LORENTZ CHERN-SIMONS TERMS IN $N = 1$ 4D SUPERGRAVITY
CONSISTENT
WITH SUPERSYMMETRY AND STRING COMPACTIFICATION

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ABSTRACT

We consider the effective $N = 1$ 4 D supergravity obtained from superstring compactification and we establish full supersymmetry of the Lorentz Chern-Simons terms induced by the modified $D = 10$ supergravity Lagrangian. Supersymmetry completion requires an extra coupling of the chiral multiplet S (containing an axion) to a chiral gravitational multiplet which includes both the Hirzebruch signature and the Euler characteristic densities. Then the consistency of the topological condition coming from superstring compactification and full off-shell supersymmetry is eventually proved.

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Among all popular candidates for a quantum theory unifying all known interactions (including gravity) the only ones which, at present, are not obviously wrong are the Superstrings [1]. They turn out to have no anomaly [2], nor ghosts or tachyons, and may be even finite, order by order, in perturbation theory [3]. A major clue towards a working theory, was the discovery by Candelas et al. [4] (arguing in the field-theoretical limit) that superstrings undergo spontaneous compactification on a 6-dimensional internal space K of the Calabi-Yau type [5], giving an effective flat 4 D Minkowsky space-time.

In that case a four dimensional supergravity theory was found, with one unbroken supersymmetry and $(1/2)|\chi(K)|$ chiral generations of fermions [4]. Moreover, there is a good evidence [4,6] that these field-theoretical solutions lift-up to solutions of the full-fledged theory of (classical) superstrings.

More recently, Dixon et al. [7] have produced an explicit, exact example of a topologically non-trivial compactification of the full string theory to 4 D, with the internal space being a so-called orbifold, which is just an extreme case of the Calabi-Yau situation.

At this point, we know that superstrings are consistent relativistic quantum theories, potentially containing all interactions, so the only question which remains to be answered is whether such a theory describes the world we live in.

The most effective way to extract experimentally detectable informations out of the string, is to construct the appropriate four dimensional effective Lagrangian. Witten [8] has explicitly written down the 4 D effective Lagrangian by dimensional reduction of the 10 D one - in a particularly manageable case - somehow related to the orbifold compactification.

However, the correct effective Lagrangian cannot, at present, be completely computed by dimensional reduction, for the very simple reason that we have not a full understanding of the 10 D case. In particular, we do not know the additional terms one has to add to the Chapline-Manton Lagrangian [9] in order to recover the supersymmetry which is violated by the Lorentz Chern-Simons 3-form added in the improved 3-index field strength of the 2-form $B_{\mu\nu}$ field (1)

$$\hat{H} = dB + (1/30)\omega_{3YM} - \omega_{3L} \quad (1)$$

in order to cancel the anomalies [2]. Some of the bosonic terms we need are at present only conjectured. Indeed, it is not even clear if there

exists a supersymmetric completion of these terms which is a local Lagrangian. Many physicists conjecture that we need terms of any order in the curvatures and their covariant derivatives, so that only the full superstring is anomaly-free and supersymmetric (2).

However, we can also argue the other way around, namely, starting with the 4 D theory which - one SUSY being unbroken - must be some sort of consistent $N = 1$ supergravity Lagrangian. It cannot be, by any means, present in the general class of Lagrangians discussed in ref.[11], simply because the analysis was confined to Lagrangians with no more than 2 derivatives, whereas from the string we must obtain higher order terms. On the other hand, a full tensor calculus [12] exists for $N = 1$ $D = 4$ SUGRA which enables us to construct full supersymmetric terms of any order in the curvatures and their derivatives. Thus, we have to look for the consistent supersymmetric completion of the Chern-Simons contribution to the effective Lagrangian in the framework of the $N = 1$ tensor calculus in $D = 4$. The results are quite amusing. First of all the supersymmetric completion is unique, and quadratic in the supercurvature multiplets. This gives hopes for attacking the same problem in the 10 D case. Moreover, the supersymmetric completion contains an $R_{\mu\nu ab} R^{\mu\nu ab}$ term with the right coefficient needed to make the configurations of ref.[4] consistent solutions to the equations of motion (in the field limit). But there are also other terms, quadratic in the Riemann tensor, which vanish on shell. The combination actually present must be

$$R_{\mu\nu ab} R^{\mu\nu ab} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \quad (2)$$

which is just proportional to the Euler characteristic density [19]

$$(1/32\pi^2) \int \epsilon_{abcd} R^{ab} \wedge R^{cd} \quad (3)$$

and so it gives no negative-norm state, in agreement with a recent analysis by Zwiebach [13]. The same multiplet contains also the Hirzebruch signature density, so that now the equations of motion of the scalars correctly imply the topological selection rule

$$v + 3\tau = 0 \quad (4)$$

$$v = (1/8\pi^2) \int (1/30) \text{Tr}(F \wedge F) \quad (-\text{Pontryagin number}) \quad (5)$$

$$\tau = -(1/24\pi^2) \int \text{Tr}(R \wedge R) \quad \text{Hirzebruch signature} \quad (6)$$

Eq. (4) is just the topological constraint from the cohomological analysis of the original theory in 10 D [10]. More crucially, it can be shown that all field configurations of the effective 4 D model which do not satisfy the topological selection rule do not contribute, by consistency, to the quantum path integral.

In this way we obtain a 4 D effective Lagrangian for the simplified case of compactification [8] which is totally consistent, both from the topological as well as from the supersymmetric point of view. But, as it will be clear in the following, the structure of the supersymmetric completion is totally general, and does not depend on the particular case envisaged.

We consider the dimensionally reduced theory from $D = 10$ to $D = 4$ and take into account the effects of the modifications given by eq.(1). It is known [8] that the effect of ω_{3YM} in eq.(1), in the four dimensional theory, is to have a coupling of a Wess-Zumino chiral multiplet S to the Yang-Mills field strength chiral multiplet [11] $W_\alpha(V)$.

The S chiral multiplet contains, as bosonic degrees of freedom, a complex scalar z whose real and imaginary parts are related to the dilaton and Calabi-Yau breathing mode, as well as to the antisymmetric field as follows [8] :

$$\text{Re } z = e^{3\sigma} \phi^{-3/4}, \quad \text{Im } z = 3\sqrt{2} D \quad (7)$$

where the scalar field D is related to the $B_{\mu\nu}$ field through a duality transformation

$$(\text{Re } z)^2 \hat{H}_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma} \partial^\sigma D \quad (8)$$

Since the $\hat{H}$ field strength of the $B_{\mu\nu}$ field contains both the Yang-Mills and the Lorentz Chern-Simons three forms, the duality transformation introduces a (linear) coupling of the axion to the combination

$$(1/30) \text{Tr}(F \wedge F) - \text{Tr}(R \wedge R) \quad (9)$$

as one immediately gets after a partial integration, by using eq. (7) and the fact that the expression in eq. (9) is actually equal to

$$(1/30) d\omega_{3YM} - d\omega_{3L}$$

The axion coupling is therefore (up to a numerical factor) equal to

$$\text{Im } z [(1/30) \text{Tr}(F \wedge F) - \text{Tr}(R \wedge R)] \quad (10)$$

The first term in eq.(10) combined with the Yang-Mills Lagrangian

$$-(1/120)\phi^{-3/4} e^{3\sigma} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) = -(1/120) \text{Re } z \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \quad (11)$$

gives rise to the bosonic part of the last component of the superfield expression [8]

$$f_{AB}(S) W^A W^B \quad (12)$$

where $f_{AB}(S) = \delta_{AB} S$.

However, as we will prove, the second term in eq.(10) has no supersymmetric completion unless we add a coupling of $\text{Re } z$ to the expression given by eq.(3); to do this, we need the appropriate multiplets containing the square curvature terms of the gravitational sector.

In the general formulation of matter-coupled supergravity Lagrangians, as described in ref.[12], one uses a tensor calculus which is entirely equivalent to the superspace geometry introduced by Wess and Zumino [14]. In this superspace geometry all the components of supercurvatures and supertorsions can be solved in terms of three basic multiplets, which were introduced, at the linearized level, in ref.[15].

The three basic multiplets are the two chiral multiplets $W_{\alpha\beta\gamma}$ and R and one (real) vector multiplet $E_{\alpha\dot{\alpha}}$ and they appear as the general solution of the Bianchi Identities of superspace geometry. These multiplets are called the Weyl, Curvature scalar and Einstein multiplets, because they contain in their components the Weyl tensor $C_{\mu\nu\rho\sigma}$ (traceless part of the Riemann tensor), the curvature scalar and the Einstein tensor of General Relativity. The tensor which appears in the Einstein multiplet is the following particular combination of the Ricci tensor and the curvature scalar

$$R_{\mu\nu} - (1/6) g_{\mu\nu} R \quad (13)$$

or, in terms of the Einstein tensor $R_{\mu\nu} - (1/2)g_{\mu\nu} R$ is

$$E_{\mu\nu} - (1/3) g_{\mu\nu} E \quad (E = E^\mu_\mu = -R) \quad (14)$$

Actually, to be more precise, the members of the supermultiplets are the same tensors with full flat indices

$$C_{\alpha\beta\gamma\delta}, \quad C_{\alpha\beta\gamma\delta}^{\cdot\cdot\cdot\cdot} \quad \text{and} \quad E_{\alpha\dot{\alpha}}$$

The Bianchi identities give the following conditions on the three multiplets [14]

$$\bar{\nabla}_{\dot{\alpha}} R = 0, \quad \bar{\nabla}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0 \quad (\text{chirality property}) \quad (15)$$

$$\nabla^{\alpha} E_{\alpha\dot{\beta}} = \bar{\nabla}_{\dot{\beta}} R^*, \quad \nabla^{\alpha} W_{\alpha\beta\gamma} = \nabla_{\beta}^{\dot{\gamma}} E_{\gamma\dot{\tau}} + \nabla_{\gamma}^{\dot{\tau}} E_{\beta\dot{\tau}} \quad (16)$$

From the three multiplets introduced above one can construct three chiral multiplets

$$W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}, \quad (\bar{\nabla}\bar{\nabla}-8R)E_{\alpha\dot{\beta}}E^{\alpha\dot{\beta}}, \quad (\bar{\nabla}\bar{\nabla}-8R)RR^* \quad (17)$$

which, in the tensor calculus notation [12], are given by

$$\Sigma(W^2_L), \quad \Sigma(E^2), \quad \Sigma(RT(R^*)) \quad (18)$$

where $T(R^*)$ is the chiral multiplet $(\bar{\nabla}\bar{\nabla}-8R)R^*$. The symbol Σ denotes the operation which associates, to a product of two multiplets, a chiral multiplet. We use standard notations A, B, χ, F and G for the (real) components of a chiral multiplet. It is then easy to see that the Hirzebruch signature density is contained in the G -component of the $\Sigma(W^2_L)$ multiplet [12], while the Euler characteristic density is contained in the F component of the following multiplet [15]

$$\hat{\Sigma}(W^2_L) = \Sigma(W^2_L) + (1/2)(\Sigma(E^2) - 4\Sigma(RT(R^*))) \quad (19)$$

The component expression of this multiplet was derived in ref. [16] (3). The above multiplet is the unique chiral multiplet with the property that its F and G components contain the two topological densities which are quadratic in the Riemann tensor.

Also, it is the multiplet which avoids ghost propagation, in the sense that its last component is a full (superspace) divergence, i.e.

$$\int d^4x \int d^2\theta \mathcal{E} \hat{\Sigma}(W^2_L) = 0$$

(Here $\int d^2\theta \mathcal{E}$ is the chiral density factor of superspace geometry [14]). We can see now that the supersymmetric completion of eq.(10) modifies eq.(12) as follows

$$S [(1/30)\Sigma(W^2_{YM}) - \hat{\Sigma}(W^2_L)] \quad (20)$$

where S is the Wess-Zumino multiplet containing the axion.

We note some feature of eq.(20). The supersymmetric completion gives new fermionic terms which couple the real and imaginary part of z to terms quadratic in the gravitino curvature (with three derivatives acting on the gravitino) as well as terms which couple the spin-1/2 partner of z to bilinear terms in the Einstein and gravitino curvatures. From the tensor calculus one can explicitly compute all these terms.

These terms show that the supersymmetric completion of the $D = 10$ theory requires new terms involving the gravitino field strength and the Einstein curvature (4).

The knowledge of these terms in the $D = 4$ $N = 1$ theory can in principle be used to deduce the supersymmetric completion to the corresponding ten dimensional theory.

We observe that on the mass-shell the last two terms in eq.(19) are proportional to the equations of motion of the supergravity sector, so one obtains

$$\hat{\Sigma}(W^2_L) = \Sigma(W^2_L) + \text{eqs. of motion terms} \quad (21)$$

Therefore, we have established that the full supersymmetric completion of the $D = 4$ $N = 1$ supergravity Lagrangian requires the extra term quadratic in the Riemann tensor which, in 10 dimensions, was added in ref.[4] to get non trivial compactifications [17], and proven to arise in the Heterotic string [6,18].

In particular in the original $D = 10$ supergravity Lagrangian, the R^2 term is part of the supersymmetric completion of the modified Chapline-Manton Lagrangian, with $\hat{H}$ containing the Lorentz Chern-Simons three form.

In addition the supersymmetric completion unavoidably requires new terms in $D = 10$ dimensions, for instance, terms quadratic in the gravitino curvature and terms which couple the λ -spin1/2 singlet of 10 D Supergravity to expressions bilinear in the gravitino and Einstein curvatures.

One important consequence of the new term in eq.(20) is that, now, the equation of motion for the axion, $\text{Im } z$,

$$\begin{aligned} -iD^\mu \{ [\partial_\mu (z-z^*)] / (z+z^*)^2 \} &= (e^{-1}/8) [(1/30)\text{Tr}(F_{\mu\nu} \tilde{F}^{\mu\nu}) - \text{Tr}(R_{\mu\nu} \tilde{R}^{\mu\nu})] \\ &+ (\text{total derivative of gauge invariant terms}) \end{aligned} \quad (22)$$

implies, by integration, the topological constraint [19]

$$(1/8\pi^2) \int [(1/30)\text{Tr}(F\wedge F) - \text{Tr}(R\wedge R)] = v + 3\tau = 0 \quad (23)$$

which corresponds to the 10-dim. Bianchi identity for the $H_{\mu\nu\rho}$ field strength [10]

$$\hat{dH} = (1/30) \text{Tr}(F\wedge F) - \text{Tr}(R\wedge R) \quad (24)$$

integrated on the particular 4-submanifold corresponding to the physical 4 dimensions. This property is sufficient for the classical consistency of our effective Lagrangian with the fundamental 10-dim. theory. However, quantum consistency requires a bit more. Indeed, eq.(24) is an identity which holds even on-shell, and so it must be for the condition (23) for the correct 4 D theory. This means that the classical field configurations violating the topological selection rule - eq.(23) - are to cancel in the quantum (Euclidean) path integral. In order to show that this is indeed the case, we assume - for definiteness - that the Euclidean 4-space is compactified in a suitable topology.

After Euclidean continuation, the relevant terms in the Lagrangian read

$$\begin{aligned} \mathcal{L}_{\text{Eucl}} = & eg^{\mu\nu} [(\partial_\mu \text{Im } z)(\partial_\nu \text{Im } z)/(z+z^*)^2] \\ & + (i/4) \text{Im } z [(1/30)\text{Tr}(F_{\rho\sigma} \tilde{F}^{\rho\sigma}) - \text{Tr}(R_{\rho\sigma} \tilde{R}^{\rho\sigma})] + \dots \end{aligned} \quad (25)$$

In the path integral, we perform first of all the integration over the coefficient x_0 of the constant mode in the expansion for $\text{Im } z$. From eq.(25) this integral gives

$$\begin{aligned} \int dx_0 \exp \{-ix_0 \int [(1/30)\text{Tr}(\tilde{F}\tilde{F}) - \text{Tr}(\tilde{R}\tilde{R})] d^4x\} = \\ = 2\pi\delta \{ \int [(1/30)\text{Tr}(\tilde{F}\tilde{F}) - \text{Tr}(\tilde{R}\tilde{R})] \} = \\ = \text{const } \delta(v + 3\tau) \end{aligned} \quad (26)$$

so the functional integral becomes

$$\int [d\phi] e^{-S(\phi)} \delta(v + 3\tau) \quad (27)$$

and the only contributing configurations are those with $v + 3\tau = 0$. As a first consequence of this, the classical symmetry

$$S \rightarrow S + ic \quad (28)$$

(where c is a real number) is not broken by instanton effects [20], whereas it would be spoiled in the truncated theory [8]. Indeed, eq.(27) just says that the gravitational instantons exactly compensate for the gauge ones.

The topological selection rule, eq.(23), puts some constraints on the possibility of having a suitable classical field configuration which contributes to the "shadow" gluino condensation. Indeed, $v \neq 0$ would require $\tau \neq 0$, so we might expect unwanted zero-modes from singlet massless fermions (for instance the gravitino) in agreement with the index theorem [19] (5).

It must be stressed that, while the specific model we discussed is a simplified version of the 4 D effective theory we get from the superstring, the topological selection rule - eq.(23) - is general, so it must be for the mechanism we found in four dimensions to enforce it. This puts some constraints on the structure of possible 4 D effective Lagrangians for superstrings.

Finally we point out that our analysis has been confined to the supersymmetric completion of the Lorentz Chern-Simons terms. Implementation of full off-shell supersymmetry in the $N = 1$ 4 D effective theory requires the R^2 terms in the tree Lagrangian added in ref.[4] to get non trivial compactification and proven to arise in the superstring theory [18,6]. The Wess-Zumino terms required in the 10 D theory to cancel the anomalies add, in principle, new terms to the quantum effective action but their supersymmetric completions is beyond the scope of the present investigation.

NOTES

(1) Our notation agrees with that of ref.[2]

(2) As it will be clear in the following, the problem with the Lagrangian of ref.[9] is that the axion field equation would enforce the topological constraint

$$\int \text{Tr}(F \wedge F) = 0$$

whereas the complete 4 D Lagrangian should imply

$$\int [\text{Tr}(F \wedge F) - 30 \text{tr}(R \wedge R)] = 0$$

as discussed by Witten (ref.[10]).

(3) We use conventions such that

$$\Sigma(W^2_{YM})_F = \text{Tr}(F^2_{\mu\nu}) + \dots, \quad \Sigma(W^2_L)_F = C^2_{\mu\nu\rho\sigma} + \dots$$

and also

$$\Sigma(W^2_{YM})_G = \text{Tr}(F_{\mu\nu} \tilde{F}_{\mu\nu}) + \dots, \quad \Sigma(W^2_L)_G = \text{Tr}(R_{\mu\nu} \tilde{R}_{\mu\nu}) + \dots$$

where

$$\tilde{T}_{\mu\nu} = (1/2) \epsilon_{\mu\nu\rho\sigma} T^{\rho\sigma}$$

(4) The component form of the full, consistent Lagrangian will be exhibited elsewhere.

(5) The situation is a bit more complex. If in one particular solution the sfermionic fields are not vanishing, we get some correction to the Dirac operator $i\not{D}$ from mass-like terms. However, for the particular model at hand, these mass-like terms for different fermions are related, so it seems very hard to obtain all wanted zero-modes without any other spurious one.

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