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QUALITATIVE FEATURES OF
THE GLUEBALL SPECTRUM*

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R. L. Jaffe, K. Johnson and Z. Ryzak**

*Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.*

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** On leave from Warsaw University, Warsaw, Poland.

ABSTRACT

The low energy meson spectrum is investigated. Quantum numbers of the lightest glueballs are obtained by studying low dimension, gauge invariant, colorless operators. A general shell model of hadrons is formulated and its predictions are specified. A comparison and criticism of existing glueball models is included.

1. INTRODUCTION

QCD predicts the existence of hadrons made all or in part from gluonic excitations.^{1,2,3} Unfortunately, both theory and experiment have yet to give precise information concerning properties of those particles – so called glueballs. In particular, there is much disagreement in the literature regarding the masses and quantum numbers of light glueballs.⁴

Phenomenological glueball models have arisen as a natural extension of models that have proved successful in the quark sector. Examples include bag models,^{3,5} duality-OZI models,^{1,6} “non-relativistic” gluon models,^{7,8} QCD sum rules⁹ and lattice calculations.¹⁰ In this paper, we analyze the spectrum of light glueballs using a pair of assumptions shared by most of the reasonable, QCD based models. Our objective is to identify model independent features of the spectrum and to clarify relations among models. We show that the low lying glueball spectrum is sparse: the lightest particles have $J^{PC} = 0^{++}$ and 2^{++} followed by 0^{-+} and 2^{-+} . There should be no light glueball with exotic J^{PC} . In the framework of our discussion, it is easy to see the origins of differences among models. For example, the fact that the lightest three gluon states are 0^{++} , 1^{+-} , 3^{+-} in the bag model, but 0^{-+} , 1^{--} and 3^{--} in the constituent glue model is obvious. The dense spectrum of the constituent glue model is discussed. The appearance of a (we believe, spurious) light 1^{-+} in lattice calculations and in the constituent glue model is explained.

We do not attempt directly to build states in the Hilbert space of color singlet hadrons. Instead, we study gauge invariant, colorless, local operators, which connect the vacuum to physical states. This approach was first suggested by Fritzsche and Minkowski¹² and by Bjorken¹¹ and we find it very convenient.

Our first assumption is that the mass of a hadron is roughly proportional to dimension of the simplest composite, local operator which can create it from the vacuum. In any constituent model (*e.g.*, bag or non-relativistic quark model) this conjecture is easily verified: higher dimensions require more, or higher excited quarks and/or gluons.

Our second assumption is that among the low lying states, selected by dimensional analysis, only the lightest can still be coupled to the vacuum when the fundamental fields are replaced by single modes in some hadronic mean field theory. We have in mind that in the low energy region there should exist an effective theory of hadrons in which objects with quark and gluon quantum numbers appear as “quasi particles”. In principle, it is derivable from QCD provided one makes a proper choice of coordinates corresponding to constituent quark and gluon states. Hadrons appear as collections of constituent quark and/or gluon excitations. The lightest particles are populated by quarks and/or gluons in their lowest modes.

All constituent quark models are based on such a premise. Indeed, the experimentally observed $SU(6) \times O(3)$ symmetry of the hadron spectrum which motivated the quark model in the first place, argues that no matter how complex the confining dynamics of QCD may be, it admits a simple, effective quark-quon shell model description at low energies.¹² In practice, the only element of this picture which we use is the existence of the lowest energy mode with some definite quantum numbers for both quarks and gluons.

The analysis proceeds in two steps. First, we determine the J^{PC} of states created from the vacuum by low dimension, color-singlet, composite operators. Then we restrict the fundamental fields to the quark and gluon ground modes, and determine which of the particles appearing in step one can still be coupled to the vacuum. This leads to an ordering of hadron masses. Two hadrons created by two operators whose dimensions differ by one have energies differing by roughly one unit (presumably of order Λ_{QCD}). Accordingly, the masses of the states interpolated by an operator of a given dimension fall within an interval whose mean value is proportional to the operator's dimension and whose width corresponds to a unit of excitation. Within a given interval, i.e., among states interpolated by operators of a given dimension, particles coupled to the vacuum by operators restricted to the lowest mode are the lightest. Therefore, we assign to them masses in the bottom of the interval. Obviously, we do not mean to treat our model too quantitatively.

It has become customary to discuss three sectors of QCD with baryon number zero: quark mesons, pure glueballs, and hybrids. Only the pure glue sector can be defined unambiguously: by considering QCD without quarks. There is no similar rigorous way of distinguishing a hybrid from an ordinary quark meson. To distinguish among all three sectors one has to adopt a constituent picture. In this picture, to the zeroth order in some effective coupling constant g , quark mesons (glueballs) are connected to the vacuum by operators built exclusively of the quark (gluon) fields. Hybrids correspond to the rest of the physical meson spectrum.

At each step of the discussion we analyze first the $q\bar{q}$ meson sector. Calculations in this well-known sector elucidate the subsequent analysis of glueballs and hybrids. The simplest meson interpolating fields are of the form $\bar{q}\Gamma q$. Flavor is suppressed, Γ is any Dirac matrix and colors are uniquely specified by the overall restriction to singlets. As we shall see, these operators create states with

$$J^{PC} = 0^{-+}, 1^{--}, 0^{++}, 1^{++}, 1^{+-}. \quad (1.1)$$

Operators of dimension 4: $\bar{q}D\Gamma q$ (D is the covariant derivative) create new quark meson states:

$$J^{PC} = 2^{++}, 2^{-+}, 2^{--}. \quad (1.2a)$$

In fact, operators $\bar{q}D\Gamma q$ create also other meson states:

$$J^{PC} = 0^{--}, 0^{+-}, 1^{-+}. \quad (1.2b)$$

In section 2, we show that in any constituent model, all of the states (1.2b) would be classified as hybrids. In those models, in the leading order in the effective coupling

constant, the states (1.2b) decouple from the operator $\bar{q}D\Gamma q$. Hybrid operators of dimension 5 are then the lowest dimension operators that can create the states (1.2b) from the vacuum. One concludes that those states are heavier than the quark meson states (1.2a). On the other hand, we note that in QCD one cannot find any obvious justification for this classification. In section 2, we comment that it may be just an artifact of the shell model approach.

The mode analysis of section 3 leads to a conclusion that the lightest $q\bar{q}$ mesons are 0^{-+} and 1^{--} . In Fig. 1, we summarize our prediction about the light quark meson spectrum. Note our analysis gives only the lightest state with each J^{PC} so "radial excitations" are missing from Fig. 1. Fig. 1 does correspond roughly to the physical $q\bar{q}$ spectrum; we regard it as a measure of accuracy we expect in the glueball and hybrid spectra.

Before going to the actual calculations let us make a few general remarks. Note that if an operator $\Theta^{\mu\nu\lambda\dots}$ is a total derivative of an operator of a lower dimension:

$$\Theta^{\mu\nu\lambda\dots} = \partial^\mu \chi^{\nu\lambda\dots} \quad (1.3)$$

then the states created by Θ are not new. They are the same ones created by χ . Consider the matrix element between some state of momentum $\vec{p}$ and mass m ($p^\mu = (m, \vec{p})$) and the vacuum:

$$\begin{aligned} \langle p | \Theta^{\mu\nu\lambda\dots} | 0 \rangle &= f^{\mu\nu\lambda\dots} \\ &= p^\mu \langle p | \chi^{\nu\lambda\dots} | 0 \rangle \\ &= p^\mu h^{\nu\lambda\dots} \end{aligned} \quad (1.4)$$

If $f^{\mu\nu\lambda\dots} \neq 0$ then $h^{\mu\nu\lambda\dots} \neq 0$ as claimed.

Another special case occurs when

$$\Gamma^{\nu\lambda\dots} = \partial_\mu \chi^{\mu\nu\lambda\dots}. \quad (1.5)$$

Suppose that the operator χ is conserved (i.e., $\Gamma \equiv 0$) and the associated symmetry is not spontaneously broken. Then some states which could *a priori* be created by χ must decouple. The J^{PC} quantum numbers of those particles can be found easily: Consider a matrix element between a state of momentum $\vec{p}$ and mass m and the vacuum:

$$\langle p | \chi^{\mu\nu\lambda\dots} | 0 \rangle = f^{\mu\nu\lambda\dots}. \quad (1.6)$$

In the particle rest frame:

$$0 = \langle m, \vec{p} = 0 | \partial_\mu \chi^{\mu\nu\lambda\dots} | 0 \rangle = m f^{0\nu\lambda\dots}. \quad (1.7)$$

Either $m = 0$, corresponding to a spontaneous symmetry breakdown, or $f^{0\nu\lambda\dots} = 0$ corresponding to a decoupling of $\chi^{0\nu\lambda\dots}$ in this frame. The rotation group decomposition of $\chi^{0\nu\lambda\dots}$ tells us which J^{PC} cannot be created by χ .

Usually, the identification of operators that are total derivatives or are conserved is complicated because of the quantum nature of the theory. Fortunately

this complication is handled automatically in our discussion since every operator of a given dimension is analyzed in its own right. Having this in mind, we will (cautiously) use classically derived relationships as a way of bypassing a double counting.

Finally, following Ref. [13], we use the fact that matrix elements of local operators $AD\psi$, ψDB or $C(D_\mu G^{\mu\nu} - ig\psi\gamma^\nu\psi)$ vanish identically between hadron states by virtue of the equations of motion of QCD. A , B , and C represent any local product of fundamental fields of quarks and/or gluons.

The remainder of the paper is organized as follows. In section 2, the states created by the simplest operators are classified in the meson, pure glue and hybrid sectors of QCD. We study the lowest dimension operators that create $q\bar{q}$ mesons:

$$\bar{q}\Gamma q \quad \dim = 3 \quad (1.8a)$$

and

$$\bar{q}\Gamma Dq \quad \dim = 4 \quad (1.8b)$$

Then we study operators creating the pure glue states:

$$\text{tr } G^{\mu\nu} G^{\delta\sigma} \quad \dim = 4 \quad (1.9a)$$

$$\text{tr } G^{\mu\nu} D^\delta G^{\alpha\beta} \quad \dim = 5 \quad (1.9b)$$

$$\text{tr } G^{\mu\nu} D^\delta D^\sigma G^{\alpha\beta} \quad \dim = 6 \quad (1.9c)$$

$$\text{tr } G^{\mu\nu} [G^{\alpha\beta}, G^{\delta\sigma}]_\pm \quad \dim = 6 \quad (1.9d)$$

$G^{\mu\nu} = G_b^{\mu\nu} \frac{\lambda_b}{2}$ where $G_b^{\mu\nu}$ is the gluon field strength and λ_b is a Gell-Mann matrix of color b . In the hybrid sector, we study the simplest operator:

$$\bar{q}\Gamma G^{\mu\nu} q \quad \dim = 5 \quad (1.10)$$

Section 3 presents the mode analysis. Once again the quark meson, pure glue and hybrid sectors are treated in turn. In each category, we conclude that only very few J^{PC} quantum numbers occur as the lightest states.

Section 4 is devoted to a summary of the results and a comparison with other models. We display the final predictions for the low lying glueball and hybrid spectra in the form of an “energy ladder” in some arbitrary energy scale. We also explore the origins of disagreements between us and other authors.

As we have mentioned our approach is an extension of the work by Fritzsche and Minkowski¹² and by Bjorken.¹¹ Some of the results of section 2 are to be found in Ref. [11]; however, the mode analysis of section 3 is new. Finally, we note that a partial summary of our conclusions has been quoted in Ref. [14].

II. OPERATORS OF THE LOWEST DIMENSION

J^{PC} Decomposition

Given an operator, we want to find all J^{PC} quantum numbers of states this operator can create from the vacuum. First, we decompose the operator into irreducible representations of the Lorentz group, and then decompose each Lorentz

irrep into irreducible representations of angular momentum, parity and charge conjugation. Here we illustrate our method by doing a J^{PC} decomposition of a few composite operators: $\bar{q}\Gamma q$, $\text{tr } GG$, $\text{tr } G\{G, G\}_\pm$. Complete results will be quoted in the next subsection.

A Lorentz decomposition of the lowest dimension meson interpolating field $\bar{q}\Gamma q$ is trivial. In the Table I we list operators and corresponding irreps. Flavor and color indices are suppressed and possible J^{PC} from the spin parity decomposition are listed alongside:

TABLE I

Operator	Lorentz Irrep	J^{PC}
$\bar{q}\gamma_5 q$	$(0, 0)$	0^{--}
$\bar{q}\gamma_\mu q$	$(\frac{1}{2}, \frac{1}{2})$	$0^{+-}, 1^{--}$
$\bar{q} q$	$(0, 0)$	0^{++}
$\bar{q}\gamma_\mu \gamma_5 q$	$(\frac{1}{2}, \frac{1}{2})$	$0^{+-}, 1^{+-}$
$\bar{q}\sigma_{\mu\nu} q$	$(1, 0) \oplus (0, 1)$	$1^{--}, 1^{+-}$

Analogously, the decomposition of the lowest dimension glueball interpolating fields $\text{tr } G^{\mu\nu} G^{\alpha\beta}$ is given in Table II:

TABLE II

Operator	Lorentz Irrep.	J^{PC}
$S = \text{tr } G_{\mu\nu} G^{\mu\nu}$	$(0, 0)$	0^{++}
$P = \text{tr } \tilde{G}_{\mu\nu} G^{\mu\nu}$	$(0, 0)$	0^{--}
$T_{\alpha\beta} = \text{tr } G_{\alpha\lambda} G_\beta^\lambda - \frac{1}{4} g_{\alpha\beta} S$	$(1, 1)$	$0^{++}, 1^{--}, 2^{++}$
$L_{\mu\nu\delta\sigma} = \text{tr } G_{\mu\nu} G_{\delta\sigma}$	$(2, 0) \oplus (0, 2)$	$2^{--}, 2^{++}$
$-(g_{\mu\delta} T_{\nu\sigma} + g_{\nu\sigma} T_{\mu\delta})$		
$-g_{\mu\sigma} T_{\nu\delta} - g_{\nu\delta} T_{\mu\sigma}$		
$+\frac{1}{2}(g_{\mu\delta} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\delta}) S$		
$+\frac{1}{24} \epsilon_{\mu\nu\delta\sigma} P$		

Here, we have classified representations of the Lorentz group using the isomorphism of its algebra to the algebra of $SU(2) \times SU(2)$: Each irrep is labelled by a pair (k, l) . Under the rotation group (k, l) transforms like a $k \otimes l$ representation (in general reducible). Parity sends (k, l) into (l, k) so to build parity eigenstates one has to consider representations of a type $(k, l) \oplus (l, k)$. Note also two operators that transform under different Lorentz irreps can create the same particle. For example, S and $T_{\alpha\beta}$ from the Table II both can create 0^{++} states.

A more powerful method is required to analyze dimension 5 and 6 glueball operators. Since a $G_a^{\mu\nu}$ field transforms like $(1, 0) \oplus (0, 1)$, a Lorentz decomposition of products of G 's is equivalent to a Cartesian tensor decomposition of direct products of 3-vectors. It is convenient to pick as a basis of the $(1, 0)$ irrep:

$$\vec{v}_a = \vec{B}_a + i\vec{E}_a \quad (11.1a)$$

and as a basis of the $(0, 1)$ irrep:

$$\vec{w}_a = \vec{B}_a - i\vec{E}_a. \quad (\text{II.1b})$$

To illustrate the method, consider once again $\text{tr } G^{\mu\nu} G^{\alpha\beta}$. This operator transforms as the symmetric part of the representation $[(1, 0) \oplus (0, 1)]^2$:

$$\begin{aligned} [(1, 0) \oplus (0, 1)]_S^2 &= [(1, 0)^2]^{(2,0)} \oplus [(0, 1)^2]^{(0,2)} \\ &\oplus [(1, 0)^2]^{(0,0)} \oplus [(0, 1)^2]^{(0,0)} \\ &\oplus [(1, 0) \otimes (0, 1)]^{(1,1)}. \end{aligned} \quad (\text{II.2})$$

The associated irreducible (Cartesian) tensor operators are given in the Table III:

TABLE III

Representation	Operator
$[(1, 0)^2]^{(0,0)}$	$L = \vec{w}_a \cdot \vec{w}_a$
$[(0, 1)^2]^{(0,0)}$	$R = \vec{w}_a \cdot \vec{w}_a$
$[(1, 0)^2]^{(2,0)}$	$L^i_j = \vec{w}_a^i \vec{w}_a^j - \frac{1}{3} \delta^i_j L$
$[(0, 1)^2]^{(0,2)}$	$R^i_j = \vec{w}_a^i \vec{w}_a^j - \frac{1}{3} \delta^i_j R$
$[(1, 0) \otimes (0, 1)]^{(1,1)}$	$\vec{w}_a^i \vec{w}_a^j + \vec{w}_a^i \vec{w}_a^j$

The tensors listed in Table III do not have definite parity. But under parity $\vec{v} \leftrightarrow \vec{w}$ so linear combinations, for example $L^i_j \pm R^i_j$ or $L \pm R$, do. This leads to the final form of the J^{PC} decomposition given in Table IV:

TABLE IV

J^{PC}	Lorentz Representation	Operator
0^{++}	$[(1, 0)^2]^{(0,0)} + [(0, 1)^2]^{(0,0)}$	$\vec{E}_a^2 - \vec{B}_a^2$
0^{++}	$[(1, 0) \otimes (0, 1)]^{(1,1)}$	$\vec{E}_a^2 + \vec{B}_a^2$
0^{-+}	$[(1, 0)^2]^{(0,0)} - [(0, 1)^2]^{(0,0)}$	$\vec{E}_a \cdot \vec{B}_a$
1^{-+}	$[(1, 0) \otimes (0, 1)]^{(1,1)}$	$\vec{E}_a \times \vec{B}_a$
2^{++}	$[(1, 0) \otimes (0, 1)]^{(1,1)}$	$\vec{E}_a^i \vec{E}_a^j + \vec{B}_a^i \vec{B}_a^j - \frac{1}{3} \delta^i_j (\vec{E}_a^2 + \vec{B}_a^2)$
2^{++}	$[(1, 0)^2]^{(2,0)} + [(0, 1)^2]^{(0,2)}$	$\vec{E}_a^i \vec{E}_a^j - \vec{B}_a^i \vec{B}_a^j - \frac{1}{3} \delta^i_j (\vec{E}_a^2 - \vec{B}_a^2)$
2^{-+}	$[(1, 0)^2]^{(2,0)} - [(0, 1)^2]^{(0,2)}$	$\vec{E}_a^i \vec{B}_a^j - \vec{B}_a^i \vec{E}_a^j - \frac{2}{3} \delta^i_j \vec{E}_a \cdot \vec{B}_a$

One may check that the results of Table IV coincide with the results of Table II. We apply the same methods to the dimension 6 operator:

$$\text{tr } G^{\mu\nu} [G^{\alpha\beta}, G^{\delta\sigma}]_- = f_{abc} G_a^{\mu\nu} G_b^{\alpha\beta} G_c^{\delta\sigma}. \quad (\text{II.3})$$

It transforms as the totally antisymmetric part of the representation $[(1, 0) \oplus (0, 1)]^3$:

$$\begin{aligned} [(1, 0) \oplus (0, 1)]_A^3 &= [(1, 0)^3]^{(3,0)} \oplus [(0, 1)^3]^{(0,3)} \\ &\oplus [(1, 0)^2]^{(1,0)} \otimes (0, 1)]^{(1,1)} \oplus [(1, 0) \otimes (0, 1)^2]^{(1,1)}. \end{aligned} \quad (\text{II.4})$$

In Table V we give a list of all possible J^{PC} quantum numbers occurring in the decomposition of (II.3) alongside examples of operators that can create states with those quantum numbers:

TABLE V

J^{PC}	Operator
0^{++}	$f_{abc} (\vec{E}_a \times \vec{E}_b) \cdot \vec{B}_c$
0^{-+}	$f_{abc} (\vec{E}_a \times \vec{E}_b) \cdot \vec{E}_c$
1^{-+}	$f_{abc} (\vec{B}_a \cdot \vec{E}_b) \vec{B}_c$
1^{++}	$f_{abc} (\vec{E}_a \cdot \vec{B}_b) \vec{E}_c$
2^{++}	$f_{abc} \left\{ (\vec{B}_a \times \vec{B}_b)^i B_c^j - (\vec{E}_a \times \vec{E}_b)^i B_c^j + 2(\vec{B}_a \times \vec{E}_b)^i E_c^j \right. \\ \left. + (i \leftrightarrow j) - \frac{2}{3} \delta^i_j [(\vec{B}_a \times \vec{B}_b) \cdot \vec{B}_c + (\vec{E}_a \times \vec{E}_b) \cdot \vec{B}_c] \right\}$
2^{-+}	$f_{abc} \left\{ (\vec{E}_a \times \vec{E}_b)^i E_c^j - (\vec{B}_a \times \vec{B}_b)^i E_c^j + 2(\vec{E}_a \times \vec{B}_b)^i B_c^j \right. \\ \left. + (i \leftrightarrow j) - \frac{2}{3} \delta^i_j [(\vec{E}_a \times \vec{E}_b) \cdot \vec{E}_c + (\vec{B}_a \times \vec{B}_b) \cdot \vec{E}_c] \right\}$

Similarly, we analyze the operator:

$$\text{tr } G^{\mu\nu} \left\{ G^{\alpha\beta}, G^{\delta\sigma} \right\}_+ = d_{abc} G_a^{\mu\nu} G_b^{\alpha\beta} G_c^{\delta\sigma}. \quad (\text{II.5})$$

It transforms as the totally symmetric part of the representation $[(1, 0) \oplus (0, 1)]^3$:

$$\begin{aligned} [(1, 0) \oplus (0, 1)]_S^3 &= [(1, 0)^3]^{(3,0)} \oplus [(0, 1)^3]^{(0,3)} \oplus [(1, 0)^2]^{(1,0)} \oplus [(0, 1)^2]^{(0,1)} \\ &\oplus [(1, 0)^2]^{(0,0)} \otimes (0, 1)]^{(0,1)} \oplus [(1, 0) \otimes (0, 1)^2]^{(0,0)}]^{(1,0)} \\ &\oplus [(1, 0)^2]^{(2,0)} \otimes (0, 1)]^{(2,1)} \oplus [(1, 0) \otimes (0, 1)^2]^{(0,2)}]^{(1,2)}. \end{aligned} \quad (\text{II.6})$$

J^{PC} quantum numbers of the states that can possibly be created and examples of operators are listed in Table VI:

TABLE VI

J^{PC}	Operator
1^{+-}	$d_{abc}(\vec{E}_a \cdot \vec{E}_b) \vec{B}_c$
1^{--}	$d_{abc}(\vec{E}_a \cdot \vec{E}_b) \vec{E}_c$
2^{+-}	$d_{abc}[\vec{E}_a^i (\vec{B}_b \times \vec{E}_c)^j + (i \leftrightarrow j)]$
2^{--}	$d_{abc}[\vec{B}_a^i (\vec{E}_b \times \vec{E}_c)^j + (i \leftrightarrow j)]$
3^{--}	$d_{abc}[\vec{E}_a^i \vec{E}_b^j \vec{E}_c^k - \frac{1}{6} \vec{E}_a \cdot \vec{E}_b (\delta^{ij} \vec{E}_c^k + \delta^{jk} \vec{E}_c^i + \delta^{ik} \vec{E}_c^j)]$
3^{+-}	$d_{abc}[\vec{B}_a^i \vec{B}_b^j \vec{B}_c^k - \frac{1}{6} \vec{B}_a \cdot \vec{B}_b (\delta^{ij} \vec{B}_c^k + \delta^{jk} \vec{B}_c^i + \delta^{ik} \vec{B}_c^j)]$

The same analysis as above has been applied to all low dimension operators: (I.8b), (I.9b,c,d) and (I.10). In the next subsection, we determine which of the J^{PC} appearing in the spin-parity decomposition of those operators do correspond to the physical states that can be created from the vacuum.

States Created from the Vacuum

(a) Quark Mesons

All the operators of Table I can create hadron states: note, however, that $\bar{q}\gamma_\mu q$ cannot create a 0^{+-} state. For equal mass quarks $\bar{q}\gamma^\mu q$ is conserved, and the J^{PC} of $\partial_\mu(\bar{q}\gamma^\mu q)$ is 0^{+-} . As discussed in Section 1, this prohibits creation of a 0^{+-} particle. For unequal mass quarks ($e.g.$, $\bar{s}\gamma^\mu u$) C -parity is not a good quantum number so the question is moot. Consequently, we expect nonets:

$$J^{PC} = 0^{-+}, 1^{--}, 0^{++}, 1^{+-}, 1^{++} \quad (\text{II.7})$$

to be the lightest in the meson spectrum.

Operators of dimension 4 interpolating $q\bar{q}$ mesons have the form:

$$\bar{q}\Gamma(\vec{D} + \vec{D})q \equiv \bar{q}\Gamma D_S q \quad (\text{II.8a})$$

or

$$\bar{q}\Gamma(\vec{D} - \vec{D})q \equiv \bar{q}\Gamma D_A q \quad (\text{II.8b})$$

$\vec{D}(\vec{D})$ is a covariant derivative acting to the right (left). Color and Lorentz indices are suppressed. Note that at the classical level, all operators (II.8a) are total derivatives of operators $\bar{q}\Gamma q$.

In the J^{PC} decomposition of operators (II.8a) and (II.8b) one finds the following new J^{PC} quantum numbers:

$$J^{PC} = 0^{+-}, 0^{--}, 1^{-+}, 2^{++}, 2^{+-}, 2^{-+}, 2^{--} \quad (\text{II.9})$$

Out of those only:

$$J^{PC} = 2^{++}, 2^{-+}, 2^{--} \quad (\text{II.10})$$

correspond to the new $q\bar{q}$ meson nonets which, without any reservations, can be coupled to the vacuum via operators of dimension 4. In fact, in the experimental

spectrum 2^{++} and 2^{-+} states do follow meson nonets listed as the lightest in (II.7). [Note the existence of two strange 2^{-} mesons: $L(1580)$ and $L(1770)$, if confirmed, implies the existence of both 2^{--} and 2^{-+} nonets, just as Q_1 and Q_2 require the existence of 1^{+-} and 1^{++} nonets.] 2^{+-} appears only in a decomposition of (II.8a). States of those quantum numbers created by operators of dimension 4 are definitely spurious.

Finally, the “exotic” nonets:

$$J^{PC} = 0^{--}, 0^{+-}, 1^{-+} \quad (\text{II.11})$$

require special attention. For those states amplitudes of creation from the vacuum by operators (II.8a,b) of dimension 4 are simply related to amplitudes of creation from the vacuum by “hybrid” operators (I.10) of dimension 5. Consider for example an operator

$$j_{0^{--}}^\mu = \bar{q}\gamma_5 D_A^\mu q \quad (\text{II.12})$$

which connects 0^{--} to the vacuum. At the classical level one has:

$$\partial_\mu j_{0^{--}}^\mu = g\Theta \quad (\text{II.13a})$$

where

$$\Theta = \frac{1}{2} \bar{q}\gamma_5 \sigma_{\mu\nu} G^{\mu\nu} q \quad (\text{II.13b})$$

In the quantum theory, the relation (II.13a) could be modified by the addition of “anomalous” terms. If such terms do arise perturbatively, they are of higher order in powers of g . If

$$\langle 0^{--} | j_{0^{--}}^\mu | 0 \rangle = f_j p^\mu \quad (\text{II.14})$$

and

$$\langle 0^{--} | \Theta | 0 \rangle = f_\Theta \quad (\text{II.15})$$

then

$$m_{0^{--}}^2 - f_j = g f_\Theta + \mathcal{O}(g^3) \quad (\text{II.16})$$

From this it is tempting to make the following (naive) argument: f_j is of the same dimension as $\frac{1}{m^2} f_\Theta$ so in terms of couplings of the same dimension Θ is more strongly coupled to 0^{--} than $j_{0^{--}}^\mu$ by a factor $\frac{1}{g}$.

In constituent models, the relation (II.16) is very important: it leads to the exclusion of the exotic states from the quark meson spectrum. This is apparent in quark models, *i.e.*, when one does not take into account any gluonic excitations. In such a case the operator $j_{0^{--}}^\mu$ is conserved and 0^{--} is decoupled all together. The same happens with 0^{+-} and 1^{-+} (for example, 1^{-+} decouples from the quark energy-momentum tensor, which is conserved). In a QCD motivated constituent model one admits both quark and gluon excitations. In such a model two independent scales are introduced. One is a confinement scale which goes into a definition of the fundamental fields. The other establishes the strength of the effective “color-like” interactions. Symmetries of the light hadron spectrum require the effective

coupling constant g to be a small parameter and all physical quantities are expressible in power series in g . Accordingly, the amplitude f_j is small because it is $O(g)$. As we have discussed in section 1, it means 0^{--} cannot be occupied by two constituent quarks. On the other hand, f_0 is presumably of the order g^0 . One concludes that 0^{--} should be classified as a hybrid and its mass must be comparable to masses of other states created by hybrid operators (II.10) of dimension 5.

In the same way, one may show that in any QCD motivated constituent model 1^{--} and 0^{+-} states from the list (II.9) would be classified as hybrids. Also, their masses would be heavier than the masses of the $q\bar{q}$ mesons (II.10) created by operators of dimension 4.

A less definite conclusion emerges from direct examination of QCD. QCD has only one parameter, Λ , which sets the scale. The coupling constant g and the operators Θ and j_0^{--} are renormalization scale and prescription dependent. Therefore, it is not possible to separate g from f_0 in any meaningful way. Moreover, g is not a small parameter of the theory (at hadron energies); the power counting argument which is used in models need not have any fundamental significance.

Exactly this kind of argument makes the classification into hybrids *vs.* quark mesons rather arbitrary within QCD. We believe that this distinction is only a byproduct of a constituent picture. We do not find any fundamental difference between exotic mesons (II.11) and quark mesons (II.10) although, as we have said, we understand why they are prohibited in quark models. Any future discovery and study of the meson states from the list (II.11) offers a very interesting opportunity to check the basic premises that lie behind most of the existing models of hadrons. Should exotics be heavier and basically different from quark mesons, and if QCD is indeed the correct description of nature, it must provide some new nonperturbative mechanism which allows for this to happen.

The present day experimental data (or a lack thereof) seems to indicate that the states (II.11) are heavier than (II.10). A natural question arises: has this occurred because experimenters missed the exotic states in their data or rather is it nonperturbative effects in QCD that justifies the reasoning of the constituent models. Addressing this question is beyond the scope of our paper. We note, however, that exotic mesons have a tendency to decouple from easily observable two or three body final state decay channels ($e.g.$, $\pi\pi$, charged 3π , etc.). For example, 0^{--} decays favorably into $\pi^+\pi^-\pi^0$ ($I^G = 0^-$) or $\pi\pi\eta$ ($I^G = 1^+$); 0^{+-} decays into $K_L K_S$ ($I^G = 0^-, 1^+$) and only 1^{--} has a charged 3π final state ($I^G = 1^-$). Note also that, outside of the e^+e^- machines, most of the new light meson states are currently being studied in the diffractive experiments and this *a priori* prohibits any observation of the "exotic" mesons.

(b) Glueballs

The simplest glueball operator $tr G^{\mu\nu} G^{\alpha\beta}$ interpolates all J^{PC} states listed in the Table II except for 1^{--} . This important exception is discussed in the next subsection. We conclude that the possible quantum numbers of the lightest glueballs are:

$$J^{PC} = 0^{++}, 0^{-+}, 2^{++}, 2^{-+} \quad (II.17)$$

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The lowest excited glueballs are created by operators of dimension 5: $tr G^{\mu\nu} D^\delta G^{\alpha\beta}$. In the J^{PC} decomposition of those operators we find new quantum numbers:

$$J^{PC} = 1^{--}, 1^{++}, 3^{--}, 3^{++} \quad (II.18)$$

Note that:

$$\delta^\delta tr G^{\mu\nu} G^{\alpha\beta} = tr G^{\mu\nu} D^\delta G^{\alpha\beta} + tr G^{\alpha\beta} D^\delta G^{\mu\nu} + O^{\delta\mu\nu\alpha\beta} \quad (II.19)$$

where $O^{\delta\mu\nu\alpha\beta}$ is a possible perturbative anomaly: an operator of the same dimension and Lorentz structure as the l.h.s. of (II.19). In the pure glue world $O^{\delta\mu\nu\alpha\beta}$ must be built of operators $tr G^{\mu\nu} D^\delta G^{\alpha\beta}$ and if it is symmetric under the exchange of indices $(\mu\nu) \leftrightarrow (\alpha\beta)$ it can be incorporated into the l.h.s. of (II.19) (i.e., simply renormalizes the classical relation). In fact, 3^{--} appears only in a decomposition of the symmetric part of $tr G^{\mu\nu} D^\delta G^{\alpha\beta}$. Likewise, using equations of motion and the Bianchi identity, all operators creating 1^{--} states can be related to the symmetric part of $tr G^{\mu\nu} D^\delta G^{\alpha\beta}$. We conclude that in the pure glue world 1^{--} and 3^{--} correspond to spurious states and the only new glueball states interpolated by operators of dimension 5 are:

$$J^{PC} = 1^{++}, 3^{++} \quad (II.20)$$

Glueball operators of dimension 6 have the form:

$$O_1 = tr G^{\mu\nu} [G^{\alpha\beta}, G^{\delta\sigma}]_- \quad (II.21a)$$

$$O_2 = tr G^{\mu\nu} \{G^{\alpha\beta}, G^{\delta\sigma}\}_+ \quad (II.21b)$$

or

$$O_3 = tr G^{\mu\nu} \{D^\delta, D^\sigma\}_+ G^{\alpha\beta} \quad (II.21c)$$

The J^{PC} decomposition of O_1 and O_2 gives quantum numbers of the lightest three gluon glueballs. In the case of O_1 those are:

$$J^{PC} = 0^{++}, 1^{+-}, 2^{+-} \quad (II.22a)$$

and in the case of O_2 :

$$J^{PC} = 1^{\pm-}, 2^{\pm-}, 3^{\pm-} \quad (II.22b)$$

Note that classically:

$$tr \left[G^{\mu\nu} \{D^\delta, D^\sigma\}_+ G^{\alpha\beta} - (\mu\nu \leftrightarrow \alpha\beta) \right] \\ = \delta^\delta tr \left[G^{\mu\nu} D^\sigma G^{\alpha\beta} - (\mu\nu \leftrightarrow \alpha\beta) \right] + \delta^\sigma tr \left[G^{\mu\nu} D^\delta G^{\alpha\beta} - (\mu\nu \leftrightarrow \alpha\beta) \right] \quad (II.23)$$

This means that operators (II.21c), antisymmetric under the exchange of G fields, create only spurious states unless they mix with other operators (II.21a,b). Therefore, to get new quantum numbers, yet unaccounted for, it suffices to study the part of O_3 symmetric under the exchange of G fields. This leads to the following possibilities:

$$J^{PC} = 1^{--}, 3^{--}, 4^{--}, 4^{++} \quad (II.24)$$

Already, we can see that the light pure glue spectrum is rather sparse. The mode analysis of Section 3 leads to more detailed predictions.

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(c) Hybrids

The lowest dimensional operators creating hybrid states are:

$$O = \bar{q}\Gamma G^{\mu\nu} q \quad (\text{II.25})$$

of dimension 5. The J^{PC} decomposition of (II.25) generates the following quantum numbers:

$$J^{PC} = 0^{\pm+}, 0^{\pm-}, 1^{\pm+}, 1^{\pm-}, 2^{\pm+}, 2^{\pm-} \quad (\text{II.26})$$

with no constraints from equations of motion or conservation laws. To select the lightest states from the list (II.26) it is necessary to perform the mode analysis.

1⁺⁻ Exotic

Some low dimension operators of a definite J^{PC} don't create hadron states of those quantum numbers. In section 1, we have outlined some general cases when this occurs. Here, we study the 1⁺⁻ glueball state as an important example and illustration of our reasoning.

In the foregoing subsection we have found that a glueball operator of dimension 4 contains a 1⁺⁻ piece. In fact, it is a part of the energy-momentum tensor of the pure glue QCD: $T^{\mu\nu}$ (see Table II). However, in the QCD without quarks, $T^{\mu\nu}$ is conserved (up to a term $\propto g^{\mu\nu} \text{tr } G^{\alpha\beta} G_{\alpha\beta}$). Therefore, a matrix element of the form:

$$\langle p, \epsilon | T^{\alpha\beta} | 0 \rangle = f_{1^{+-}} (p^\alpha \epsilon^\beta + \epsilon^\alpha p^\beta) \quad (\text{II.27})$$

must obey

$$0 = \langle p, \epsilon | \partial_\alpha T^{\alpha\beta} | 0 \rangle = \epsilon^\beta M^2 f_{1^{+-}}. \quad (\text{II.28})$$

Assuming $M^2 \neq 0$ (no spontaneous breakdown of Lorentz invariance), this gives $f_{1^{+-}} = 0$ corresponding to no 1⁺⁻ state in the spectrum of this operator.

Also, a glueball operator of dimension 5 (II.9b) contains a 1⁺⁻ piece. In this case, all operators of 1⁺⁻ quantum numbers can be related to operators symmetric under the exchange of G fields, i.e., it is only total derivative fields that interpolate 1⁺⁻ states.

We have shown that there is no light 1⁺⁻ state in QCD without quarks. On the other hand, the operator $\bar{q}\Gamma G^{\mu\nu} q$ can create a 1⁺⁻ state so we believe such a state is properly classified as a hybrid, not a glueball. In section 4, we examine the glueball models which do predict 1⁺⁻ states and show which of our assumptions they do not satisfy.

III. MODE ANALYSIS

Quantization

In order to obtain more detailed information about the spectrum, we formulate a generic "mean field theory" description of hadrons. Dynamical variables in such a theory include fields with quark and gluon quantum numbers. Interactions among those fields are "QCD-like": the same ones as in a color gauge theory. The effective coupling constant g is a small parameter. All results quoted hereafter are true in

the leading order of g . The Hamiltonian must have a g independent piece H_0 which provides for the confinement of quarks and gluons. Note since gluon self-interaction is a higher order effect, our model is "quasi-Abelian" in the leading order.

One may quantize this theory in the interaction picture where the dynamics of the fundamental fields is given by H_0 alone. Then, the effective quark and gluon fields create simple one-particle excitations: constituent quarks and gluons. H_0 ought to be gauge, parity and rotation invariant and we require that quark and gluon eigenvalues diagonalize simultaneously H_0 , J^2 , J_z and P (parity). A particular choice of H_0 selects a reference frame so the eigenstates are localized. In this model, a hadron state being a superposition of the constituent quark and gluon modes is normalized, localized and represents a particle at rest.*

The most general wave function of a quark mode is:

$$\psi_{\omega JLM}(\vec{r}) = \begin{bmatrix} f_\omega(\vec{r}) \Omega_{JLM}(\hat{r}) \\ g_\omega(\vec{r}) \vec{\sigma} \cdot \hat{r} \Omega_{JLM}(\hat{r}) \end{bmatrix} \quad (\text{III.1})$$

where $\hat{r} = \vec{r}/r$, $\vec{\sigma}$ is a triplet of Pauli matrices and Ω_{JLM} are spinor spherical harmonics¹⁵ (hereafter $t=0$). ω , $J(J+1)$, M and $(-1)^L$ are the eigenvalues of H_0 , J^2 , J_z and P , respectively.

From charge conjugation invariance it follows that for every quark mode of energy ω there is an antiquark mode of the same energy. A wave function of this eigenmode is:

$$\phi_{\omega JLM}(\vec{r}) = i\gamma^2 \psi_{\omega JLM}^*(\vec{r}) = \begin{bmatrix} g_\omega^*(\vec{r}) \Theta_{JLM}(\hat{r}) \\ f_\omega^*(\vec{r}) \vec{\sigma} \cdot \hat{r} \Theta_{JLM}(\hat{r}) \end{bmatrix} \quad (\text{III.2})$$

where $\Theta_{JLM} = -i\sigma^2 \Omega_{JLM}^*$ and ω , $J(J+1)$, M and $(-1)^{L+1}$ are the eigenvalues of H_0 , J^2 , J_z and P , respectively.

Let a particular choice of ω , J , and L in (III.1) (and III.2) correspond to a quark (antiquark) ground mode. The only degeneracy is $M = -J, \dots, J$. A quark field restricted to the ground mode has the form:

$$q = \sum_{M=-J}^J (b_{\omega JLM} \psi_{\omega JLM} + d_{\omega JLM}^\dagger \phi_{\omega JLM}) \quad (\text{III.3})$$

where $b(d)$ annihilates a quark (antiquark) in a ground mode state. All color, flavor and spacetime labels have been suppressed.

When quantizing gluon fields one must be careful not to introduce unphysical gluon modes. Physical modes obey simultaneously the equations of motion and gauge constraint. Fixing a gauge is a precondition of going to the Hamiltonian picture. Without losing generality one gauge condition can be chosen as:

$$A^0 \equiv A_a^0 = 0 \quad (\text{III.4})$$

* We also assume $SU(3)_F$ flavor symmetry. $SU(3)_F$ flavor breaking effects are of the order of the hyperfine splittings, i.e., not relevant in our discussion.

(so called Weyl gauge). A wave function of a gluon mode is then a vector-valued function $\vec{A}(\vec{r}, t)$. The second gauge condition need not be specified. It suffices to note that it must be parity invariant if gluon modes are to be parity eigenstates.

Let a gluon eigenmode simultaneously diagonalize H_0 , $\vec{J}^2$, J_z and P . Eigenfunctions of $\vec{J}^2$, J_z and P have either of the two forms:

$$\vec{A}_{jm}^{TE}(\vec{r}) = G(r)\vec{Y}_{jm}(\hat{r}), \quad (\text{III.5a})$$

or

$$\vec{A}_{jm}^{TM}(\vec{r}) = H(r)\vec{Y}_{j-1m}(\hat{r}) + F(r)\vec{Y}_{j+1m}(\hat{r}) \quad (\text{III.5b})$$

where $\vec{Y}_{jm}(\hat{r})$ are vector spherical harmonics¹⁵ and j, m are the eigenvalues of $\vec{J}$ and J_z . The parity of (III.5a) is $(-1)^{j+1}$ and the parity of (III.5b) is $(-1)^j$. By the analogy with electrodynamics we call (III.5a) and (III.5b), TE and TM , respectively. The radial functions H, F, G depend on the actual form of H_0 and the second gauge condition. For given j and m , the H_0 eigenvalue ω is in general different for (III.5a) and (III.5b).

For a given set of the eigenvalues ω , j and parity π the wave functions (III.5 or 6) correspond to a $2j+1$ -fold degenerate gluon mode. The eight restricted gluon fields have the form:

$$\vec{A}_a = \sum_{m=-j}^j (c_{ajm\lambda}^a \vec{A}_{ajm} + c_{ajm\lambda}^{a\dagger} \vec{A}_{ajm}^*) \quad (\text{III.6})$$

where $\omega, j(j+1), \lambda$ are the eigenvalues of H_0 , $\vec{J}^2$, P in the ground mode and $c^\dagger$ creates a given gluon ground state.

Matrix Elements of Interpolating Fields

We wish to discuss a production from the vacuum of a given particle by an interpolating field $\Theta(x)$. In general, we need the matrix element:

$$\langle \psi(p) | \Theta(0) | 0 \rangle = f(p^2). \quad (\text{III.7})$$

[Here we consider a scalar operator and an 0^{++} state $|\psi(p)\rangle$ – the generalization to more complex cases is straightforward].

In our model, one may represent only localized states so construction of the momentum eigenstate is precluded. It turns out to be sufficient for our purposes to evaluate instead:

$$\int d^3x \langle \psi | \Theta(\vec{x}, t=0) | 0 \rangle \equiv F \quad (\text{III.8})$$

where $|\psi\rangle$ is a localized hadron state, normalized to unity. In fact, we will only ever need to evaluate the angular integral in Eq. (III.8) since the radial integral requires a detailed knowledge of the quark and gluon wavefunctions.

We assume that the localized state is a superposition of momentum eigenstates sharply peaked about $\vec{p}^2 = 0^{16}$:

$$|\psi\rangle = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \phi(\vec{p}) |\psi(p)\rangle \quad (\text{III.9})$$

where

$$E_p = p^0 = (\vec{p}^2 + m^2)^{1/2}$$

and

$$\langle \psi(p) | \psi(p') \rangle = 2E_p (2\pi)^3 \delta^3(\vec{p} - \vec{p}').$$

Inserting Eq. (III.9) into (III.8) we find:

$$F = \frac{(2\pi)^{3/2}}{(2\pi)^{1/2}} \phi^*(0) f(m^2). \quad (\text{III.10})$$

In general, our uncertainty about $\phi(\vec{p})$ precludes learning anything about $f(m^2)$ from the Eq. (III.10). However, if for some reason F vanishes we can conclude with certainty that $f(m^2)$ is also zero. If it happens that the angular integration in the Eq. (III.8) gives zero for certain states when only the lowest quarks and/or gluon modes in $\Theta(\vec{x})$ are retained, and if this happens for all operators capable of connecting a state of given J^{PC} to the vacuum, then we conclude those quantum numbers do not appear in the physical spectrum to a given order.

The Lightest Hadrons

a) $q\bar{q}$ Mesons.

To learn about the quantum numbers of the lightest mesons we study a matrix element of the form:

$$\begin{aligned} F &= \int d^3x \langle \psi | \bar{q}(\vec{x}) \Gamma q(\vec{x}) | 0 \rangle \\ &= \sum_{M, M'=-J}^J \langle \psi | b_M^\dagger d_{M'}^\dagger | 0 \rangle \int d^3x \psi_M^\dagger(\vec{x}) \gamma^0 \Gamma \phi_{M'}(\vec{x}) \end{aligned} \quad (\text{III.11})$$

where all quantum numbers of the quark ground mode but eigenvalues of J_z , have been suppressed. Since ϕ_M and ψ_M are related by charge conjugation, the angular integral in (III.11) can be evaluated explicitly (see Appendix for exemplary calculations).

The angular integration gives $F = 0$ for $\Gamma = 1, \bar{\gamma}\gamma$ s and $\sigma^{\gamma i}$. This holds for arbitrary energy, angular momentum and parity of the ground mode. We conclude in shell models $0^{++}, 1^{++}, 1^{+-}$ meson nonets, even though interpolated by dimension 3 operators, cannot be the lightest in the $q\bar{q}$ spectrum.

On the other hand, $F \neq 0$ for $\Gamma = \gamma_s, \bar{\gamma}, \gamma^0\gamma_s, \sigma^{\gamma i}$ independent of the quantum numbers of the ground mode. Hence, 0^{-+} and 1^{--} are the lightest meson states.

b) Two-Gluon Glueballs

The lightest glueballs are created by an operator $\text{tr } G^{\mu\nu} G^{\alpha\beta}$. In the gauge (III.4), the field strength operators are given by:

$$\vec{E}_a = -\vec{A}_a, \quad (\text{III.12a})$$

$$\vec{B}_a = \vec{\nabla} \times \vec{A}_a - \frac{1}{2} g f_{abc} \vec{A}_b \times \vec{A}_c. \quad (\text{III.12b})$$

Because we are interested only in the lightest states, and the estimates of masses are roughly to the zeroth order in g , the second term in (III.12b) will be disregarded (as mentioned our model is “quasi-Abelian”).

One may calculate the form of $\vec{E}_a$ and $\vec{B}_a$ fields in the ground mode using (III.12a,b) and (III.5a,b). For the TE polarization:

$$\begin{aligned} \vec{E}_{\omega jm}^{TE}(\vec{r}) &= G_{\omega jm}^{TE}(\vec{r}) \vec{Y}_{jm}, \\ \vec{B}_{\omega jm}^{TE}(\vec{r}) &= F_{\omega jm}^{TE}(\vec{r}) \vec{Y}_{j+1m} + H_{\omega jm}^{TE}(\vec{r}) \vec{Y}_{j-1m}. \end{aligned} \quad (\text{III.13a})$$

and for the TM polarization:

$$\begin{aligned} \vec{E}_{\omega jm}^{TM}(\vec{r}) &= F_{\omega jm}^{TM}(\vec{r}) \vec{Y}_{j+1m} + H_{\omega jm}^{TM}(\vec{r}) \vec{Y}_{j-1m}, \\ \vec{B}_{\omega jm}^{TM}(\vec{r}) &= G_{\omega jm}^{TM}(\vec{r}) \vec{Y}_{jm}. \end{aligned} \quad (\text{III.13b})$$

In the expressions above, ω , j , m are the quantum numbers of the gluon ground mode, and radial functions F , G , H are derivable from radial functions in (III.5a,b).

In fact, the analysis of the operator $\text{tr } G^{\mu\nu} G^{\alpha\beta}$ is independent of ω , j and polarization π of the ground mode. For example:

$$\langle 0^{++} | \int d^3x (\vec{E}_a^2 - \vec{B}_a^2) | 0 \rangle = \sum_{m, m'=-j}^j \langle 0^{++} | c_{am}^+ c_{am'}^+ | 0 \rangle \delta_{m, -m'} \int d\vec{r} \mathcal{F}(\vec{r}) \quad (\text{III.14})$$

where all the indices but m have been suppressed and $\mathcal{F}(\vec{r})$ is some combination of $F_{\omega}^{\lambda}(\vec{r})$, $G_{\omega}^{\lambda}(\vec{r})$, $H_{\omega}^{\lambda}(\vec{r})$ ($\lambda = TE$ or TM) from (III.13a or b). On the other hand:

$$\begin{aligned} \langle 0^{++} | \int d^3x \vec{E}_a \cdot \vec{B}_a | 0 \rangle &= \sum_{m, m'=-j}^j \langle 0^{++} | c_{am}^+ c_{am'}^+ | 0 \rangle \int d\vec{r} r^2 G_{\omega}^{\lambda}(\vec{r}) \\ &\int d\vec{r} \left[F_{\omega}^{\lambda}(\vec{r}) \vec{Y}_{jm} \cdot \vec{Y}_{j+1m'} + H_{\omega}^{\lambda}(\vec{r}) \vec{Y}_{jm} \cdot \vec{Y}_{j-1m'} \right] \equiv 0. \end{aligned} \quad (\text{III.15})$$

In this way, one proves that only 0^{++} and 2^{++} are the lightest glueballs in any “mean field theory” model. As before, this result is independent of the quantum numbers of the gluon ground mode.

c) Three-Gluon Glueballs

For the $\text{tr } G^{\mu\nu} [G^{\alpha\beta}, G^{\beta\sigma}]_-$ and $\text{tr } G^{\mu\nu} \{G^{\alpha\beta}, G^{\beta\sigma}\}_+$ operators the mode analysis is more complex. Note the three facts:

- parity of a three gluon operator depends on the number of $\vec{E}$ fields in it;
- the angular form of $\vec{E}$ and $\vec{B}$ in the ground mode is uniquely given by (III.13a or b);
- for spherical harmonic $\int d\Omega Y_{LM} Y_{L'M'} Y_{L''M''} = 0$ if $L + L' + L'' = \text{odd}$,

combined together put restrictions on the parities of the states created from the vacuum by the three-gluon operators. The allowed parities are listed in the Table VII.

TABLE VII

j	ODD	EVEN
TE	Only positive parity states	Only negative parity states
TM	Only negative parity states	Only positive parity states

To get more specific information, one has to choose the angular momentum j and parity π of the gluon ground mode. A natural guess is that j is the lowest possible. A choice $j = 0$ is excluded because the equation of motion $\vec{D} \cdot \vec{E} = 0$ requires the radial function of this gluon mode to be singular at $r = 0$. The next choice $j = 1$ appears in the existing glueball models. The bag model predicts $j^{\pi} = 1^+$ (i.e., TE polarization)^{3,5} and the constituent glue model assumes $j^{\pi} = 1^-$ (i.e., TM polarization).^{7,8}

There are three independent, degenerate gluon states for a $j = 1$ mode. Field strengths in those states are:

$$\begin{aligned} \vec{B}_i^{TM}(\vec{r}) &= G_i^{TM}(\vec{r}) \hat{r} \times \vec{e}_i, \\ \vec{E}_i^{TM}(\vec{r}) &= F_i^{TM}(\vec{r}) \hat{r} \cdot \vec{e}_i + H_i^{TM}(\vec{r}) \vec{e}_i, \end{aligned} \quad (\text{III.16a})$$

in the 1^- case, and

$$\begin{aligned} B_i^{TE}(\vec{r}) &= F_i^{TE}(\vec{r}) \hat{r} \cdot \vec{e}_i + H_i^{TE}(\vec{r}) \vec{e}_i, \\ \vec{E}_i^{TE}(\vec{r}) &= G_i^{TE}(\vec{r}) \hat{r} \times \vec{e}_i. \end{aligned} \quad (\text{III.16b})$$

in the 1^+ case, where $\vec{e}_i$ are the Cartesian unit vectors and $i = 1, 2, 3$. From now on we will discuss the $j^{\pi} = 1^+$ case with a trivial generalization to $j^{\pi} = 1^-$.

As indicated in Table VII, only parity plus states have a chance to be created by the three gluon operators. According to Section 2, these are:

$$J^{PC} = 0^{++}, 1^{++}, 2^{++}, 3^{++}. \quad (\text{III.17})$$

To determine whether, for example, 0^{++} can be produced with all three gluons in the ground mode we must calculate:

$$\begin{aligned} \langle 0^{++} | \int d^3x \sum_{a,b,c} f_{abc} (\vec{E}_a \times \vec{E}_b) \cdot \vec{E}_c | 0 \rangle \\ = \sum_{i,j,k=1}^3 \left(\sum_{a,b,c} \langle 0^{++} | f_{abc} c_a^+ c_b^+ c_c^+ | 0 \rangle \right) \hat{\tau}_{0^{++}}^{ijk} \end{aligned} \quad (\text{III.18d})$$

where

$$\hat{\tau}_{0^{++}}^{ijk} = \int dr r^2 G_1^{TE^2} \int d\Omega [(\hat{r} \times \vec{e}_i) \times (\hat{r} \times \vec{e}_j)] \cdot [\hat{r}_1^T E \hat{r} \cdot \vec{e}_k + H_1^T E \vec{e}_k] \cdot (\text{III.18b})$$

and c_a^+ creates the i^{th} gluon ground state of the color a . Only a totally antisymmetric part of $\hat{\tau}_{0^{++}}^{ijk}$ contributes to (III.18a). But:

$$\hat{\tau}_{0^{++}}^{ijk} = \frac{4\pi}{3} \epsilon^{ijk} \int dr r^2 G_1^{TE^2} (F_1^{TE} + H_1^{TE}) \quad (\text{III.18c})$$

i.e., it contains only a totally antisymmetric part. We conclude that 0^{++} is indeed among the lightest three gluon glueballs.

Analogously other matrix elements may be studied (see Appendix for exemplary calculations). For the 1^+ ground mode the quantum numbers of the lightest three-gluon mesons created by the operators (II.21a) and (II.21b) are:

$$J^{PC} = 0^{++}, 1^{+-}, 3^{+-} \quad (\text{III.19})$$

For the 1^- ground mode the lightest three gluon mesons are:

$$J^{PC} = 0^{-+}, 1^{--}, 3^{--} \quad (\text{III.20})$$

d) Hybrids

Mode analysis in this case requires assumptions about j^* of both the quark and gluon ground modes. The only sensible prediction about the quark ground mode is $1/2^+$ (on which all existing mean-field models agree). As before, we consider two possible cases for the gluon ground mode: 1^+ and 1^- . In the first case we conclude that the operator (II.25) can create four meson nonets:

$$J^{PC} = 0^{-+}, 1^{-+}, 1^{--}, 2^{-+} \quad (\text{III.21})$$

(III.21) is then a list of the lightest hybrid nonets.

If the gluon ground mode has $j^* = 1^-$ we conclude that the lightest hybrid nonets are:

$$J^{PC} = 0^{++}, 1^{++}, 1^{+-}, 2^{++} \quad (\text{III.22})$$

e) Excited States

For operators that are not of the lowest dimension in their category and involve derivatives ((1.8b) for $q\bar{q}$ mesons, (1.9b,c) for two-gluon glueballs) the mode analysis gets more complicated. Still, the easiest (and the most plausible) case of the $1/2^+$ quark ground mode and the 1^- gluon ground mode leads to definite conclusions, namely that none of these operators, restricted to the ground modes only, can create new meson states from the vacuum.

IV. CONCLUSIONS

We have investigated the spectrum of mesons using a simple, minimal model. We have applied this model to the $q\bar{q}$ meson sector at first. The result is that the general features of the spectrum are satisfactorily reproduced. The obvious limitations of the model are apparent e.g.: masses of the 0^{-+} and 1^{--} are indistinguishable. Also, in the nonrelativistic quark models, 2^{++} belongs to the lightest quark mesons, whereas, in our classification it is an excited state. The shortcomings may be attributed to the lack of the resolution of the fine and hyperfine structure.

Presumably, with the same level of accuracy as in the $q\bar{q}$ sector (see Fig. 1) our model predicts the light glueball spectrum (in Fig. 2). In a constituent model, the states in Fig. 2 would correspond to two- and three-gluon states. The three-gluon states are underlined and their ordering assumes $j^* = 1^+$ of the gluon ground mode.

Analogously, we show our predictions for the lightest hybrid states in Fig. 3. The ordering of the states in Fig. 3 assumes $j^* = 1^+$ of the gluon ground mode and $j^* = 1/2^+$ of the quark ground mode.

The interesting feature of the spectra is their sparsity for the lowest energies and the fact that the lightest glueball states do not depend on the quantum numbers of the gluon ground mode. There is an analogous situation in the $q\bar{q}$ sector, so we conclude that it is very probable that the lightest glueballs are 0^{++} and 2^{++} with 0^{-+} and 2^{-+} among the lowest excitations. Our discussion has the disadvantage of not addressing the problems of fine and hyperfine structure, mixing of the 0^{++} with the condensate, mixing among particles with the same quantum numbers, etc. These effects, properly taken into account, could modify the predicted ordering of states in the physical spectrum. On the other hand, more sophisticated models agree on the existence of the light 0^{++} , 2^{++} and 0^{-+} glueballs. We are quite confident that 2^{-+} should be added to this short list.

Note the lack of exotics among the lightest pure glue mesons. In the physical spectrum, as predicted by shell models, the lightest exotics are hybrids and are created by operators of dimension 5. This suggests they are heavier than the low excited mesons (II.10) or the lightest glueballs (II.17). On the other hand, QCD does not distinguish between hybrids and $q\bar{q}$ mesons. Since composite operators (1.8b) of dimension 4 are capable of creating exotics (II.11), it is more probable, we believe, that the states (II.11) themselves belong to the low excited meson states and their masses are in the same region as the masses of the "ordinary" excited $q\bar{q}$ mesons (II.10).

Now let us look more closely at the predictions of models. The bag model^{3,5} and the constituent glue model^{7,8} belong to the class of the "mean-field" models and have been studying. In the bag, both the confining potential and the gauge constraints are specified and yield the $j^* = 1^+$ gluon ground mode.³ [A non-standard confining potential may change this prediction. See Ref. 17 where $j^* = 1^-$ is the gluon ground mode]. Not surprisingly, the bag model results⁵ coincide with ours (if we assume the $j^* = 1^+$ ground mode). Our approach has the advantage of the unambiguous identification of the spurious states. This offers more confidence in the study of excited states.

The constituent glue model⁷ may seem to be a particular realization of our model with the $j^\pi = 1^-$ ground mode. In fact, the lightest two- and three-gluon states predicted in Ref. 7 mostly coincide with ours. Yet, in general, the “non-relativistic” gluon models predict a much denser spectrum for the excited states. The reason is that those models possess too many degrees of freedom: ones that are not present in the QCD theory. In a constituent model number of helicity states for each gluon mode increases from 2 to 3. [In QCD that additional longitudinal helicity state is prescribed by gauge invariance.] This allows for the existence of unphysical gluon modes and consequently leads one to consider unphysical hadron states. One example is the light 1^{--} glueball found in the non-relativistic gluon models.^{7,8} It seems to us that such a state is an artifact of breaking the local color gauge invariance.

Since in the lattice and QCD sum rule approach one studies states connected to the vacuum by the lowest dimension composite operators, we may compare predictions of those models with our results. QCD sum rules⁹ predict 0^{++} and 2^{++} to be the lightest glueballs followed by 0^{--} . In the Ref. 9, the authors make an interesting claim that different mass scales are associated with different sectors of QCD. It can be easily accommodated in our model: it would mean that the scales of the energy axis in Fig. 1, Fig. 2 and Fig. 3 are different in each case.

The lattice calculation, contrary to our results, shows the existence of the 1^{--} state among the lightest glueballs.¹⁰ Our demonstration that 1^{--} is absent from the low lying spectrum relies on translation invariance, i.e., $\partial_\mu T^{\mu\nu} = 0$, so it would seem that the appearance of this state is an artifact of breaking translation invariance when the lattice spacing is finite. A careful analysis of the dependence of the 1^{--} mass on the lattice spacing should reveal that it does not scale according to the renormalization group and disappears from the light glueball spectrum in the continuum limit. A very recent high statistics $SU(3)$ study¹⁸ gives support to this. In Ref. [18] the authors find explicitly that the mass of 1^{--} scales differently than predicted by the renormalization group analysis. This result, although welcome, is obvious in a sense that, as we have shown, mathematical structure of the theory prohibits the smoothest operators of dimension 4 from creating any 1^{--} state. Thus, we cannot agree with the conclusion of Ref. [18] that a higher precision calculation, with a smaller lattice spacing, will eventually reveal the physical parameters of the 1^{--} meson. To attain information about a possible 1^{--} meson, it is necessary to use higher dimension, nonsmooth operators or, if possible, include quark fields into the calculations.

APPENDIX A

Let us consider a matrix element of an operator $\bar{q}\gamma^5 q$ between the vacuum and a 0^{--} state:

$$\begin{aligned} F &= \langle 0^{--} | \int d^3x \bar{q}\gamma^5 q | 0 \rangle \\ &= \sum_{M, M'=-J}^J \langle 0^{--} | b_M^\dagger d_{M'}^\dagger | 0 \rangle \int dr r^2 (f_\omega^{*2} - g_\omega^{*2}) \int d\Omega \Omega_{JLM}^\dagger \Theta_{JLM} \\ &\neq 0. \end{aligned} \quad (\text{A.1})$$

But a matrix element of an operator $\bar{q}\gamma\gamma^5 q$ between the vacuum and a 1^{++} state:

$$\begin{aligned} F &= \langle 1^{++} | \int d^3x \bar{q}\gamma\gamma^5 q | 0 \rangle \\ &= \sum_{M, M'=-J}^J \langle 1^{++} | b_M^\dagger d_{M'}^\dagger | 0 \rangle \int dr r^2 f^{*} g^{*} \int d\Omega \Omega_{JLM}^\dagger \hat{r} \Theta_{JLM'} \\ &\equiv 0. \end{aligned} \quad (\text{A.2})$$

As an example of calculations in the three-gluon sector, consider a matrix element of an operator $f_{abc}(\vec{E}_a \cdot \vec{B}_b)\vec{E}_c$ between the vacuum and a 1^{++} state:

$$\begin{aligned} & \langle 1^{++} | \int d^3x \sum_{a,b,c} f_{abc}(\vec{E}_a \cdot \vec{B}_b)\vec{E}_c | 0 \rangle \\ &= \sum_{i,j,k=1}^3 \left(\sum_{a,b,c} \langle 1^{++} | f_{abc} c_a^\dagger c_b^\dagger c_c^\dagger | 0 \rangle \right) \mathcal{F}_{1^{++}}^{ijk} \end{aligned} \quad (\text{A.3})$$

where

$$\begin{aligned} \mathcal{F}_{1^{++}}^{ijk} &= \int dr r^2 G^2 \int d\Omega \left[(\hat{r} \times \vec{e}_i) \cdot (\hat{r} \hat{r} \hat{r} \vec{e}_j + H \vec{e}_j) \right] (\hat{r} \times \vec{e}_k) \\ &= \frac{4\pi}{3} (\delta^{ik} \delta^{jp} - \delta^{ip} \delta^{jk}) \vec{e}_p \int dr r^2 G^2 H. \end{aligned} \quad (\text{A.4})$$

But $\mathcal{F}_{1^{++}}^{ijk}$ does not contain a piece totally antisymmetric in the indices i, j, k and (A.3) is zero.

On the other hand, consider a matrix element of an operator $d_{abc}(\vec{E}_a \cdot \vec{E}_b)\vec{E}_c$ between the vacuum and a 1^{+-} state:

$$\begin{aligned} & \langle 1^{+-} | \int d^3x \sum_{a,b,c} d_{abc}(\vec{E}_a \cdot \vec{E}_b)\vec{E}_c | 0 \rangle \\ &= \sum_{i,j,k=1}^3 \left(\sum_{a,b,c} \langle 1^{+-} | d_{abc} c_a^\dagger c_b^\dagger c_c^\dagger | 0 \rangle \right) \mathcal{F}_{1^{+-}}^{ijk}. \end{aligned} \quad (\text{A.5})$$

Only a totally symmetric piece of $\bar{f}_{1+}^{ijk}$ contributes to (A.5). But

$$\begin{aligned} \bar{f}_{1+}^{ijk} = & \frac{4\pi}{15} \int dr r^2 [\delta_{ij}\delta_k(5HG^2 + 5FG^2) \\ & - \bar{c}_p(\delta_{ij}\delta_{pk} + \delta_{ik}\delta_{pj} + \delta_{ji}\delta_{ik})FG^2] . \end{aligned} \quad (\text{A.6})$$

and we see explicitly that it contains a part totally symmetric under the exchange of i, j, k and in general (A.5) is not zero.

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FIGURE CAPTIONS

1. The lightest quark meson spectrum.
2. The lightest glueball spectrum. Three gluon states are underlined and their ordering assumes $j^{\pi} = 1^+$ of the gluon ground mode.
3. The lightest hybrid spectrum. The ordering assumes $j^{\pi} = 1^+$ of the gluon ground mode and $j^{\pi} = 1/2^+$ of the quark ground mode.