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ON THE EXPONENTIALLY INCREASING LEVEL DENSITY IN STRING MODELS
AND THE TACHYON SINGULARITY

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ABSTRACT

We consider the scalar (and the Neveu-Schwarz) string models. We show that for these models the "singularity" which arises when the Boltzmann factor $e^{-\beta E_n}$ can no longer compete with the exponential level density, gives rise to the well-known tachyon singularity. We also discuss the situation for the Ramond model.

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1. - INTRODUCTION

The physics of a possible deconfining phase transition is not very well understood. It is therefore of interest that some string models are able to provide a simple, intuitively attractive picture of the deconfining transition. Alvarez and Pisarski¹⁾ have shown that in the scalar string model one obtains an effective temperature-dependent string tension $\sigma(\beta)$,

$$\sigma(\beta) = \sigma \sqrt{1 - \beta_H^2 / \beta^2}, \quad (1.1)$$

where β is the inverse temperature and β_H^{-1} is the Hagedorn temperature,

$$\beta_H = \sqrt{\frac{\pi(d-2)}{3\sigma}}. \quad (1.2)$$

Furthermore, σ is the zero temperature string tension and d is the space-time dimension. Equation (1.1) is valid for an open string with fixed end-points separated by a distance R , where $R \gg \beta$.

Equation (1.1) was originally derived in the $d \rightarrow \infty$ saddle point. The author has presented a plausible argument that this equation is valid for all d for the Nambu-Goto string. Of course, the calculations are of lowest order in the string coupling constant, and higher order corrections can change Eq. (1.1). However, it should be remembered that in order to obtain this equation, one uses a string with end-points fixed at a large distance. In string models one may expect that non-perturbative effects will be most important for short distances, and hence there is reasonable hope that Eq. (1.1) correctly represents at least the qualitative features of the large-distance behaviour of open strings^{*)}.

Physically, the behaviour (1.1) comes about in the following way¹⁾. At small temperatures, the very long string is approximately at rest. As the string is heated up, it starts to oscillate more and more, thus becoming longer, which costs energy. However, the increased entropy, in the form of disorder, pays for

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*) Furthermore, it is not clear how to define an effective string tensions when loops are included, because then the string can "break" (this is analogous to the case of dynamical fermions).

the energy increase. As the phase transition point β_H is reached, the length of the string becomes infinite, but the transverse oscillations remain bounded. Thus it is only possible for the string to have infinite length if it is folded in a completely chaotic manner; then the entropy also becomes infinite.

For $\beta = \beta_H$, one has $\sigma(\beta_H) = 0$, i.e., there is no effective string tension. Also, in Ref. 2), we have argued that if the string is given a finite transverse extension, (like a vortex line), then for $\beta \approx \beta_H$ the phase is essentially a "particle" phase where the string has disappeared. Thus, β_H is most likely a deconfinement temperature¹⁾.

In Ref. 2), we showed that the behaviour exhibited in Eq. (1.1) can be related to the zero point energy of the harmonic oscillators in the string model. This energy is tachyonic in the case of the scalar string. It turns out that from this point of view, the quantity β_H is directly proportional to the mass of the tachyon, which is indeed very strange. Usually the Hagedorn temperature is connected with the exponential increase of the density of states in string models^{3),4)}, and apparently this behaviour has nothing to do with the tachyon.

The central point of the present paper is that at least for some string models (the scalar, the Neveu-Schwarz, and the Ramond model with antiperiodic boundary conditions for the two-dimensional spinor field in the temperature "direction"), the exponential Hagedorn density of states is related to the tachyon. Clearly, this also appears as the most logical explanation why β_H can occur in Eq. (1.1) as a result of the existence of a tachyon.

This result can be explained as follows. When one computes the string partition function $Z(\beta, R)$, it is a calculation carried out for a two-dimensional field theory. The boundary conditions are very important. In general, we compute $Z(\beta, R)$ for a rectangle with sides β and R . Let us introduce the notation of "channels":

- 1) The temperature channel. Here $Z(\beta, R)$ is computed for $R \gg \beta$, and one uses periodic boundary conditions in the "time" (i.e., β) direction.
- 2) The potential channel. Here $Z(\beta, R)$ is computed for $\beta \gg R$, and one uses boundary conditions such that the end-points of the string are fixed. This was done first by Alvarez⁵⁾ for the scalar model in general, and the result is completely analogous to Eq. (1.1):

$$V(R) = \sigma(R) R, \quad (1.3)$$

$$\sigma(R) = \sigma \sqrt{1 - R_c^2/R^2}. \quad (1.4)$$

Here $V(R)$ is the potential between the end-points, and

$$R_c^2 = \sqrt{\frac{\pi(d-2)}{12\sigma}}. \quad (1.5)$$

This result was originally derived for $d \rightarrow \infty$. However, using operator methods the results was found by Arvis⁶⁾ to be valid for all d in the Nambu-Goto model. Again, the "shortest" distance (1.5) is proportional to the tachyon mass⁷⁾. When expanded for large R , Eq. (1.4) reproduces Lüscher's large-distance "Coulomb" potential⁸⁾.

In the present paper we shall investigate $Z(\beta, R)$ (ignoring loop corrections). For the scalar model we find that in the temperature channel the tachyon gives the smallest $\beta = \beta_H$ (i.e., the "maximum" temperature), whereas the Hagedorn exponentially-increasing spectrum gives the "smallest" distance $R = R_c$. In the potential channel it is just the opposite: the tachyon produces the "smallest" distance, and the level spectrum is responsible for the "smallest" β .

Thus, in other words, what appears to be a tachyon in one channel becomes the singularity due to the exponentially-increasing mass spectrum in the other channel. One can formulate this by saying that there is a complete duality between the tachyon and the exponentially-increasing level density. Hence, the scalar model has (in this duality sense) a tachyon because the level density increases exponentially, and vice versa.

In this connection it should be mentioned that in Refs. 2) and 7) we pointed out that the inconsistencies (e.g., an imaginary potential and lack of rotational symmetry) associated with the tachyon (or, alternatively, the level density) disappear at large distances and small temperatures. So even if the model has a

tachyon, it can still potentially represent an effective theory in some restricted kinematical range.

In this paper we shall discuss in detail the duality between the tachyon and the experimental level density for some string models. In Section 2, we discuss the large distance/small temperature case; in Section 3 we then show the "duality" between the level density and the tachyon for the scalar model.

In Refs. 2) and 7), the author has shown that the ("dual") results (1.1) and (1.4) can be extended to the Neveu-Schwarz model. Since this model has a tachyon with a different mass, one obtains again Eqs. (1.1), (1.3) and (1.4), but now

$$\beta_H^{N-s} = \sqrt{\frac{\pi(d-2)}{2\sigma}}, \quad (1.6)$$

$$R_c^{N-s} = \sqrt{\frac{\pi(d-2)}{8\sigma}}. \quad (1.7)$$

Therefore, the Neveu-Schwarz model has qualitatively the same features as the scalar model. The Ramond model, however, has a rather different (and more complicated) behaviour, which we shall discuss in Section 4. In Section 5 we conclude with a discussion of our results.

2. - THE LARGE-DISTANCE/SMALL-TEMPERATURE APPROXIMATION

It was made plausible by Lüscher some time ago⁸⁾ that at large distances (or, alternatively, for small temperatures) the dominant string action contains the lowest possible number of derivatives. Thus for the scalar model one expects an effective action of the form

$$S = \sigma R\beta + \frac{\sigma}{2} \int d^2\xi \partial_i x \partial_i x, \quad (2.1)$$

where x is a $(d-2)$ -component vector corresponding to the transverse degrees of freedom of the string. In writing Eq. (2.1) it has been assumed that there are no other scalar degrees of freedom. The integral in this equation should be

performed for a $\beta \times R$ rectangle, where we assume a periodic boundary condition in the "time" direction, and where the end-points are fixed in the spatial directions. The determinant of $-\partial^2$ should thus be compared under these restrictions. The eigenfunctions can therefore be taken to be

$$\begin{aligned} X_n^0(r, t) &= \sqrt{\frac{2}{\beta R}} \sin n \frac{\pi r}{R}, \\ X_{nm}^1(r, t) &= \sqrt{\frac{4}{\beta R}} \cos\left(2n \frac{\pi t}{\beta}\right) \sin\left(n \frac{\pi r}{R}\right), \\ X_{nm}^2(r, t) &= \sqrt{\frac{4}{\beta R}} \sin\left(2n \frac{\pi t}{\beta}\right) \sin\left(n \frac{\pi r}{R}\right), \end{aligned} \quad (2.2)$$

where $n, m = 1, 2, \dots$. The corresponding eigenvalues are given by

$$\left(\frac{n\pi}{R}\right)^2 \text{ for } x^0, \left(\frac{2\pi n}{\beta}\right)^2 + \left(\frac{n\pi}{R}\right)^2 \text{ for } x^1 \text{ and } x^2. \quad (2.3)$$

Hence, the partition function is given by

$$\begin{aligned} Z(R, \beta) &= \int \mathcal{D}x e^{-S(x)} \\ &= \text{const.} e^{-\frac{1}{2}\beta R} \prod_{n=1}^{\infty} \left(\frac{n\pi}{R}\right)^{-(d-2)} \\ &\quad \times \prod_{n,m=1}^{\infty} \left[\left(\frac{2\pi n}{\beta}\right)^2 + \left(\frac{n\pi}{R}\right)^2 \right]^{-(d-2)}. \end{aligned} \quad (2.4)$$

Using a generalized ξ -function regularization method, Eq. (2.4) has been evaluated by Minami⁹⁾, whose result is

$$\begin{aligned}
 Z(R, \beta) &= e^{-\sigma R \beta + \frac{\pi}{24}(\alpha-2) \frac{\beta}{R}} \\
 &\times \prod_{n=1}^{\infty} \left(1 - e^{-n\pi \frac{\beta}{R}}\right)^{-(\alpha-2)} \\
 &\equiv Z_{\text{pot}}(R, \beta).
 \end{aligned} \tag{2.5}$$

Equation (2.5) is especially useful in the situation where $\beta \gg R$, since then the product becomes exponentially close to 1, and hence we denote this form by Z_{pot} , corresponding to the potential "channel" introduced in Section 1. For the temperature "channel" $R \gg \beta$, Eq. (2.5) can be written in the more convenient form

$$\begin{aligned}
 Z(R, \beta) &= e^{-\sigma \beta R + \frac{\pi}{2}(\alpha-2) \frac{R}{\beta}} \\
 &\times \prod_{n=1}^{\infty} \left(1 - e^{-\frac{4n\pi R}{\beta}}\right)^{-(\alpha-2)} e^{\frac{\alpha-2}{2} \ln(\beta/2R)} \\
 &\equiv Z_{\text{temp}}(R, \beta).
 \end{aligned} \tag{2.6}$$

Equation (2.6) can be obtained^{*} from Eq. (2.5) by noticing that Z can be expressed in terms of $\theta_1'(0|\tau)$ which satisfies

$$\theta_1'(0|\tau) = \frac{1}{(-i\tau)^{3/2}} \theta_1'(0|\frac{-1}{\tau}). \tag{2.7}$$

The transition between (2.5) and (2.6) represents a "duality" transformation between the potential and the temperature "channels". The factors

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^{*}) The equality between (2.5) and (2.6) can also be obtained by considering a $\beta \times R$ rectangle with the same boundary condition on all four sides (e.g., periodic boundary conditions in t as well as r). The result must obviously be invariant under interchange of β and R . This method gives a "physicist's derivation" of Eq. (2.7).

$$e^{\frac{\pi}{24}(d-2)\frac{\beta}{R}} \quad \text{and} \quad e^{\frac{\pi}{6}\frac{R}{\beta}(d-2)}$$

in Eqs. (2.5) and (2.6) represent the tachyon contribution. From Eq. (2.5), one obtains the potential

$$V(R) = \sigma R - \frac{\pi(d-2)}{24 R}, \quad (2.8)$$

which is indeed the large R expansion of Alvarez's square root formula (1.4). The second term on the left-hand side is Lüscher's long-distance "Coulomb" term⁸⁾ which thus appears because there is a tachyon. From Eq. (2.6), one similarly obtains the effective string tension

$$\sigma(\beta) = \sigma - \frac{\pi}{6} \frac{d-2}{\beta^2}, \quad (2.9)$$

which agrees with the expansion of Eq. (1.1). Hence, it is clear how the tachyon produces the large-distance/low-temperature corrections.

We shall now relate these results to the Hagedorn temperature. First we notice that the function $\prod(1-x^n)^{-(d-2)}$ is the generating function for the level density^{3),4)}, i.e.,

$$\sum_{n=0}^{\infty} d(n) x^n = \prod_{n=1}^{\infty} (1-x^n)^{-(d-2)}. \quad (2.10)$$

Using this result in Eq. (2.6), we obtain^{*)}

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*) Here we ignore the logarithmic factor in Eq. (2.6). It plays no role in the following, since it is not linear in β or R .

$$Z_{\text{temp}}(R, \beta) = \sum_{n=0}^{\infty} d(n) e^{-\beta E_n} \quad (2.11)$$

$$E_n = \sigma \beta - \frac{\pi}{6} (d-2) \frac{1}{\beta} + \frac{4\pi n}{\beta} \quad (2.12)$$

with $n = 0, 1, 2, \dots$. Equation (2.12) represents the energy of the n^{th} oscillator. Thus, Eq. (2.11) gives the usual Boltzmann expression for the partition function. In the next section we shall explicitly verify that this equation is the large β (i.e., small temperature) expansion of the energy of the n^{th} level.

From Eq. (2.5), we obtain similarly for the Wilson loop

$$Z_{\text{pot}}(R, \beta) = \sum_{n=0}^{\infty} d(n) e^{-\beta E_n}, \quad (2.13)$$

$$E_n = \sigma R - \frac{\pi}{24} (d-2) \frac{1}{R} + \frac{\pi n}{R}, \quad (2.14)$$

which has an interpretation similar to that of Eq. (2.11). The behaviour of the level density in Eqs. (2.11) and (2.14) is given by^{3),4)}

$$\begin{aligned} d(n) &\approx n^{-\frac{d+1}{4}} e^{2\pi \sqrt{\frac{d-2}{6}} n} \\ &= (\sqrt{\alpha' m^2})^{-\frac{d+1}{2}} e^{\beta_H m}, \end{aligned} \quad (2.15)$$

where $\alpha' m^2 = n$, and where n is large. β_H is given by Eq. (1.2) with $\sigma = 1/2\pi\alpha'$.

Starting from Eq. (2.13), one would derive (2.8) by noticing that the higher states are exponentially damped by the factor $e^{-\beta E_n}$ ($n \neq 0$) relative to the lowest state $e^{-\beta E_0}$. Thus the lowest state is expected to dominate. From Eq. (2.5) we know that this argument is correct.

Hence, the usual "singularity" due to the Hagedorn temperature does not arise in this case. We shall see in the next section that this is due to the use of the low temperature expansion. In spite of what has been said above, it is, however, clear that the expression (2.11) converges more and more slowly as R/β decreases. Let us therefore investigate what happens when R/β is very small (i.e., going from the temperature "channel" to the potential channel). The large n contribution to Eq. (2.11) can be approximated by the integral^{*}

$$\begin{aligned}
 Z(R, \beta) &\approx 2 e^{-\sigma \beta R + \frac{\pi}{6} \frac{R}{\beta} (d-2)} I, \\
 I &= \int_{\sqrt{N_0}}^{\infty} dx x^{-(d+1)/2} e^{-4\pi \frac{R}{\beta} x^2 + 2\pi \sqrt{\frac{d-2}{6}} x} \\
 &= e^{\frac{\pi}{24} (d-2) \frac{R}{\beta}} \int_{\sqrt{N_0}}^{\infty} dx x^{-(d+1)/2} e^{-4\pi \frac{R}{\beta} (x - \frac{1}{4} \sqrt{\frac{d-2}{6}} \frac{\beta}{R})^2}.
 \end{aligned} \tag{2.16}$$

So, for $R/\beta \rightarrow 0$, the dominant contribution to Z is

$$Z(R, \beta) \approx e^{-\sigma \beta R + \frac{\pi}{24} (d-2) \frac{\beta}{R}}, \tag{2.17}$$

and we see that for $R/\beta \rightarrow 0$ the Hagedorn "singularity" becomes dominant and correctly reproduces the tachyon singularity contained in Eq. (2.5). Therefore one can say that the tachyon in Eq. (2.17) exists because the level density in the channel dual to the tachyon increases exponentially; and thus, even if the usual Hagedorn singularity is absent, the exponentially increasing level density manages to produce the right result in crossing over from one channel to the other.

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^{*}) Note that if $R/\beta = 0$ the integral diverges exponentially, due to the exponentially increasing level density.

So far, we have only discussed the scalar model. The Neveu-Schwarz model can, however, be discussed in a similar manner. Again, there exists a tachyon, and it can be understood as coming from the Hagedorn exponential spectrum in the crossed channel. We shall not give the details, since they are very similar to the scalar case.

3. - THE EMERGENCE OF A "SHORTEST" DISTANCE/"HIGHEST" TEMPERATURE AND THE HAGEDORN TEMPERATURE

We shall now discuss the case where we are not at large distances/small temperatures. In this case, the tachyon shows up quite clearly in the square root formulae (1.1) and (1.4). We see, for example, that from Eq. (1.4) the potential becomes imaginary when R is less than R_c , which is a clear indication of the tachyonic instability.

Our first goal is to generalize the Boltzmann formulae (2.11) and (2.13) to the case of arbitrary distances and temperatures. In so doing, we need the energies E_n . To find E_n , we shall follow a method due to Arvis⁶⁾. However, in order to simplify the discussion, we shall present a simple intuitive argument which leads to the right result without the use of much formalism [this argument is a simple extension of the one presented by the author in Ref. 7)].

In the case of the Nambu-Goto string with free ends (the normal derivative vanishes at the end-points), the equations of motion are solved by

$$\begin{aligned} X^i(\sigma, \tau) &= x^i + 2\alpha' p^i \tau + \sqrt{2} \sum_{n \neq 0} \frac{a_n^i}{n} \cos n\sigma e^{-in\tau}, \\ X^0(\sigma, \tau) &= x^0 + 2\alpha' p^0 \tau + i\sqrt{2} \sum_{n \neq 0} \frac{a_n^0}{n} \cos n\sigma e^{-in\tau}. \end{aligned} \quad (3.1)$$

One then obtains the mass formula

$$\alpha' p_0^2 = \alpha' \tilde{p}^2 - \frac{d-2}{24} + n, \quad (3.2)$$

with $n = 0, 1, 2, \dots$. The second term on the right-hand side is the zero point energy, i.e., the tachyon. For strings with boundaries fixed at $X^1 = 0$ and $X^1 = R^1$, one obtains instead of the first Eq. (3.1)

$$X^i(\sigma, \tau) = \frac{R^i}{\pi} \sigma + \sqrt{2} \sum_{n \neq 0} \frac{a_n^i}{n} \sin n\sigma e^{-in\tau}, \quad (3.3)$$

whereas the second Eq. (3.1) is unchanged. Thus we see that the spatial part of the momentum has been replaced by R^1 and σ by τ , i.e.,

$$2\alpha' p^i \rightarrow \frac{R^i}{\pi} \quad \text{and} \quad \sigma \rightarrow \tau \quad (3.4)$$

Due to the symmetry in the action between σ and τ , one obtains instead of Eq. (3.2)

$$\alpha' p_0^2 = \alpha' \left(\frac{R}{2\pi\alpha'} \right)^2 - \frac{d-2}{24} + n, \quad (3.5)$$

that is,

$$p_0 = E_n = \sigma R \sqrt{1 - \frac{d-2}{12} \frac{\pi}{\sigma R^2} + \frac{2n\pi}{\sigma R^2}} \quad (3.6)$$

In the finite temperature case, R is replaced by β and one has to take into account the periodicity in the "time" direction. The result analogous to Eq.(3.6) is

$$E_n = \sigma \beta \sqrt{1 - \frac{\beta_H^2}{\beta^2} + \frac{8\pi n}{\sigma \beta^2}}, \quad (3.7)$$

where β_H^{-1} is again the Hagedorn temperature given by Eq. (1.2).

As promised in Section 2, we can now understand the energy formulae given by Eqs. (2.12) and (2.14) as the low temperature/large distance approximations of Eqs. (3.7) and (3.6) respectively.

From Eq. (3.7), we expect the partition function to be given by the Boltzmann formula

$$Z_{\text{temp}}(R, \beta) = \sum_{n=0}^{\infty} d(n) \times e^{-\sigma \beta R \sqrt{1 - \beta n^2 / \beta^2 + 8\pi n / \sigma \beta^2}} \quad (3.8)$$

Of course, when β is very large the results of Section 2 are reproduced. Now let us investigate in general the large n behaviour of Eq. (3.8). For $n \rightarrow \infty$, the sum behaves as

$$\sum_{n=0}^{\infty} d(n) e^{-R \sqrt{8\pi \sigma n}} \quad (3.9)$$

Thus we see that in this limit the Boltzmann factor is independent of β ! So the Hagedorn temperature does not in this way play the role of a "limiting" temperature!

This is somewhat surprising. However, we see that the exponential increase of $d(n)$ does create a "singularity", or potential trouble with respect to the convergence of the expansion (3.9). Using Eq. (2.15) for the level density, we see that the exponential decrease of the Boltzmann factor is cancelled by the exponential increase of the level density for $R = R_c$, where R_c is determined by

$$2\pi \sqrt{\frac{d-2}{6} n} = R_c \sqrt{8\pi \sigma n}$$

i.e.,

$$R_c = \sqrt{\frac{\pi(d-2)}{12\sigma}}. \quad (3.10)$$

This is precisely the "minimum" distance in Alvarez's formula (1.1)! So we see that now the minimum distance does not arise from the tachyon, but from the "singularity" caused by the exponentially increasing level density. Loosely speaking, the tachyon is caused by the exponential increase of the density.

It is of interest to derive Alvarez's formula (1.1) directly from the exponential increase of the level density. Let us approximate the larger n behaviour of (3.8) by the integral

$$Z_{\text{temp}} \approx \int_{N_0}^{\infty} dn n^{-\frac{d+1}{4}} \exp \left[-\sigma \beta R \times \right. \\ \left. \times \sqrt{1 - \beta_H^2 / \beta^2 + 8\pi n / \sigma \beta^2} + 2\pi \sqrt{(d-2)n/6} \right]. \quad (3.11)$$

Introducing the variable $t = (1/\beta) \sqrt{8\pi n / \sigma}$, we obtain

$$Z_{\text{temp}} \approx 2 \left(\beta \sqrt{\frac{\sigma}{8\pi}} \right)^{-\frac{d+1}{2}} \int_{\sqrt{8\pi N_0 / \sigma} / \beta}^{\infty} dt t^{-\frac{d-1}{2}} \\ \times \exp \left[-\sigma \beta R \sqrt{1 + t^2} + \beta R_c \sigma t \right], \quad (3.12)$$

where we have introduced R_c from Eq. (1.5). For $\beta \rightarrow \infty$, R finite (i.e., the potential "channel") the integrand in Eq. (3.12) has a saddle point for

$$t = \frac{1}{\sqrt{(R/R_c)^2 - 1}}. \quad (3.13)$$

The behaviour of Z is then given by

$$Z_{\text{temp}} \approx \exp \left[-\sigma \beta \sqrt{R^2 - R_c^2} \right] \quad (3.14)$$

to the leading order. Thus we have derived Alvarez's formulae (1.3)-(1.4) from the exponentially increasing level density, whereas this equation is usually obtained from the tachyon energy! Again, the tachyon and the level density appear essentially to be equally unstable!

So, to sum up the results obtained, we see that if we start from the partition function written in the temperature "channel" $\beta \ll R$, we can then obtain the Wilson loop by continuing to $\beta \rightarrow \infty$ and the tachyon in the Wilson loop originates from the exponential level density in the temperature "channel".

Finally, we shall see how the Hagedorn "singularity" as a limiting temperature does enter after all. To do this, we should start in the potential "channel". There one has from Eq. (3.6)

$$Z_{\text{pot}}(R, \beta) = \sum_{n=0}^{\infty} d(n) \times e^{-\sigma \beta R \sqrt{1 - \frac{\pi(d-2)}{12\sigma R^2} + \frac{2\pi n}{\sigma R^2}}} \quad (3.15)$$

Using Eq. (2.15) for the asymptotic behaviour of $d(n)$, we see that the sum ceases to converge exponentially for

$$2\pi \sqrt{\frac{d-2}{6}} n \geq \sigma \beta R \sqrt{1 - \frac{\pi(d-2)}{12\sigma R^2} + \frac{2\pi n}{\sigma R^2}}$$

$$\underset{n \rightarrow \infty}{\approx} \sigma \beta \sqrt{\frac{2\pi n}{\sigma}},$$

i.e., for

$$\sqrt{\epsilon} \beta \leq \sqrt{\frac{\pi(d-2)}{3}} = \beta_H \sqrt{\epsilon}. \quad (3.16)$$

Hence, when the temperature exceeds the Hagedorn temperature, the sum diverges! Proceeding as in Eqs. (3.11)-(3.14), one finds once more a saddle point for large values of R , and the resulting partition function is

$$Z_{\text{pot}} \approx \exp \left[-\epsilon R \sqrt{\beta^2 - \beta_H^2} \right]. \quad (3.17)$$

which is the same as Eq. (1.1).

Let us end this section by summarizing our results: there are two equivalent ("dual") descriptions of the partition function, namely:

- 1) The "potential channel": the potential for $\beta \rightarrow \infty$, R fixed, can be obtained from Z_{pot} in (3.15) by keeping only the lowest state in the Boltzmann formula. The higher states are exponentially damped. (This is valid as long as $\beta > \beta_H$.) The lowest state is the tachyon and one can therefore say that the tachyon governs the non-leading terms in the potential. However, when β approaches β_H and $R \rightarrow \infty$, the exponential level density causes the square root behaviour (3.17).
- 2) The "temperature channel": the square root behaviour (3.17) now comes as the tachyon contribution (i.e., the lowest energy) to Z given by Eq. (3.8). The square root behaviour (1.4), on the other hand, now comes from the exponentially increasing level density, as seen in Eqs. (3.11)-(3.14).

It therefore appears as if the tachyon and the exponential level density "bootstrap" one another.

We mention, without going into detail, that a very similar analysis can be performed for the Neveu-Schwarz model, which has a tachyon and an exponentially increasing level density.

4. - THE RAMOND MODEL

In the last section, we saw that for the scalar (and the Neveu-Schwarz) model the tachyon and the exponential level density "bootstrap" one another. Thus, one would expect that if there is no tachyon, then there is no exponentially increasing level density. This appears to be incorrect, however, since it is well known that the Ramond model (and various superstrings) have no tachyon, but nevertheless there is an exponentially increasing level density. On the other hand, it is hard to see why the correlation between tachyon and level density worked out in the last section should break down.

In the present section we shall therefore analyze the Ramond model from the point of view discussed in Section 2. The action at large distances/small temperatures is given by

$$S = \sigma \beta R + \sigma \int d^2 \xi \left[\frac{1}{2} (\partial x)^2 + \frac{i}{2} \psi \not{\partial} \psi \right].$$

(4.1)

Again, we consider an $R \times \beta$ rectangle. For the spinor variable we impose the boundary conditions

$$\psi_2 = 0 \quad \text{for } r=0 \text{ and } r=R.$$

(4.2)

In the "time" direction, we impose the condition of antiperiodicity,

$$\psi_{1,2}(r, t) = -\psi_{1,2}(r, t + \beta).$$

(4.3)

This model is therefore really a hybrid of the Ramond and Neveu-Schwarz models. The functional integration over ψ produces the result

$$\prod_{m,n=1}^{\infty} \left[\left(\frac{2\pi}{\beta} \right)^2 \left(m - \frac{1}{2} \right)^2 + \left(\frac{\pi}{R} \right)^2 n^2 \right]^{d-2} \times \prod_{m=1}^{\infty} \left(\frac{2\pi}{\beta} \left(m - \frac{1}{2} \right) \right)^{d-2},$$

(4.4)

which should be multiplied by the scalar result (2.4). Using a generalized ζ function regularization, one obtains⁹⁾

$$Z(R, \beta) = e^{-\sigma R \beta} \prod_{n=1}^{\infty} \left(\frac{1 + e^{-n\pi\beta/R}}{1 - e^{-n\pi\beta/R}} \right)^{d-2} \quad (4.5)$$

Notice that the tachyon factor has disappeared. In the potential channel $\beta \gg R$, one obtains the potential

$$V(R) = \sigma R. \quad (4.6)$$

So, as pointed out before by the author⁷⁾, there is no Lüscher's long-range "Coulomb" potential in this model. Ignoring loop corrections, the potential is exactly linear, due to the lack of a tachyon factor in (4.5). From Eq. (4.5), we get the Boltzmann formula

$$Z_{\text{pot}}(R, \beta) = \sum_{n=0}^{\infty} d_{\text{pot}}(n) e^{-\beta E_n}, \quad (4.7)$$

where E_n is the tachyon-free energy,

$$E_n = \sigma R + \frac{\pi n}{R}, \quad (4.8)$$

and $d_{\text{pot}}(n)$ is the level density with asymptotic behaviour

$$d_{\text{pot}}(n) \underset{n \rightarrow \infty}{\sim} e^{\pi \sqrt{(d-2)n}} \quad (4.9)$$

All these results are consistent with the fact that there is no tachyon, except for the mysterious exponential increase of $d_{\text{pot}}(n)$.

Let us now ask for the consequences of the behaviour (4.9). Using the fact that the partition function can be expressed in terms of θ_4 ,

$$\Theta_4(0|\tau) = \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n-1})^2 = \prod_{n=1}^{\infty} \frac{1-q^n}{1+q^n} \quad (4.10)$$

as well as

$$\Theta_4(0|\tau) = \frac{1}{\sqrt{-i\tau}} \Theta_2(0|\frac{1}{\tau}), \quad (4.11)$$

we see that Eq. (4.5) can also be expressed as

$$\begin{aligned} Z(R, \beta) &= e^{-\sigma\beta R + \frac{\pi(d-2)}{4} \frac{R}{\beta}} \\ &\times \prod_{n=1}^{\infty} \left(1 - e^{-\frac{4n\pi R}{\beta}}\right)^{-(d-2)} \left(1 + e^{-\frac{4n\pi R}{\beta}}\right)^{-2(d-2)} \\ &= \sum_{n=0}^{\infty} d_{\text{temp}}(n) e^{-R E_n}, \end{aligned} \quad (4.12)$$

with

$$E_n = \sigma\beta - \frac{\pi(d-2)}{4} \frac{1}{\beta} + \frac{4\pi n}{\beta}. \quad (4.13)$$

In Eq. (4.12), we have ignored a $\ln \beta/R$ which is not linear in β or R .

The level density $d_{\text{temp}}(n)$ gives the number of states actively coupling in the temperature "channel". Using the asymptotic relations

$$\prod_{n=1}^{\infty} (1-x^n) \underset{x \rightarrow 1}{\sim} e^{-\frac{\pi^2}{6(1-x)}},$$

$$\prod_{n=1}^{\infty} \frac{1+x^n}{1-x^n} \underset{x \rightarrow 1}{\sim} e^{\frac{\pi^2}{4(1-x)}}$$

(4.14)

we see that

$$\left[\prod_{n=1}^{\infty} (1-x^n)(1+x^n)^2 \right]^{-1} \underset{x \rightarrow 1}{\sim} e^0$$

(4.15)

times non-leading (i.e., non-exponential) terms. Thus, using

$$d_{\text{temp}}(n) = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} \frac{1}{\left(\prod_s (1-z^s)(1+z^s)^2 \right)^{(d-2)}} \quad (4.16)$$

we see in the usual way⁴⁾ that there is no exponential increase of $d_{\text{temp}}(n)$. Hence the sum (4.12) is always exponentially damped, and one obtains the effective temperature-dependent string tension from the lowest term in Eq. (4.12):

$$\sigma(\beta) = \sigma - \frac{\pi(d-2)}{4\beta^2} = \sigma \left(1 - \frac{1}{2} \frac{\beta_H^2}{\beta^2} \right). \quad (4.17)$$

There is thus a tachyonic correction to the string tension.

The reason for the tachyon-like singularity is due to the exponentially increasing level density $d_{\text{pot}}(n)$, as can be seen from Eqs. (4.7)-(4.9), in the same way as in Section 2 [see Eqs. (2.16) and (2.17)]. In the limit where β is large, Eq. (4.7) can be approximated by

$$\begin{aligned}
 Z &\sim \sum_{N_0}^{\infty} e^{-\sigma\beta R - \frac{\pi n\beta}{R} + \pi \sqrt{(d-2)n}} \\
 &\sim e^{-\sigma\beta R} \int_{N_0}^{\infty} dn e^{-\frac{\pi n\beta}{R} + \pi \sqrt{(d-2)n}} \\
 &\approx 2 e^{-\sigma\beta R + \frac{\pi(d-2)R}{4\beta}} \int_{\sqrt{N_0}}^{\infty} dx x e^{-\frac{\pi\beta}{R} (x - \frac{1}{2}\sqrt{d-2}\frac{R}{\beta})^2} \quad (4.18)
 \end{aligned}$$

So we see that for β large, the exponentially increasing level density produces a singularity $\exp(\pi(d-2)R/4\beta)$, which is precisely the tachyonic singularity in Eq. (4.17). However, in the temperature channel, the level density does not increase exponentially, and hence no tachyon singularity occurs in the potential $V(R)$.

One can also extend these considerations beyond the large distance/low temperature case. Proceeding as in Section 3, one finds that Eqs. (4.12)-(4.13) should be replaced by

$$Z(R, \beta) = \sum_{n=0}^{\infty} d_{\text{pot}}(n) e^{-\sigma\beta R \sqrt{1 + 2\pi n / \sigma R^2}} \quad (4.19)$$

The sum converges as long as

$$\sigma\beta R \sqrt{1 + \frac{2\pi n}{\sigma R^2}} > \pi \sqrt{(d-2)n},$$

i.e., for large n for

$$\beta > \sqrt{\frac{\pi(d-2)}{2\sigma}} = \beta_H, \quad (4.20)$$

where again β_H is the Hagedorn temperature for the Ramond model. If one continues as in Eqs. (3.11)-(3.14), one finds

$$Z_{\text{pot}}(R, \beta) \approx \exp \left[-\sigma \beta R \sqrt{1 - \beta_H^2 / \beta^2} \right]. \quad (4.21)$$

The exponentially increasing level density thus gives rise to a tachyonic singularity at $\beta = \beta_H$, just like in the scalar case.

However, due to the fact that $d_{\text{temp}}(n)$ does not increase exponentially, the potential behaves as

$$V(R) = \sigma R. \quad (4.22)$$

So, in the present model, we again find the feature that the exponentially increasing level density gives rise to a tachyonic behaviour. Furthermore, if there is no exponential increase of the level density, then we do not find any tachyon. In this remark, the level density always refers to the number of states which actively participate in a given "channel".

5. - CONCLUSIONS

The main conclusion of this paper is that the "singularity" arising when the Boltzmann factor $e^{-\beta E_n}$ can no longer compete with the exponential level density $d(n)$ gives rise to a singularity which is of a tachyonic character. This conclusion holds both for the scalar and for the Neveu-Schwarz models.

In the Ramond model, the situation is more complicated. We have discussed a model where the eigenfunctions depending on r have integer modes of the form

$$\sin \quad \text{or} \quad \cos \left(n \frac{\pi r}{R} \right). \quad (5.1)$$

However, due to the fact that we took antiperiodic boundary conditions in the "time" direction, the eigenfunctions have a time dependence of the form

$$\sin \quad \text{or} \quad \cos\left(\left(n - \frac{1}{2}\right) \frac{2\pi t}{\beta}\right). \quad (5.2)$$

This mixture of even and odd modes causes the tachyon to reappear in the temperature-dependent effective string tension. In the potential, on the other hand, there is no tachyon, as one would expect for the Ramond model. This result is not surprising; in the potential case one takes the limit $\beta \rightarrow \infty$. Thus, the boundary conditions in the time direction are of no relevance, and hence only the integer modes are important. In the time direction, it is just the opposite situation, where only the half-integer modes are important.

In supersymmetric models one would always expect (ignoring loop corrections) that

$$V(R) = \sigma R, \quad (5.3)$$

since there is no non-vanishing vacuum energy. However, if one goes to the finite temperature case, the situation becomes more complicated because it is not easy to reconcile finite temperatures with supersymmetry.

One may hope that it is possible to construct a supersymmetric model without any tachyon in the "temperature channel". If this can be done, one would expect the effective string tension to be temperature-independent,

$$\sigma(\beta) = \sigma \quad (5.4)$$

In such a case it would be of much interest to investigate the Boltzmann formula

$$Z = \sum d(n) e^{-\beta E_n} \quad (5.5)$$

to see how the potential singularity is avoided.

In this connection we mention that it is possible to construct a Ramond type of model where the antiperiodic boundary condition (4.3) for the spinor field is replaced by a periodic boundary condition,

$$\psi_{1,2}(r,t) = + \psi_{1,2}(r,t). \quad (5.6)$$

It is then easy to see that the scalar degrees of freedom exactly cancel the spinor degrees of freedom in the partition function [this was pointed out a long time ago in Ref. 10)], and instead of Eqs. (4.5) and (4.12), one obtains the simple result

$$Z(R,\beta) = e^{-\sigma\beta R}, \quad d(n)=0 \quad \text{for } n > 0, \quad (5.7)$$

for the large distance/low temperature approximation to the partition function. Thus, this model has the feature that there is no exponentially increasing level density! In accordance with the previous analysis, there is therefore no tachyon instability in the potential or in the finite temperature string tension, for which one simply obtains

$$V(R) = \sigma R \quad \text{and} \quad \sigma(\beta) = \sigma. \quad (5.8)$$

This model thus supports the point of view emerging from the analysis in Sections 2, 3 and 4, in the sense that absence of an exponentially increasing level density in both "channels" implies the absence of a tachyonic instability in both "channels".

Finally, let us mention that in the present paper we have considered only open strings. This has the advantage that one can look at large distances by fixing the end-points. Hence, higher loop corrections may not be of crucial importance, since for strings non-perturbative effects are expected to be of importance at short distances. A likely exception to this expectation is the case when $R \rightarrow R_c$ or $1/\beta \rightarrow 1/\beta_H$, because then the string oscillates violently (with an infinite length). Hence the string probably has many self-crossings, and this means that a lot of closed strings can be "burned off" the violently-oscillating string. This situation presumably involves non-perturbative effects, or perhaps even a transition to a different "deconfined" theory^{1),2)}.

For closed strings the situation is generally much more complicated because, for example, one closed string can go into two closed strings. This process necessarily involves short distances, i.e., non-perturbative effects. Hence the

case of closed strings is in general more difficult than the case of open strings.

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REFERENCES

- 1) R.D. Pisarski and O. Alvarez, Phys. Rev. D26 (1982) 3735.
- 2) P. Olesen, Phys. Lett. B, to be published (1985).
- 3) S. Fubini, D. Gordon and G. Veneziano, Phys. Lett. 29B (1969) 679.
- 4) K. Huang and S. Weinberg, Phys. Rev. Lett. 25 (1970) 895.
- 5) O. Alvarez, Phys. Rev. D24 (1981) 440.
- 6) J.F. Arvis, Phys. Lett. 127B (1983) 106.
- 7) P. Olesen, Phys. Lett. B, to be published (1985).
- 8) M. Lüscher, Nucl. Phys. B180 [FS2] (1981) 317.
- 9) M. Minami, Prog. Theor. Phys. 59 (1978) 1709.
- 10) L. Brink and D. Fairlie, Nucl. Phys. B74 (1974) 321.