where

$$(N+2)\beta d/6 = 1 \quad (4.3)$$

Reinstating the lattice spacings a_τ and a_s we hence see a clear second order phase transition at the critical temperature $T_{\text{ch}} = 1/a_\tau \beta_{\text{ch}}$ determined by

$$\tilde{\beta} = \left(\frac{a_s}{a_\tau} \right)^2 \beta_{\text{ch}} = \frac{6}{(N+2)d} \quad (4.4)$$

For temperatures $T < T_{\text{ch}}$ chiral symmetry is always broken, whereas for $T > T_{\text{ch}}$ we have chiral symmetry restoration.

In the limit $N \rightarrow \infty$ the situation is different. Expanding the large- N effective potential (3.31) for small σ we find:

$$V_{\text{eff}} = N\beta d \left(-\frac{1}{2} \sigma + \frac{1}{4} \sigma^2 + \mathcal{O}(\sigma^3) \right) \quad (4.5)$$

which shows that in this limit the phase transition disappears. Hence, in the $N \rightarrow \infty$ limit chiral symmetry is always spontaneously broken. Note that this agrees with the $N \rightarrow \infty$ limit of Eq. (4.4).

What is the effect of introducing a bare quark mass term? From Eq. (3.15), we know that this is equivalent to making a shift $\sigma \rightarrow \sigma + 2M/d$ in the second term

of the effective potential Eq. (3.36). A typical plot of the effective potential now looks as shown in Fig. 3. Again expanding out the potential for small σ we get, up to order M^2 :

$$V_{\text{eff}} = - \frac{N(N+2)}{24} M d \beta^2 \sigma + \frac{N \beta d}{4} \left(1 - \frac{(N+2) \beta d}{6} \right) \sigma^2 + \mathcal{O}(\sigma^3) \quad (4.6)$$

It is clear that in this case the phase transition is completely washed away. This phenomenon has an intuitive physical interpretation if we make a comparison with the standard Landau-Ginzburg theory of magnetization. The order parameter $\langle \bar{\chi} \chi \rangle$ (represented here by σ) then plays the rôle of spontaneous magnetization. As can be seen from Eq. (4.6), the bare quark mass couples directly and linearly to the order parameter σ . This means that it acts as an external magnetic field. Generally external magnetic fields tend to align the order parameter and prohibit the appearance of long-distance correlations: second order phase transition will be smoothed out. Note that within this analogy with ferromagnetic systems, the magnetized, ordered, state corresponds to the phase of broken chirality, whereas the disordered state corresponds to the chirally symmetric phase. (Of course, even in the presence of an external field - i.e., a bare quark mass term - one should still witness a rapid crossover region in moving from a state of strongly broken chirality to a state of very weakly broken chirality. This crossover region should then be the only remnant of the underlying chiral phase transition).

We would like to emphasize that our effective potential has exactly the same features as the standard σ -model or, equivalently, as the Landau-Ginzburg potential. The important difference is that the temperature-dependent coefficients of the potential actually can be explicitly derived in our case, giving us the opportunity to calculate explicitly the "pion mass" m_π and the corresponding pion decay constant f_π as a function of temperature.

By analogy with the σ -model Lagrangian, we expect the following qualitative features near the chiral symmetry restoration temperature:

$$\begin{aligned}
 & \left. \begin{aligned} m_{\pi}^2 &= 0 \\ f_{\pi}^2 &\propto (\beta - \beta_{ch}) \end{aligned} \right\} \text{ for } \beta \geq \beta_{ch} \\
 & \left. \begin{aligned} m_{\pi}^2 &\propto (\beta - \beta_{ch}) \\ f_{\pi}^2 &= 0 \end{aligned} \right\} \text{ for } \beta < \beta_{ch}
 \end{aligned} \tag{4.7}$$

Thus, in the low temperature region ($\beta > \beta_{ch}$) chiral symmetry is spontaneously broken and the massless pion appears as the associated Goldstone boson. Similarly, f_{π} becomes non-zero through the corresponding vacuum expectation value.

In the high temperature region ($\beta < \beta_{ch}$) chiral symmetry is restored, the pion becomes massive and we obtain $f_{\pi} = 0$. We have explicitly verified the relations of Eq. (4.7) using our effective Lagrangian. In order to obtain the pion mass one needs to consider fluctuations $\delta\sigma(\tau, \mathbf{x})$ around the mean field σ . Assuming that temporal fluctuations of $\sigma(\tau, \mathbf{x})$ in the periodic time direction can be ignored, the $\mathbf{x}$ -dependent effective Lagrangian follows trivially from Eq. (3.29). The pion mass can now be computed from the exponential damping factor of the σ Green function at spacelike separation, $G(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \sim e^{-m|\vec{\mathbf{x}} - \vec{\mathbf{y}}|}$. This mass behaves precisely as indicated in Eq. (4.7).

As we see, the critical exponent associated with m_{π} turns out to be $\frac{1}{2}$. This is, of course, just a consequence of the mean field techniques used. Only in sufficiently large dimensions do we expect this prediction to be reliable.

In order to calculate f_{π} we again use the correspondence to the σ -model, and identify $f_{\pi} = 2\langle\sigma_{\text{norm}}\rangle$ where σ is the appropriately normalized σ -field. Without computing this normalization, we notice that $\langle\sigma\rangle$ can be calculated near β_{ch} by expanding (3.30) up to order $O(\sigma^6)$:

$$V_{\text{eff}} = \frac{Nd\beta}{4} \sigma^2 - \frac{U(N+2)}{24} d^2 \beta^2 \sigma^2 - \frac{N(U+2)}{2880} (U^2 + 2N + 2) \beta^4 d^4 \sigma^4 + O(\sigma^6) \tag{4.8}$$

which gives

$$\frac{1}{N^2} \langle \bar{\chi} \chi \rangle^2 = \langle \sigma \rangle^2 = \frac{5(N+2)^3 d}{18(N^2+2N+2)} (\beta - \beta_{ch}) \quad (4.9)$$

near the phase transition ($\beta > \beta_{ch}$). Through $f_\pi \propto \langle \sigma \rangle$ we hence recover the result (4.7).

5. INTRODUCING A CHEMICAL POTENTIAL

Several strong arguments indicate that even at zero temperature, chiral symmetry should be restored at sufficiently high baryon density. As is well known, the effects of a finite baryon (or quark) density can be incorporated in the Lagrangian by the introduction of a chemical potential μ which couples to the quark number density $\psi^\dagger \psi$. (The corresponding physical density is then found by differentiation of the partition function with respect to μ .) Therefore, at a sufficiently large $\mu = \mu_{ch}$, we expect chiral symmetry to be restored. This leads to a conjectured phase diagram as shown in Fig. 4, a diagram which is completely analogous to the conjectured phase diagram for deconfinement.

In the ultra-strong coupling limit $g^2 \rightarrow \infty$ which we consider in this paper there is no deconfinement. Furthermore, if we restrict ourselves to the gauge group $U(N)$, so that the effective Lagrangian will contain only meson fields (obviously carrying zero net quark number), we would expect that the introduction of the chemical potential μ should have no effect whatsoever on the effective Lagrangian. Since we are bound to stay within the confined phase, no single quark lines can exist on the lattice. In terms of strong coupling diagrams we are therefore led to consider only "tree graphs" of the type depicted in Fig. 5. Any cut of such diagrams will correspond to states of zero quark number.

The lack of dependence on the chemical potential μ can be shown explicitly from the effective Lagrangian. Consider first the zero temperature case. Introducing a chemical potential^{14),29)}, our strong coupling lattice action takes the form

$$S = \sum_x \left[-\bar{\chi}(x) M \chi(x) + \frac{1}{2} \sum_j \eta_j(x) \{ \bar{\chi}(x) U_j(x) \chi(x+j) - \bar{\chi}(x+j) U_j^\dagger(x) \chi(x) \} \right. \\ \left. + \frac{1}{2} \{ \bar{\chi}(x) e^M U_0(x) \chi(x+\delta) - \bar{\chi}(x+\delta) e^{-M} U_0^\dagger(x) \chi(x) \} \right] \quad (5.1)$$

where $\eta_j(x)$ is as defined in Eq. (2.6). The $U_j(x)$ integrations can now be performed, precisely as before. Furthermore, in this zero temperature case the U_0 -integrations can be done using the same formula [Eq. (3.9)]. All terms in the expansion appear as powers of $J^\dagger J$, where

$$J^\dagger J = \bar{\chi}(x) e^M \chi(x+\delta) \bar{\chi}(x+\delta) e^{-M} \chi(x) = \bar{\chi}(x) \chi(x+\delta) \bar{\chi}(x+\delta) \chi(x) \quad (5.2)$$

As can be seen, although the action (5.1) has an apparent μ -dependence, this dependence cancels when the gauge integrations are performed. In this way we recover the standard zero temperature effective Lagrangian.

In the case of finite temperature the cancellation between μ -dependent terms is far from trivial, and this cancellation in fact provides us with a nice check on the calculation. Let us therefore show explicitly how this cancellation takes place.

We use the same gauge fixing as in Section 2, so that our action with a chemical potential for the quark number density is given by Eq. (3.5) with the following replacement:

$$D \rightarrow e^M D, \quad D^* \rightarrow e^{-M} D^* \quad (5.3)$$

This corresponds to the following replacement:

$$e^{i\theta_b/\beta} \rightarrow e^{i(\theta_b - i\beta\mu)/\beta}, \quad e^{-i\theta_b/\beta} \rightarrow e^{-i(\theta_b - i\beta\mu)/\beta} \quad (5.4)$$

for the elements of the gauge fixing matrix D and D^* . From this point the calculation with a chemical potential is completely straightforward. The corresponding expression for (3.21) in Section 2 is now

$$Z = \int d[\sigma] d[u_b] e^{-\frac{Nd}{4} \beta \sum_x \sigma^2} \prod_x \prod_b [2 \cosh(\beta s) + 2 \cos(\theta_b - i\beta\mu)] \quad (5.5)$$

In the present case the I_n 's are different from Eq. (3.24);

$$\begin{aligned}
 I_0 &= 2 \cosh(\beta s) \\
 I_{-1} &= \cosh(\beta \mu) + \sinh(\beta \mu) \\
 I_{+1} &= \cosh(\beta \mu) - \sinh(\beta \mu) \\
 I_{+n} &= I_{-n} = 0, \quad n \geq 2
 \end{aligned}
 \tag{5.6}$$

Note that now $I_{+1} \neq I_{-1}$ so that we have to extend our formula Eq. (3.22) to the case $I_n = I_{-n}$.

The recursion formula in the present case is given by

$$B_N = I_0 B_{N-1} - I_{-1} I_{+1} B_{N-2} \tag{5.7}$$

where, by Eq. (5.6)

$$I_{-1} I_{+1} = 1 \tag{5.8}$$

The recursion formula (5.7) therefore coincides with Eq. (3.26) (where $I_1 = I_{-1} = 1$). Hence the solution is the same, and we obtain the same effective action even in the case of a non-vanishing chemical potential^{*)}.

We have thus explicitly demonstrated that the quark chemical potential μ does not influence the effective potential in the case of the $U(N)$ colour gauge group. Of course, for an $SU(N)$ gauge group, where baryons (carrying non-zero quark number) appear, a dependence on the chemical potential will emerge. This case, QCD at finite baryon density (but zero temperature), can be analyzed within the same effective Lagrangian framework as considered here³¹⁾.

6. FLAVOUR DEPENDENCE

Finally, we wish to investigate what happens when we no longer restrict ourselves to the case $N_f = 1$. The derivation of the effective Lagrangian for an arbitrary number of flavours N_f is essentially identical to the $N_f = 1$ case up to

*) In Ref. 30) the effect of a non-zero chemical potential was also considered, and there an effect was found. This is due to the fact that these authors integrate the thermal links U_0 over the full $SU(N)$ Haar measure, thus incorporating the effect of "baryons" in that direction. This will, of course, introduce a non-trivial μ -dependence. However, ignoring the baryon terms from the $SU(N)$ integrations in the spatial directions [as is done in Ref. 30)] is only a valid approximation when $E^{N_f} \gg d$.

Eq. (3.12). For the general N_f case. $\sigma(\tau, \vec{x})$ is now a matrix field $\bar{\sigma}_{\alpha\beta}(\tau, \vec{x})$ with α and β being flavour indices:

$$\bar{\sigma}_{\alpha\beta}(\tau, x) = \delta_{\alpha\beta} \bar{\sigma}_0(\tau, x) + \sum_{A=1}^{N_f^2-1} \left(\frac{\lambda^A}{2}\right)_{\alpha\beta} \bar{\sigma}^A(\tau, x) \quad (6.1)$$

As the vacuum does not violate flavour symmetry, we use the mean field ansatz $\bar{\sigma}^B(\tau, \vec{x}) = \delta_{B,0} \bar{\sigma}$ and hence obtain

$$S_{flav.} = \sum_{\tau} \sum_x \left\{ -\frac{Nd}{4} N_f \bar{\sigma}^2 - \frac{d}{2} \bar{\chi}_\alpha^a(\tau, x) \bar{\sigma} \chi_\alpha^a(\tau, x) \right. \\ \left. + \frac{1}{2} \left\{ \bar{\chi}_\alpha^a(\tau, x) D_{aa}(x) \chi_\alpha^a(\tau+\delta, x) - \bar{\chi}_\alpha^a(\tau+\delta, x) D_{aa}^*(x) \chi_\alpha^a(\tau, x) \right\} \right\} \quad (6.2)$$

instead of (3.12). Performing the Grassmann integrations, we then find the partition function

$$Z_{flav.} = \int d[\sigma] d[U_0] e^{-\frac{Nd}{4} N_f \sum_x \bar{\sigma}^2} \prod_x \prod_b (2 \cosh(\beta \bar{s}) + 2 \cos(\theta_b))^{N_f} \quad (6.3)$$

with $\bar{s} = \sinh^{-1}(\bar{\sigma}d/2)$. Thus the bulk of the flavour dependence arises from the fermion determinant

$$W_{flav.}(\theta_b) = (W(\theta_b))^{N_f}$$

where $W(\theta_b)$ is given by Eq. (3.23).

Introducing

$$\bar{I}_n \equiv \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (2 \cosh(\beta \bar{s}) + 2 \cos(\theta))^{N_f} e^{in\theta} \quad (6.4)$$

the corresponding $\bar{C}_N$ is given by

$$\bar{C}_N = \det \begin{vmatrix} \bar{I}_0 & \bar{I}_1 & \dots & \bar{I}_{N-1} \\ \bar{I}_1 & \bar{I}_0 & & \\ \vdots & & \ddots & \\ \bar{I}_{N-1} & & & \bar{I}_0 \end{vmatrix} \quad (6.5)$$

The effective potential for this flavour dependent case is then

$$V_{\text{eff, flav.}} = \frac{Nd}{4} N_f \beta \bar{\sigma}^2 - \ln \bar{C}_N \quad (6.6)$$

For the special case $N_f = 2$ we can obtain an analytic expression for $\bar{C}_N$ for all N . In this case $\bar{C}_N$ is given by

$$\bar{C}_N = \det \begin{vmatrix} \bar{I}_0 & \bar{I}_1 & 1 & & 0 \\ \bar{I}_1 & \bar{I}_0 & \bar{I}_1 & 1 & \\ 1 & \bar{I}_1 & \ddots & \ddots & 1 \\ & 1 & \ddots & \ddots & \bar{I}_1 \\ 0 & & \ddots & 1 & \bar{I}_1 & \bar{I}_0 \end{vmatrix} \quad (6.7)$$

which can be solved by recursion. To this end, introduce

$$\bar{D}_N = \det \begin{vmatrix} \bar{I}_1 & 1 & & & 0 \\ \bar{I}_1 & \bar{I}_0 & \bar{I}_1 & 1 & \\ 1 & \bar{I}_1 & \bar{I}_0 & \bar{I}_1 & 1 \\ & 1 & \bar{I}_1 & \bar{I}_0 & \bar{I}_1 & 1 \\ 0 & & \ddots & \ddots & \ddots & \ddots \end{vmatrix} \quad (6.8)$$

The recursion relations for $\bar{C}_N$ and $\bar{D}_N$ then take the following form:

$$\begin{aligned}\bar{C}_N &= \bar{I}_0 \bar{C}_{N-1} - \bar{I}_1 \bar{D}_{N-1} + \bar{I}_1 \bar{D}_{N-2} - \bar{I}_0 \bar{C}_{N-3} + \bar{C}_{N-4} \\ \bar{D}_N &= \bar{I}_1 \bar{C}_{N-1} - \bar{D}_{N-1}\end{aligned}\quad (6.9)$$

Inserting the latter recursion relation into the former, and making use of $\bar{I}_1^2 = 4\bar{I}_0 - 8$, we arrive at

$$\bar{D}_{N+5} - (\bar{I}_0 - 1)\bar{D}_{N+4} + (3\bar{I}_0 - 8)\bar{D}_{N+3} - (3\bar{I}_0 - 8)\bar{D}_{N+2} + (\bar{I}_0 - 1)\bar{D}_N = 0 \quad (6.10)$$

After a tedious but straightforward calculation (see Appendix A) we obtain

$$\begin{aligned}\bar{C}_N &= l_1 + l_2 N + l_3 N^2 + l_4 \frac{\sinh\{2(N-1)\beta\bar{S}\}}{\sinh\{2\beta\bar{S}\}} \\ &+ l_5 \frac{\sinh\{2N\beta\bar{S}\}}{\sinh\{2\beta\bar{S}\}} + l_6 \frac{\sinh\{2(N+1)\beta\bar{S}\}}{\sinh\{2\beta\bar{S}\}}\end{aligned}\quad (6.11)$$

The explicit forms of $l_1 \sim l_6$ are given in Appendix A.

Qualitatively, the chiral phase transition is not fundamentally altered by going from $N_f = 1$ to $N_f = 2$. In Figs. 6 and 7, we show the effective potential [Eq. (6.6) with $\bar{C}_N$ given by Eq. (6.11)] for the number of colours N equal to three and five, respectively. The phase transition is clearly second order, occurring for $N = 2$ at $\beta \approx 1.0$ and for $N = 5$ at $\beta \approx 0.9$. The phase transition presumably remains of second order for all N . (We have explicitly verified this up to $N = 20$.)

For $N_f > 3$ the analytic solution for $\bar{C}_N$ becomes formidably complicated for arbitrary N . Instead, we shall here restrict ourselves the most interesting case of $N = 3$, and consider the flavour dependence for $N_f = 3, 4, 5, 6$ respectively. The formula for $\bar{C}_N$ ($N = 3$) reads

$$\bar{C}_3 = \det \begin{vmatrix} \bar{I}_0 & \bar{I}_1 & \bar{I}_2 \\ \bar{I}_1 & \bar{I}_0 & \bar{I}_1 \\ \bar{I}_2 & \bar{I}_1 & \bar{I}_0 \end{vmatrix} = \bar{I}_0^3 + 2\bar{I}_1^2 \bar{I}_2 - 2\bar{I}_0 \bar{I}_1^2 - \bar{I}_0 \bar{I}_2^2 \quad (6.12)$$

where

$$\bar{I}_n = 2^{N_f} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (\cosh(\beta \bar{s}) + \cos(\theta))^{N_f} e^{in\theta} \quad (6.13)$$

More explicitly, the $\bar{I}_n$'s are here given by

$$\bar{I}_0 = 2^{N_f} \sum_{l=0}^{\left[\frac{N_f}{2}\right]} \binom{N_f}{2l} [\cosh(\beta \bar{s})]^{N_f-2l} \frac{(2l-1)!!}{(2l)!!} \quad (6.14)$$

$$\bar{I}_1 = 2^{N_f} \sum_{l=1}^{\left[\frac{N_f+1}{2}\right]} \binom{N_f}{2l} [\cosh(\beta \bar{s})]^{N_f-2l+1} \frac{(2l-1)!!}{(2l)!!} \quad (6.15)$$

$$\bar{I}_2 = 2^{N_f} \sum_{l=0}^{\left[\frac{N_f}{2}\right]} \binom{N_f}{2l} [\cosh(\beta \bar{s})]^{N_f-2l} \left\{ 2 \cdot \frac{(2l-1)!!}{(2l+2)!!} - \frac{(2l-1)!!}{(2l)!!} \right\} \quad (6.16)$$

The $N_f = 3$ case looks as follows:

$$\left. \begin{aligned} \bar{I}_0 &= 2^3 \left\{ \cosh^3(\beta \bar{s}) + \frac{3}{2} \cosh(\beta \bar{s}) \right\} \\ \bar{I}_1 &= 2^3 \left\{ \frac{3}{2} \cosh^2(\beta \bar{s}) + \frac{3}{8} \right\} \\ \bar{I}_2 &= 2^3 \left\{ \frac{3}{4} \cosh(\beta \bar{s}) \right\} \end{aligned} \right\} N_f = 3 \quad (6.17)$$

The effective potential corresponding to Eq. (6.17) is shown in Fig. 8. The chiral transition is still clearly second order, now occurring at $\beta \approx 1.1$.

A second order phase transition at $N_f = 3$ is somewhat against naive

expectations: one can here imagine a cubic invariant of the form (all indices being flavour indices) $\epsilon_{ijk} \phi^i \phi^j \phi^k \epsilon^{\lambda mn} \bar{\phi}_\lambda \bar{\phi}_m \bar{\phi}_n \sim \det[\sigma]$ induced in the effective action for σ^{32}). Such a cubic term would presumably drive the transition to first order. However, as we have shown, such a term is not induced at strong coupling.

We give the explicit formulae for $N_f = 4, 5$ and 6 in Appendix B. We have checked that the phase transition remains second order for these numbers of flavours, with only slightly shifted values of β_{ch} .

7. DISCUSSION

Having presented these results from the strong coupling Lagrangian formulation of QCD, we feel that a critical reassessment of the calculation is appropriate. Working in the very strong coupling region of QCD on the lattice is of course in itself a crude approximation to continuum physics. Could the picture change as $1/g^2$ is increased? There are several possibilities. When $1/g^2$ grows, the effects due to "deconfinement" of heavy quark sources (i.e., the interactions with Wilson-Polyakov lines) becomes important, and the two transitions could show a non-trivial interplay, stabilizing each other "driving" each other, or changing the order of the transitions.

However, even without going to the intermediate coupling region (where in fact most Monte Carlo computations are performed), we can reconsider some of the approximations involved. For example, the exclusion of baryons in the effective Lagrangian [by considering the gauge group $U(N)$ instead of $SU(N)$] is a somewhat subtle problem. Within the context of the $1/d$ expansion the baryon contribution is heavily suppressed (by extra powers of d^{-N}), but the experience from zero temperature calculations have taught us that baryon terms tend to distort the potential $v(\sigma)$ slightly near the origin. The distortion will be small due to the extra powers of $1/d$ or, more physically, because the baryons are much heavier than the mesons. [At strong coupling $M_B \sim N \ln(d)$.] Nevertheless, since second order phase transitions generically do not show stability towards small perturbations, it cannot be ruled out that the inclusion of baryons could possibly change the order of the transition from second order to very weakly first order. In practice, however, the transition should appear almost indistinguishable from a second order transition. [If another free parameter, such as a non-zero chemical potential for the quark number density is introduced, this picture is of course

subject to change³¹⁾.]

An important point which we would like to discuss is the way the $1/d$ expansion is implemented in our calculation. Our $1/d$ expansion is, in effect, a $1/(d-1)$ expansion^{*)} since we single out the periodic time direction and treat it differently from the remaining spatial dimensions. Correspondingly, the zero temperature effective potential derived as the $\beta \rightarrow \infty$ limit of our effective potential²¹⁾ has an overall factor d instead of $(d+1)$. Let us clarify this point a little further. One could imagine performing all dU_j ($j = 0, \dots, d$) symmetrically, keeping only the leading order term [corresponding to Eq. (3.10)] in the $1/d$ expansion for all directions, including the time direction. Subsequently, one could use identities like (3.11) to partially eliminate the fermion fields by trading them with one scalar field σ . If one did this, one would essentially recover the zero temperature effective action, only differing by a trivial overall factor. This effective action, being completely τ -independent, would of course not display chiral symmetry restoration. This is not a surprising result, since clearly the periodicity (antiperiodicity) in the timelike direction does not even enter in such a calculation. (Only one link is probed in each direction.) Physically, it is not surprising either. It corresponds to expanding around $d = \infty$, where, of course, finite temperature loses importance since the periodicity in just one direction out of $d = \infty$ other directions is completely irrelevant.

Instead, the appearance of a chiral phase transition in our calculation can be traced back to the fact that we integrate out exactly the gauge field U_0 and the fermions in the periodic (antiperiodic) time direction, without truncating the fermion series at a finite number of terms, and without introducing the identification (3.11) for the fields propagating in the time direction. It is only in this way that we recover sensitivity to the periodicity (antiperiodicity) in the time direction. A symmetric treatment of time and space directions (and subsequent space-time independent mean field ansätze) will always fail to achieve this sensitivity to periodicity. It is by integrating out exactly (and gauge-invariantly) the fermions and gauge fields propagating along the time direction

*) If we expand our results to leading order in $1/d$, the two answers are, of course, identical.

before applying the mean field techniques that we obtain the phase transition^{*)}.

Let us finally mention that two other groups^{30),33)} have done work similar in spirit to the treatment of this paper, but also including interactions due to Wilson-Polyakov lines. Reference 30) has nicely supplemented and confirmed our results.

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^{*)} Mean field theory of gauge theories in the Lagrangian formulation of finite temperature is a long-standing problem which has not yet found a satisfactory solution. Perhaps our technique used here, first integrating out exactly the time-like fields and only then using mean field techniques for the space directions, may be of use there also.

APPENDIX A

In this appendix we show how to solve the recursion relation

$$\bar{D}_{N+5} - (\bar{I}_0 - 1)\bar{D}_{N+4} + (3\bar{I}_0 - 8)\bar{D}_{N+3} - (3\bar{I}_0 - 8)\bar{D}_{N+2} + (\bar{I}_0 - 1)\bar{D}_{N+1} - \bar{D}_N = 0 \quad (A.1)$$

of Eq. (6.10).

The characteristic equation for (A.1) reads

$$x^5 + (\bar{I}_0 - 1)x^4 + (3\bar{I}_0 - 8)x^3 - (3\bar{I}_0 - 8)x^2 + (\bar{I}_0 - 1)x - 1 = 0 \quad (A.2)$$

which can be rewritten as

$$(x-1)^3(x^2 - (\bar{I}_0 - 4)x + 1) = 0 \quad (A.3)$$

Using this relation, the general form of the recursion coefficient $\bar{D}_N$ is given by

$$\bar{D}_N = k_1 + k_2 N + k_3 N^2 + k_4 a^N + k_5 b^N \quad (A.4)$$

where a and b are the solutions of the equation

$$x^2 - (\bar{I}_0 - 4)x + 1 = 0 \quad (A.5)$$

and $k_1 \sim k_5$ are N -independent constants determined from the explicit forms of the determinants of the small matrices $\bar{D}_0$, $\bar{D}_1$, $\bar{D}_2$, $\bar{D}_3$ and $\bar{D}_4$.

For later convenience, denote $\bar{I}_1 = 2c$, which gives $\bar{I}_0^2 = c^2 + 2$. Then a and b are given by

$$a = \frac{c^2 - 2 + \sqrt{c^4 - 4c^2}}{2}, \quad b = \frac{c^2 - 2 - \sqrt{c^4 - 4c^2}}{2} \quad (A.6)$$

For $\bar{D}_0 \sim \bar{D}_4$ we find

$$\bar{D}_0 = 0 \quad (A.7)$$

$$\bar{D}_1 = \bar{I}_1 = 2c \quad (A.8)$$

$$\bar{D}_2 = (\bar{I}_0 - 1)\bar{I}_1 = 2c^3 + 2c \quad (A.9)$$

$$\bar{D}_3 = (\bar{I}_0 - 1)\bar{I}_0\bar{I}_1 - \bar{I}_1^3 + \bar{I}_1 \quad (A.10)$$

$$\bar{D}_4 = \bar{I}_0^2(\bar{I}_0 - 1)\bar{I}_1 - 2\bar{I}_0\bar{I}_1^2 + 3\bar{I}_1^3 - \bar{I}_1 \quad (A.11)$$

The coefficients $k_1 \sim k_5$ are then determined:

$$k_1 = \frac{1}{(c^2 - 4)^2} (-16c^7 + 8c^6 + 16c^5 - 8c^4 + 94c^3 - 48c^2 + 6c) \quad (A.12)$$

$$k_2 = \frac{1}{(c^2 - 4)^2} (24c^7 - 12c^6 - 48c^5 + 24c^4 - 195c^3 + 96c^2 + 12c) \quad (A.13)$$

$$k_3 = \frac{1}{(c^2 - 4)^2} (-8c^7 + 4c^6 + 16c^5 - 8c^4 + 63c^3 - 32c^2 + 4c) \quad (A.14)$$

$$k_4 = \frac{1}{(c^2 - 4)^{5/2}} \left\{ (-16c^6 + 8c^5 + 16c^4 - 8c^3 + 94c^2 - 48c + 6)b \right. \\ \left. + (18c^4 - 8c^3 + 22c^2 - 16c + 10) \right\} \quad (A.15)$$

$$k_5 = \frac{1}{(c^2 - 4)^{5/2}} \left\{ (-16c^6 + 8c^5 + 16c^4 - 8c^3 + 94c^2 - 48c + 6)a \right. \\ \left. - (18c^4 - 8c^3 + 22c^2 - 16c + 10) \right\} \quad (A.16)$$

Noting that $a \cdot b = 1$, we can rewrite part of the expression for $\bar{D}_N$ as follows:

$$k_4 a^N + k_5 b^N = k_6 (a^{N-1} - b^{N-1}) + k_7 (a^N - b^N) \quad (\text{A.17})$$

where

$$k_6 = \frac{1}{(c^2-4)^{5/2}} (-16c^6 + 8c^5 + 16c^4 - 8c^3 + 94c^2 - 48c + 6) \quad (\text{A.18})$$

$$k_7 = \frac{1}{(c^2-4)^{5/2}} (18c^4 - 8c^3 + 22c^2 - 16c + 10) \quad (\text{A.19})$$

Using these relations, we obtain $\bar{C}_N$ by substitution of $\bar{D}_N$:

$$\begin{aligned} \bar{C}_N &= \frac{\bar{D}_{N+1} + \bar{D}_N}{2c} \\ &= \frac{1}{2c} \{ 2k_1 + k_2 + k_3 + (2k_2 + 2k_3)N + 2k_3 N^2 \\ &\quad + k_6 (a^{N-1} - b^{N-1}) + (k_6 + k_7)(a^N - b^N) + k_7 (a^{N+1} - b^{N+1}) \} \end{aligned} \quad (\text{A.20})$$

We can parametrize $a = e^\phi$. By noticing that $b = 1/a$ we obtain

$$\phi = \ln \left\{ \frac{c^2 - 2 + \sqrt{c^4 - 4c^2}}{2} \right\} \quad (\text{A.21})$$

with $c = \bar{I}_1/2 = 2\cosh(\beta\bar{s})$. Thus, in fact

$$\phi = 2\beta\bar{s} \quad (\text{A.22})$$

Using the relation $a-b = c\sqrt{c^2-4} = 2\sinh(2\beta\bar{s})$ we then finally arrive at the following expression:

$$\begin{aligned} \bar{C}_N &= l_1 + l_2 N + l_3 N^2 + l_4 \frac{\sinh\{2(N-1)\beta\bar{s}\}}{\sinh\{2\beta\bar{s}\}} \\ &\quad + l_5 \frac{\sinh\{2N\beta\bar{s}\}}{\sinh\{2\beta\bar{s}\}} + l_6 \frac{\sinh\{2(N+1)\beta\bar{s}\}}{\sinh\{2\beta\bar{s}\}} \end{aligned} \quad (\text{A.23})$$

where $l_1 \sim l_6$ are given by

$$l_1 = \frac{1}{(c^2-4)^2} (-8c^6 + 4c^5 + 28c^2 - 16c + 14) \quad (\text{A.24})$$

$$l_2 = \frac{1}{(c^2-4)^2} (16c^6 - 8c^5 - 32c^4 + 16c^3 - 132c^2 + 64c + 16) \quad (\text{A.25})$$

$$l_3 = \frac{1}{(c^2-4)^2} (-8c^6 + 4c^5 + 16c^4 - 8c^3 + 63c^2 - 32c + 4) \quad (\text{A.26})$$

$$l_4 = \frac{1}{(c^2-4)^2} (-8c^6 + 4c^5 + 8c^4 - 4c^3 + 47c^2 - 24c + 3) \quad (\text{A.27})$$

$$l_5 = \frac{1}{(c^2-4)^2} (-8c^6 + 4c^5 + 17c^4 - 8c^3 + 58c^2 - 32c + 8) \quad (\text{A.28})$$

$$l_6 = \frac{1}{(c^2-4)^2} (9c^4 - 4c^3 + 11c^2 - 8c + 5) \quad (\text{A.29})$$

APPENDIX B

In this appendix we show the explicit form of $\bar{I}_0$, $\bar{I}_1$ and $\bar{I}_2$ for $N_f = 4, 5, 6$ and $N = 3$. Using Eqs. (6.14)-(6.16) we get

$$\bar{I}_0 = 2^4 \left\{ \cosh^4(\beta\bar{s}) + 3 \cosh^2(\beta\bar{s}) + \frac{3}{8} \right\} \quad (B.1)$$

$$\bar{I}_1 = 2^4 \left\{ 2 \cosh^3(\beta\bar{s}) + \frac{3}{2} \cosh(\beta\bar{s}) \right\} \quad (B.2)$$

$$\bar{I}_2 = 2^4 \left\{ \frac{3}{2} \cosh^2(\beta\bar{s}) + \frac{1}{4} \right\} \quad (B.3)$$

for $N_f = 4$. Similarly, for $N_f = 5$ we obtain

$$\bar{I}_0 = 2^5 \left\{ \cosh^5(\beta\bar{s}) + 5 \cosh^3(\beta\bar{s}) + \frac{15}{8} \cosh(\beta\bar{s}) \right\} \quad (B.4)$$

$$\bar{I}_1 = 2^5 \left\{ \frac{5}{2} \cosh^4(\beta\bar{s}) + \frac{15}{4} \cosh^2(\beta\bar{s}) + \frac{5}{16} \right\} \quad (B.5)$$

$$\bar{I}_2 = 2^5 \left\{ \frac{5}{2} \cosh^3(\beta\bar{s}) + \frac{5}{4} \cosh(\beta\bar{s}) \right\} \quad (B.6)$$

Finally,

$$\bar{I}_0 = 2^6 \left\{ \cosh^6(\beta\bar{s}) + \frac{15}{2} \cosh^4(\beta\bar{s}) + \frac{45}{8} \cosh^2(\beta\bar{s}) + \frac{5}{16} \right\} \quad (B.7)$$

$$\bar{I}_1 = 2^6 \left\{ 3 \cosh^5(\beta\bar{s}) + \frac{15}{2} \cosh^3(\beta\bar{s}) + \frac{15}{8} \cosh(\beta\bar{s}) \right\} \quad (B.8)$$

$$\bar{I}_2 = 2^6 \left\{ \frac{15}{4} \cosh^4(\beta\bar{s}) + \frac{15}{4} \cosh^2(\beta\bar{s}) + \frac{15}{64} \right\} \quad (B.9)$$

for the case $N_f = 6$.

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FIGURE CAPTIONS

- Fig. 1: A Susskind-flavour interpretation of the staggered fermions in two space-time dimensions, where $(x_1, x_2) = (E, 0)$ means $x_1 = \text{even}$, $x_2 = \text{odd}$ and so on.
- Fig. 2: The effective potential V_{eff} for $N = 3$ in $(3+1)$ dimensions as a function of $\tilde{\sigma} = \sigma/a_s$ for various $\tilde{\beta} = \beta(a_s/a_\tau)^2$. The phase transition occurs at $\tilde{\beta}_{\text{ch}} = 6/15$.
- Fig. 3: The addition of a small mass term to the action makes the effective potential V_{eff} asymmetric. There will then always be a global minimum at a non-zero value of σ ; the second order phase transition disappears.
- Fig. 4: The hypothesized phase diagram for chiral symmetry restoration in the μ - T plane.
- Fig. 5: When baryons are excluded [e.g., by considering the colour gauge group $U(N)$ instead of $SU(N)$], all strong coupling diagrams will be built of meson lines. Since therefore any cut of the diagrams will involve zero quark number, the effective action should be insensitive to the addition of a chemical potential for the quark number density $\psi^\dagger\psi$ in the original action.
- Fig. 6: The effective potential for $N_f = 2$, and various β . The derivation of the analytic expression for V_{eff} ($N_f = 2$) at an arbitrary number of colours N is described in the text. The effective potential here corresponds to $N = 3$.
- Fig. 7: Same as Fig. 6, but now with $N = 5$. The $N_f = 2$ phase transition remains of second order for all N .
- Fig. 8: V_{eff} corresponding to $N = 3$ and $N_f = 3$. The phase transition is still of second order. For $N = 3$ it remains of second order for $N_f = 4, 5$ and 6. (See Appendix B for details of the cases $N_f = 4, 5$ and 6.)

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8