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THE COMPLETE $N = 3$ MATTER COUPLED SUPERGRAVITY

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A B S T R A C T

The complete $N = 3$ matter coupling to supergravity is obtained in a geometrical framework. This coupling always exists if the $3n$ complex scalars of the n vector multiplets are co-ordinates of the Kähler-Grassmannian manifold $SU(3,n)/SU(3) \times SU(n) \times U(1)$. Subgroups of $SO(3,n)$ $SU(3,n)$ of dimension $3+n$ can be gauged and give rise to a non-trivial scalar potential. The techniques used in this paper allow for the calculation of scalar potentials of extended supergravities in any dimension without explicit construction of the Lagrangian. This opens the possibility of discussing patterns of partial supersymmetry and gauge symmetry breaking on a purely group-theoretical ground.

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1)- Introduction

Whether supersymmetry is implemented in Nature is far from being an answered question. In any case, supersymmetry must be spontaneously broken since physical particles do not combine into supermultiplets of equal mass. Accordingly, all phenomenological applications of supergravity ¹⁾ have been based on the so-called superHiggs phenomenon which takes place when matter multiplets involving scalar fields are coupled to supergravity ²⁾.

In particular it appears that the low energy world corresponding to scales between 100 GeV and say 1 TeV should be described in terms of just one supersymmetry (N=1 theory ³⁾). This is mainly due to the fact that the only supersymmetric theory compatible with a chiral gauge theory has precisely one supercharge.

There is, however, a general consensus that the N=1 description is just an effective approximation of a more fundamental one, since it does not provide a finite theory of gravity and since it involves too many arbitrary functions and representation choices. Specifically the N=1 matter coupled supergravity involves, besides the choice of a gauge group G and of the G-representations to which the chiral multiplets $(\mathbf{z}^{\alpha}, \lambda^{\alpha})$ are assigned, also the arbitrary selection of a Kähler manifold with Kähler potential $\mathcal{G}(\mathbf{z}, \mathbf{z}^*)$ determining the σ -model of the scalar fields. Furthermore the kinetic term of the vector fields $V_{\mu}^{\hat{a}}$ contains an arbitrary analytic symmetric function $\mathcal{F}_{ij}(\mathbf{z})$, and one can introduce an optional "potential" term determined by $\mathcal{G}$.

In general, supersymmetric theories are more restricted if formulated in higher dimensions or with a larger number of supercharges. It would then be desirable to obtain the effective N=1 model through the spontaneous breakdown of a larger theory. In the case $N > 1$ one is led to consider the question of partial supersymmetry breaking via the superHiggs mechanism, while for higher dimensional supergravities one deals with the issue of spontaneous compactification ⁴⁾: here the number of killing spinors admitted by the manifold of the compactified dimensions ⁵⁾ equals the number of unbroken supersymmetries of the 4-dimensional theory.

Higher-dimensional supergravities could, in turn, be a low energy limit of superstring theory ⁶⁾, and might first compactify to a 4-dimensional theory with $N > 1$ and then break to $N=1$ via a suitable extremum of its scalar field potential.

Therefore, any attempt to link supersymmetric phenomenology to more fundamental theories requires a firm control on the properties of matter coupled extended supergravities in four dimensions.

Specifically one needs the explicit form of the scalar field potential and the fermion supersymmetry transformation rules in order to study the problem of partial supersymmetry breaking.

The most relevant models have $N=2,3,4$ since, starting with $N=5$, up to $N=8$ one cannot couple supergravity to gauge multiplets and thus obtain theories with the phenomenological $SU_3 \otimes SU_2 \otimes U_1$.

The complete $N=2$ theories have been constructed ⁷⁾ using the tensor calculus approach as in the $N=1$ case. The $N=4$ matter coupling to supergravity has been recently constructed with the Noether procedure ⁸⁾. In this paper we couple $N=3$ matter to supergravity using an entirely geometrical method, the group manifold approach.

Let us review some of the features of the $N=2$ and $N=4$ couplings.

i) For $N=2$ there are two kinds of scalar manifolds because there are two independent multiplets containing scalar fields, the vector and the hypermultiplets. Scalar fields belonging to vector multiplets are associated to a restricted class of Kähler manifolds, whose Kähler potential can be computed in terms of a scalar analytic function $\phi(z^i)$ ⁷⁾. On the other hand, hypermultiplet scalars are co-ordinates of quaternionic manifolds which in general do not have a Kähler structure.

In contrast to the $N=1$ case a scalar potential is possible in $N=2$ supergravity only if some group is gauged. This feature still holds in all $N > 2$ extended supergravities.

ii) In $N=4$ matter coupling the scalar manifold seems uniquely determined to be ⁹⁾ $(SO(n,6) / SO(n) \otimes SO(6)) \otimes (SU(1,1) / U(1))$ where n is the

number of vector multiplets. This manifold is non Kählerian, a feature shared by the scalar manifolds for $N=2$ hypermultiplets and $N=8$ supergravity. Non - Kählerian manifolds are associated to scalar fields satisfying a reality condition due to the self-conjugation property of the multiplet.

For $N=3$ the matter multiplet is not self-conjugate and the scalar manifold turns out to be Kähler. As in the $N=4$ case there is a natural candidate, the homogeneous Kähler manifold $SU(3,n) / SU(3) \otimes SU(n) \otimes U(1)$. As explained later, the lack of arbitrariness in the coupling of $N=3$ and $N=4$ matter to supergravity may be traced back to the absence of auxiliary fields in these theories. Indeed arbitrary interactions are already absent in the global version of the theory, independently of the coupling to supergravity.

We notice that both $N=3$ and $N=1$ supergravities couple to Kählerian scalars. This gives reasonable hope for a partial breaking of $N=3$ to $N=1$. Another remarkable fact is that an $N=3$ theory in four dimensions can be obtained from a compactification of $D=11$ supergravity¹⁰⁾. This is also true for $N=1$ and $N=2$, but not for $N=4$ ¹¹⁾.

The method we use to construct the $N=3$ matter coupled supergravity is the so called Group-Manifold Approach (4a) which turns out to be particularly economical and efficient for theories involving non linear - models. Given the scalar manifold G/H , one can introduce the vielbein and connections of G/H , together with the supervielbein (V^a, ψ_A) of the N -extended superspace and solve the Bianchi identities of the complete system under the assumption of rheonomy (4a)²⁰⁾. This yields the complete set of supersymmetry transformation rules, closing the on-shell algebra from which the potential can already be computed, as shown in¹²⁾ and¹³⁾. Indeed the extra terms in the fermion transformation rules, due to the gauging of a suitable subgroup of the isometry group of the scalar manifold, are all one needs to work out the potential and therefore to study the possible supersymmetry and gauge symmetry breakings of these theories. We emphasize that these extra terms are obtained, in our approach, from pure group-theory without any reference to the Lagrangian.

Actually the scalar potential of all $N \geq 2$ extended supergravities depends only on the gauge structure constants and the coset space structure of the scalar manifold.

Our results are organized as follows:

- In section 2 we discuss the $N=3$ vector multiplet, the absence of auxiliary fields and of self-interactions in the global supersymmetry limit. Turning supergravity on we expect the scalars to become the co-ordinates of a coset manifold G/H , where G is the group of duality rotations on the vector field strengths. We find the unique answer: $G = SU(3,n)$ and $H = SU(3) \otimes SU(n) \otimes U(1)$, n being the number of vector multiplets involved.
- In section 3 we introduce the $SU(3,n) / SU(3) \otimes SU(n) \otimes U(1)$ formalism and we solve the Bianchi identities both with and without gauging of the $\mathcal{G}$ -model. In particular we show that the possible gauge groups are either $SO(3) \times K_n$ or $SO(3,1) \times K_{n-3}$, K_m being any compact group of dimension m .
- In section 4 we derive the Lagrangian, including the 4-fermi terms, for the non-gauged case.
- In section 5 we consider the additional terms due to the gauging and calculate the scalar field potential.
- Section 6 contains our conclusions.

2) - The $N=3$ vector multiplet and the G / H structure of the supergravity coupling

We begin by fixing our conventions for the indices:

- a, b, c, d = Lorentz flat indices: take 4 values
- A, B, C = $SU(3)$ indices: take 3 values and enumerate the gravitinos
- i, j, k, l = $SU(n)$ indices, n being the number of vector multiplets

Λ, Σ, Π = $SU(3,n)$ indices will take the values A, B, C plus the values i, j, k, l .

No curved indices are needed since we shall use the language of forms throughout the whole paper. In order to discuss the $N=3$ supersymmetric

vector multiplet, we need the flat coframe $(\omega^{ab}, V^a, \psi_A)$ describing N=3 superPoincaré algebra in the dual language of 1-forms and Maurer-Cartan equations. Let V^a be the vierbein, ω^{ab} the spin connection, A_A a triplet of real 1-forms associated to the 3 central charges and ψ_A the 3 chiral gravitinos:

$$\psi_A = \gamma_5 \psi_A \quad ; \quad \bar{\psi}^A = \psi_A^\dagger \gamma_0 \quad (2.1a)$$

$$\psi^A = -\gamma_5 \psi^A \quad ; \quad \psi^A = G(\bar{\psi}^A)^T \quad ; \quad \bar{\psi}^A = (\psi^A)^\dagger \gamma_0 \quad (2.1b)$$

The N=3 superPoincaré algebra reads:

$$R^a_{\text{def}} = \mathcal{D}V^a - i \bar{\psi}^A \gamma^a \psi_A = 0 \quad (2.2a)$$

$$F_A \stackrel{\text{def}}{=} dA_A - \frac{1}{2} \epsilon_{ABC} \bar{\psi}^B \wedge \psi^C - \frac{1}{2} \epsilon^{ABC} \bar{\psi}_B \wedge \psi_C = 0 \quad (2.2b)$$

$$\rho_A \stackrel{\text{def}}{=} \mathcal{D}\psi_A = 0 \quad (2.2c)$$

$$R^{ab}_{\text{def}} = d\omega^{ab} - \omega^{ac} \wedge \omega_c^{b} = 0 \quad (2.2d)$$

where the Lorentz covariant derivatives are ^(*):

$$\mathcal{D}V^a = dV^a - \omega^{ab} \wedge V_b \quad ; \quad \mathcal{D}\psi_A = d\psi_A - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi_A \quad (2.3)$$

This algebra admits an SO(3) automorphism group which can be extended to SU(3) only at the price of introducing, besides the "electric" potential A_A , a "magnetic" one B_A defined by the following equation:

$$\tilde{F}_A = dB_A - \frac{i}{2} \epsilon_{ABC} \bar{\psi}^B \wedge \psi^C + \frac{i}{2} \epsilon^{ABC} \bar{\psi}_B \wedge \psi_C \quad (2.4)$$

(*) The conventions for gamma matrices are given in Appendix A.

The dual Lie Algebra formed by equations (2.2) and (2.4) is now invariant under the following SU(3) infinitesimal transformations:

$$\delta_{SU_3} \psi_A = \Omega_{AB} \psi_B + i \mathcal{J}_{AB} \psi_B \quad (2.5a)$$

$$\delta_{SU_3} A_A = \Omega_{AB} A_B - \mathcal{J}_{AB} B_B \quad (2.5b)$$

$$\delta_{SU_3} B_A = \Omega_{AB} B_B + \mathcal{J}_{AB} A_B$$

where $\Omega_{AB} = -\Omega_{BA}$ are the parameters of the SO(3) subgroup and $\mathcal{J}_{AB} = \mathcal{J}_{BA}$ those of SU_3 / SO_3 .

This is different from what happens in the N=2 case where the analogue of eq. (2.2b)

$$F = dA - \frac{1}{2} \epsilon^{AB} \bar{\psi}_A \wedge \psi_B - \frac{1}{2} \epsilon_{AB} \bar{\psi}^A \wedge \psi^B = 0 \quad (2.6)$$

is already invariant under the SU_2 -transformation:

$$\delta_{SU_2} \psi_A = \Omega_{AB} \psi_B + i \mathcal{J}_{AB} \psi_B \quad \delta_{SU_2} A = 0 \quad (2.7)$$

without the need to introduce a magnetic potential B. This may explain why N=3 multiplets do not admit auxiliary fields and hence do not allow for an arbitrary choice of the σ -model structure contrary to what happens for N=1 and N=2. Indeed, for N=3 there are no hypermultiplets and the only available matter multiplets are the vector ones. Each of them has the following field content

$$[(1), 3(\frac{1}{2}) \oplus 3(\frac{1}{2}), 3(0^+) \oplus 3(0^-)]$$

which corresponds to a 1-form A, a 0-form left handed triplet of spinors

$$\gamma_5 \lambda_A = \lambda_A \quad (2.8)$$

a right handed singlet spinor

$$\gamma_5 \lambda = - \lambda \quad (2.9)$$

and a complex 0-form triplet of scalars $\bar{z}^A$ ($\bar{z}_A = (\bar{z}^A)^*$).

Consider now n of these multiplets labelled by an index i . Following the procedure already utilized for the $N=1$ ¹⁴⁾, $N=2$ ¹⁵⁾ and $N=4$ ¹⁶⁾ vector multiplets, the most general rheonomic parametrization of the exterior derivatives is:

$$d\bar{z}_i^A = \bar{z}_i^A{}_{1a} V^a + \bar{\lambda}_{iB} \psi_C \epsilon^{ABC} + \bar{\lambda}_i \psi^A \quad (2.10a)$$

$$\begin{aligned} F_i = dA_i = & F_i^{ab} V_a \wedge V_b + b_1 \bar{\lambda}_{iA} \gamma^a \psi^A \wedge V_a - \\ & - b_2^* \bar{\lambda}^{Ai} \gamma_a \psi_A \wedge V^a + \frac{1}{2} \bar{z}_i^A \bar{\psi}^B \psi^C \epsilon_{ABC} + \frac{1}{2} \bar{z}_A^i \bar{\psi}_B \psi_C \epsilon^{ABC} \end{aligned} \quad (2.10b)$$

$$\begin{aligned} D\lambda_{iA} = & D_a \lambda_{iA} V^a + b_2 \bar{z}_i^B{}_{1a} \gamma^a \psi^C \epsilon_{ABC} + \\ & + b_3 G_i^{ab} \gamma_{ab} \psi_A + \mathcal{M}_{iA}{}^B \psi_B \end{aligned} \quad (2.10c)$$

$$D\lambda_i = D_a \lambda_i V^a + b_4 \bar{z}_i^A{}_{1a} \gamma^a \psi_A + \mathcal{N}_{iA} \psi^A \quad (2.10d)$$

where b_1, b_2, b_3, b_4 are coefficients to be determined from Bianchi identities, G_i^{ab} is defined by:

$$G_i^{ab} = \frac{1}{2} \left(F_i^{ab} - \frac{i}{2} \epsilon^{ab\text{cd}} F_i^{\text{cd}} \right) \quad (2.11)$$

and $\mathcal{M}_{iA}{}^B, \mathcal{N}_{iA}$ are functions of the scalars which we can call the "auxiliary fields" (i.e., the value taken on shell by the auxiliary fields).

Implementing the Bianchi identities:

$$d^2 z_i^A = dF_i = D^2 \lambda_{iA} = D^2 \lambda_i = 0 \quad (2.12)$$

in the flat background described by eqs. (2.2), we find:

$$b_1 = \frac{i}{2} \quad b_2 = -i \quad b_3 = 1 \quad b_4 = i \quad (2.13)$$

and

$$\mathcal{M}_{iA}^{\dots B} = \mathcal{N}_{iA} = 0 \quad (2.14)$$

This last result follows from:

$$\mathcal{M}_{iB}^{\dots R} \bar{\Psi}_C \Psi_R \epsilon^{ABC} = 0 \quad (2.15a)$$

$$\bar{\Psi}^A \Psi^B \mathcal{N}_{iB} = 0 \quad (2.15b)$$

These equations, because of the integrability condition

$$d^2 z_i^A = 0 \quad (2.16)$$

only have the trivial solution (2.14). The analogous equation one obtains for the N=2 multiplet¹⁵⁾

$$\epsilon^{BC} \bar{\Psi}_C \Psi_R \mathcal{M}_{iB}^{\dots R} = 0 \quad (2.17)$$

admits instead the general solution

$$\mathcal{M}_{iB}^{\dots R} = SU_2 \text{ Lie Algebra element} \quad (2.18)$$

The difference is due to the fact that SU_2 is an automorphism group for equation (2.6) while $SU(3)$ is not an automorphism group for eq. (2.2b).

In other words, the $N=2$ scalars are SU_2 singlets while in the second case of $N=3$ they transform under SU_3 . We stress that the transformation properties of the scalars are connected to those of the central charges, since the scalar fields show up as coefficients of the $\bar{\Psi}_A \wedge \Psi_B$ terms in the expansion of the vector field strength F_i (see eq. (2.10b)).

The conclusion, therefore, is that in the global supersymmetry limit the $N=3$ vector multiplet has no arbitrary self-interaction and gives rise only to a trivial σ -model (=flat manifold). Hence the non-linear σ -model structure is entirely determined by the coupling to supergravity and must correspond to a unique choice of the scalar manifold $\mathcal{M}$. Let us determine $\mathcal{M}$. First of all, the connection of $\mathcal{M}$ must act linearly on the $SU(3)$ index of Ψ_A ; therefore the holonomy group $\mathcal{H}(\mathcal{M})$ must be of the form

$$\mathcal{H}(\mathcal{M}) = SU(3) \otimes \mathcal{H}' \quad (2.19)$$

where $\mathcal{H}'$ is some other group acting on the tangent space indices.

Secondly, if we assume that $\mathcal{M}$ is a homogeneous space we have

$$\mathcal{M} = \mathcal{G} / \mathcal{H}(\mathcal{M}) \quad (2.20)$$

The isometry group $\mathcal{G}$ is the group of duality rotations of the vector field strengths; according to ref. ¹⁷⁾ it satisfies the following constraints:

- i) it is a subgroup of $Sp[2x(3+n)]$, the maximal duality group;
- ii) it has a $3+n$ complex representation or equivalently a $2x(3+n)$ real representation $\mathcal{D}$ ($3+n$ is the total number of vector fields of supergravity $\oplus$ matter theory).

Finally, $\mathcal{G}/\mathcal{H}$ must have $3n$ complex dimensions corresponding to the $3n$ complex scalars.

One may easily verify that these requirements are satisfied, independently of n , by the choice

$$\mathcal{G} = SU(3, n) \quad (2.21a)$$

$$\mathcal{H} = SU(3) \times SU(n) \times U(1) \quad (2.21b)$$

Let us now discuss the possible gaugings of the vector fields. The basic idea is that, irrespectively of the specific theory, when we deal with a σ -model, the symmetries we can gauge are the isometries of the scalar manifold. Hence in our case the gauge group G must be a subgroup of $\mathcal{G}$ = $SU(3, n)$. In supergravity, however, we have the further restriction that the isometry group $\mathcal{G}$ of the scalar manifold is also the duality group of the vector field strengths. This means that the available gauge fields, which are by definition in the adjoint representation of G , are also assigned, together with their duals, to an irreducible representation of $\mathcal{G}$. In our case this is the fundamental $\underline{3+n}$ of $SU(3, n)$.

Let

$$\mathbb{D}(T)_{\Lambda}{}^{\Sigma}$$

be the $(\underline{3+n}) \times (\underline{3+n})$ matrix representation of the generator $T \in SU(3, n)$ and let

$$J_{\Lambda\Sigma} = J^{\Lambda\Sigma} = \begin{pmatrix} \delta_{AB} & 0 \\ 0 & -\delta_{ij} \end{pmatrix} \quad (2.22)$$

be the $SU(3, n)$ - invariant metric; then we have:

$$\mathbb{D}^{\dagger}(T) J + J \mathbb{D}(T) = 0 \quad (2.23)$$

Decomposing $\mathbb{D}(T)$ into its real and imaginary parts:

$$\mathbb{D}(T) = X(T) + i Y(T) \quad (2.24)$$

eq. (2.23) becomes

$$X^T(T) J + J X(T) = 0 \quad (2.25)$$

$$-Y^T(T) J + J Y(T) = 0 \quad (2.26)$$

The elements $T \in SU(3,n)$ for which $Y(T)=0$ form, because of equation (2.25), the subgroup $SO(3,n) \subset SU(3,n)$. On the other hand the vector fields are arranged into a $\underline{3+n}$ of $SU(3,n)$ by writing

$$H_{\Sigma} = A_{\Sigma} + i B_{\Sigma} \quad (2.27)$$

$$\delta H_{\Sigma} = D(T)_{\Sigma}^{\pi} H_{\pi} \quad (2.28)$$

where A_{Σ} is the "electric" potential (the physical field) and B_{Σ} is the "magnetic" potential defined from A_{Σ} via a duality transformation. The symmetry under duality is broken by the gauging since we can gauge only the electric potential which is the real part of the $SU(3,n)$ vector $\underline{3+n}$. This means that we can gauge only those generators of $SU(3,n)$ transforming A_{Σ} into itself. Hence G must be a subgroup of $SO(3,n)$:

$$G \subset SO(3,n) \quad (2.29)$$

Now the $3+n$ complex representation D must decompose into $\text{adj} \oplus \text{adj}$ when G is restricted to the gauge group $G \subset SO(3,n)$

$$D \xrightarrow{G} \text{adj} \oplus \text{adj} \quad (2.30)$$

This condition selects as possible gauge groups G either

$$G = SO(3) \times K_m \quad (2.31)$$

or

$$G = SO(3,1) \times K_{m-3} \quad (2.32)$$

(K_n meaning compact group of dimension n) canonically embedded in the $SO(3,n)$ subgroup of $SU(3,n)$. Indeed the fundamental $(\underline{3+n})^{\text{complex}}$ of $SU(3,n)$ decomposes under $SO(3,n)$ into two $(\underline{3+n})^{\text{real}}$

$$(\underline{3+m})^{\text{complex}} \xrightarrow{SO(3,m)} (\underline{3+m})^{\text{real}} \oplus (\underline{3+m})^{\text{real}} \quad (2.33)$$

and each of the two $(3+n)^{\text{real}}$ can be identified with the adjoint of G given by either (2.31) or (2.32). The last statement follows from the fact that the adjoint matrices of an n -dimensional compact Lie Algebra are anti-symmetric, and therefore also elements of the $SO(n)$ Lie Algebra. Furthermore the action of the subgroup $G \subset SO(3,n) \subset SU(3,n)$ on A_z must be the same whether we consider it as an element of the adjoint representation of G or as a vector of $SO(3,n)$: if

$$T_\Omega \in G \subset SO(3,m) \subset SU(3,m) \quad (2.34)$$

is a generator of G , we must have

$$X(T_\Omega)_\Lambda{}^\Sigma = f_{\Omega\Lambda}{}^\Sigma \quad (2.35)$$

where $f_{\Omega\Lambda}{}^\Sigma$ are the G structure constants

$$[T_\Omega, T_\Lambda] = f_{\Omega\Lambda}{}^\Sigma T_\Sigma \quad (2.36)$$

Comparing eq. (2.36) with eq. (2.23) we see that $J_{\Lambda\Sigma}$ is the Killing metric of G and that $f_{\Lambda\Sigma\Gamma}$, defined by

$$f_{\Lambda\Sigma\Gamma} = f_{\Lambda\Sigma}{}^{\Gamma'} J_{\Gamma\Gamma'} \quad (2.37)$$

is completely antisymmetric in its three indices. This last equation will be essential to prove the supersymmetry of the Lagrangian.

3 - $SU(3,n) / SU(3) \times SU(n) \times U(1)$ formalism and the solution of Bianchi identities.

We now introduce the G/H coset representatives in terms of which we construct the full theory: our procedure is similar to that applied in ref. ⁸⁾ to the $N=4$ coupling; the essential difference is that we consider the composite vielbein and connections as forms over super-

space rather than ordinary space.

Let $L_{\Lambda}^{\Sigma}(z)$ be the $SU(3,n)$ element which represents the equivalence class (i.e. the point of the $\mathcal{G} / \mathcal{H}$ manifold) identified by the co-ordinates z (i.e. the scalar fields z_{Λ}^i)¹⁸⁾.

By definition L is an $SU(3,n)$ matrix and hence satisfies the condition

$$L^{\dagger} J L = J \quad (3.1)$$

or in matrix notation:

$$(L^{-1})_{\Lambda}^{\Sigma} = J_{\Lambda\pi} J^{\pi\Delta} (L_{\Delta}^{\pi})^{*} \quad (3.2)$$

Here $J_{\Lambda\pi}$ is the $SU(3,n)$ - invariant metric defined in (2.22).

L can now be used to write down the vielbein P and the connection Q of the gauged σ -model.

We first define

$$\Omega_{\Lambda}^{\pi} = (L^{-1})_{\Lambda}^{\Sigma} dL_{\Sigma}^{\pi} + g(L^{-1})_{\Lambda}^{\Sigma} f_{\Sigma}^{\Omega\Gamma} A_{\Omega} L_{\Gamma}^{\pi} \quad (3.3)$$

where g is the gauge coupling constant and we see that when $g=0$, Ω reduces to the ordinary left-invariant 1-form over the coset manifold. When $g \neq 0$, Ω no longer satisfies the standard Cartan-Maurer equations and develops a non-vanishing $SU(3,n)$ curvature. Indeed we find

$$d\Omega_{\Lambda}^{\pi} + \Omega_{\Lambda}^{\Gamma} \wedge \Omega_{\Gamma}^{\pi} = g(L^{-1})_{\Lambda}^{\Sigma} f_{\Sigma}^{\Omega\Gamma} J_{\Sigma}^{\pi} L_{\Gamma}^{\pi} \quad (3.4)$$

where

$$J_{\Sigma}^{\pi} \equiv dA_{\Sigma} + f_{\Sigma}^{\Omega\Delta} A_{\Omega} \wedge A_{\Delta} \quad (3.5)$$

is the field strength of the gauge group G ..

Decomposing Ω into $SU(3) \otimes SU(n) \otimes U(1)$ tensors we obtain both the scalar vielbein P and the composite connections Q .

We set

$$\Omega_i^A = P_i^A \quad ; \quad \Omega_A^i = P_A^i = (P_i^A)^* \quad (3.6)$$

$$\Omega_A^B = \varphi_A^B - m \delta_A^B \varphi^\otimes \quad (3.7)$$

$$\Omega_i^j = \varphi_i^j + 3 \delta_i^j \varphi^\otimes \quad (3.8)$$

where φ_A^B ($\varphi_A^A = 0$) and φ_i^j ($\varphi_i^i = 0$) are the SU(3) and SU(n) connections respectively, while $\varphi^\otimes$ is the U(1) connection.

The geometric structure of the matter coupled N=3 supergravity theory is now obtained by merging together the coframe of the pure supergravity sector (ω^{ab}, V^a, ψ_A) with the coframe of the σ -model ($P_A^i, \varphi_A^B, \varphi_i^j, \varphi^\otimes$) via the definition of a new set of curvatures. Let us recall eqs. (2.2), valid in the global limit, and replace them with

$$R^a \equiv DV^a - i \bar{\psi}^A \gamma^a \psi_A \quad (3.9a)$$

$$\rho_A \equiv \nabla \psi_A = D\psi_A + \varphi_A^B \psi_B + p \varphi_A^\otimes \psi_A \quad (3.9b)$$

$$F_A \equiv \mathcal{F}_A - \frac{1}{2} L_A^A \epsilon_{ABC} \bar{\psi}^B \psi^C - \frac{1}{2} L_A^{-1} \pi J_{A\pi} \epsilon^{ABC} \bar{\psi}_B \psi_C \quad (3.9c)$$

$$R^{ab} \equiv d\omega^{ab} - \omega^a_c \omega^c_b \quad (3.9d)$$

where p is the U(1) charge of the gravitino which we shall shortly determine. Equation (3.9c) encompasses in a single structure both the vectors pertaining to supergravity and those pertaining to the matter multiplets. We now supplement the list of curvatures with the SU(3) $\times$ SU(n) $\times$ U(1) covariant derivatives of the spin $\frac{1}{2}$ fermions. These are the λ_{iA} and λ_i spinors already introduced in eqs. (2.10) plus the spin $\frac{1}{2}$ field $\chi_\otimes$ of the supergravity multiplet, which is a singlet under SU(3) and SU(n):

$$\gamma_5 \chi_\otimes = \chi_\otimes \quad \bar{\chi}^\otimes = \chi_\otimes^\dagger \gamma_0 \quad (3.10a)$$

$$\chi^\otimes = c(\bar{\chi}^\otimes)^\top \quad \bar{\chi}_\otimes = \chi^\otimes{}^\dagger \gamma_0 \quad (3.10b)$$

Hence we write:

$$\nabla \chi_\otimes \equiv \mathcal{D} \chi_\otimes + q \varphi^\otimes \chi_\otimes \quad (3.11a)$$

$$\nabla \lambda_i \equiv \mathcal{D} \lambda_i + \varphi_i{}^j \lambda_j + r \varphi^\otimes \lambda_i \quad (3.11b)$$

$$\begin{aligned} \nabla \lambda_{iA} \equiv & \mathcal{D} \lambda_{iA} + \varphi_i{}^j \lambda_{jA} + \varphi_A{}^B \lambda_{iB} \\ & + s \varphi^\otimes \lambda_{iA} \end{aligned} \quad (3.11c)$$

where the U(1) charges q, r, s will be determined by consistency with the charge p of the gravitino.

From equations (3.11) we see that the spinors λ_i, λ_{iA} have been assigned to the fundamental representation of SU(n).

From equations (3.9) and (3.11), by taking a second covariant derivative we obtain the Bianchi identities:

$$\mathcal{D} R^a + R^{ab} \wedge V_b + i \bar{\psi}^\Lambda \gamma^a \psi_A - i \bar{\psi}^\Lambda \gamma^a \rho_\Lambda = 0 \quad (3.12a)$$

$$\nabla \rho_A + \frac{1}{4} R^{ab} \gamma_{ab} \psi_A - R_A{}^B \psi_B - p R^\otimes \psi_A = 0 \quad (3.12b)$$

$$\begin{aligned} \nabla F_\Lambda + \frac{1}{2} L_\Lambda{}^i \rho_i{}^A \epsilon_{ABC} \bar{\psi}^B \psi^C - \frac{1}{2} \rho_A{}^i (L^{-1})_i{}^\pi J_{\Lambda\pi} \epsilon^{ABC} \bar{\psi}_B \psi_C \\ - L_\Lambda{}^A \epsilon_{ABC} \bar{\psi}^B \rho^C - (L^{-1})_A{}^\pi J_{\Lambda\pi} \epsilon^{ABC} \bar{\psi}_B \rho_C = 0 \end{aligned} \quad (3.12c)$$

$$\mathcal{D} R^{ab} = 0$$

(3.12d)

$$\nabla^2 \chi_{\otimes} = -\frac{1}{4} R^{ab} \gamma_{ab} \chi_{\otimes} + q R^{\otimes} \chi_{\otimes} \quad (3.12e)$$

$$\begin{aligned} \nabla^2 \lambda_{iA} = & -\frac{1}{4} R^{ab} \gamma_{ab} \lambda_{iA} + R_i^j \lambda_{jA} \\ & + R_A^B \lambda_{iB} + s R^{\otimes} \lambda_{iA} \end{aligned} \quad (3.12f)$$

$$\nabla^2 \lambda_i = -\frac{1}{4} R^{ab} \gamma_{ab} \lambda_i + R_i^j \lambda_j + \kappa R^{\otimes} \lambda_i \quad (3.12g)$$

where the curvatures of the composite $SU(3) \times SU(n) \times U(1)$ connections have been introduced. They are defined by

$$R_A^B \equiv d\varphi_A^B + \varphi_A^M \wedge \varphi_M^B \quad (3.13)$$

$$R_i^j \equiv d\varphi_i^j + \varphi_i^m \wedge \varphi_m^j \quad (3.14)$$

$$R^{\otimes} \equiv d\varphi^{\otimes} \quad (3.15)$$

These curvatures are completely determined by the structure of the coset manifold, specifically by eq. (3.4).

We now introduce "boosted structure constants" which are functions of the scalar fields:

$$C_{\Lambda}^{\pi\pi} = C_{\Lambda}^{\pi\pi}(z) = (L^{-1})_{\Lambda}^{\Lambda'} L_{\pi'}^{\pi} L_{\pi'}^{\pi'} f_{\Lambda'}^{\pi'\pi'} \quad (3.16a)$$

$$C_{\pi\pi}^{\Lambda} = C_{\pi\pi}^{\Lambda}(z) = L_{\Lambda'}^{\Lambda} (L^{-1})_{\pi}^{\pi'} (L^{-1})_{\pi'}^{\pi'} f_{\pi'\pi'}^{\Lambda'} \quad (3.16b)$$

$$C_{\Lambda}^{\pi\pi} = J_{\Lambda\Lambda'} J^{\pi\pi'} J^{\pi'\pi'} (C_{\pi'\pi'}^{\Lambda'})^* \quad (3.16c)$$

and from eq. (3.4) we obtain

$$\begin{aligned}
 R_A^B = & g \left\{ (L^{-1})_A^{\cdot \wedge} L_Z^B - \frac{1}{3} \delta_A^B (L^{-1})_M^{\cdot \wedge} L_Z^M \right\} f_{\Lambda}^{\cdot \Omega \Sigma} F_{\Omega} \\
 & + \frac{1}{2} g \left(C_A^{\cdot PB} - \frac{1}{3} \delta_A^B C_M^{\cdot PM} \right) \epsilon_{PRS} \bar{\Psi}^R \wedge \Psi^S + \\
 & + \frac{1}{2} g \left(C_{AP}^{\cdot \cdot B} - \frac{1}{3} \delta_A^B C_{MP}^{\cdot \cdot M} \right) \epsilon^{PRS} \bar{\Psi}_R \wedge \Psi_S - \\
 & - \left(P_A^m \wedge P_m^B - \frac{1}{3} \delta_A^B P_M^m \wedge P_m^M \right)
 \end{aligned} \tag{3.17}$$

$$\begin{aligned}
 R_{ij} = & g \left\{ (L^{-1})_i^{\cdot \wedge} L_Z^j - \frac{1}{m} \delta_i^j (L^{-1})_m^{\cdot \wedge} L_Z^m \right\} f_{\Lambda}^{\cdot \Omega \Sigma} F_{\Omega} \\
 & + \frac{1}{2} g \left\{ C_i^{\cdot Pj} - \frac{1}{m} \delta_i^j C_m^{\cdot PM} \right\} \epsilon_{PRS} \bar{\Psi}^R \wedge \Psi^S \\
 & + \frac{1}{2} g \left\{ C_{ip}^{\cdot \cdot j} - \frac{1}{m} \delta_i^j C_{mp}^{\cdot \cdot m} \right\} \epsilon^{PRS} \bar{\Psi}_R \wedge \Psi_S \\
 & - \left(P_i^M \wedge P_M^j - \frac{1}{m} \delta_i^j P_m^M \wedge P_M^m \right)
 \end{aligned} \tag{3.18}$$

$$\begin{aligned}
 R^{\otimes} = & - \frac{g}{3m} (L^{-1})_M^{\cdot \wedge} L_Z^M f_{\Lambda}^{\cdot \Omega \Sigma} F_{\Omega} + \\
 & - \frac{g}{6m} C_M^{\cdot \cdot AM} \epsilon_{ABC} \bar{\Psi}^B \wedge \Psi^C - \frac{g}{6m} C_{MA}^{\cdot \cdot M} \epsilon^{ABC} \bar{\Psi}_B \wedge \Psi_C \\
 & + \frac{1}{3m} P_M^m \wedge P_m^M .
 \end{aligned} \tag{3.19}$$

where we have added and subtracted the spinor terms necessary to complete

the definition of F_Λ . If we set $g = 0$, eqs. (3.17), (3.18) and (3.19) reduce to the ordinary Maurer-Cartan equations of $SU(3,n)$. Furthermore from eq. (3.4) we also obtain

$$\nabla P_i^A = dP_i^A + Q_i^j P_j^A + P_i^B Q_B^A + (3+n) Q^0 P_i^A \quad (3.20)$$

which we can call the "internal torsion equation" since it states that the covariant derivative of the composite vielbein P_i^A vanishes. Eq. (3.20) fixes the $U(1)$ weight of P_i^A from which we deduce all the others. At this point, we write down the parametrization of all the curvatures, or exterior derivatives, and compute the coefficients by implementation of the Bianchi identities. We stress that our base space is superspace, for which an independent basis is provided by (V^a, ψ_A) : the vielbein P_i^A is not an independent form, being given in terms of the scalar field differentials. Hence similarly to dz_A^i in the global case, P_i^A must be expanded along the complete basis (V^a, ψ_A) .

In full analogy to eq. (2.10a) we write

$$P_i^A = P_i^A{}_{1a} V^a + \bar{\lambda}_{iB} \psi_c e^{ABC} + \bar{\lambda}_i \psi^A \quad (3.21)$$

identifying the spinor fields λ_{iB}, λ_i with the spinor derivatives of the scalar fields.

By comparison with (3.20) and (3.9c), eq. (3.21) fixes all the $U(1)$ weights of the spinor fields except q :

$$p = \frac{m}{2} \quad ; \quad r = 3 \left(1 + \frac{m}{2}\right) \quad ; \quad s = 3 + \frac{m}{2} \quad (3.22)$$

The weight q is fixed by writing the most general rheonomic parametrization of the F_Λ -curvature:

$$\begin{aligned}
 F_{\Lambda} = & F_{\Lambda}^{ab} V_a V_b + b_{\perp} L_{\Lambda}^i \bar{\lambda}_{iA} \gamma^a \psi^A \wedge V_a \\
 & + b_{\perp}^* (L^{-1})_i^{\pi} J_{\Lambda\pi} \bar{\lambda}^{iA} \gamma^a \psi_A \wedge V_a + \\
 & + i L_{\Lambda}^A \bar{\chi}^{\otimes} \gamma^a \psi_A \wedge V_a + \\
 & + i (L^{-1})_A^{\pi} J_{\Lambda\pi} \bar{\chi}_{\otimes} \gamma^a \psi^A \wedge V_a
 \end{aligned} \tag{3.23}$$

and is the analogue of eq. (2.10b). It implies

$$q = \frac{3}{2} m \tag{3.24}$$

In order to write the rheonomic parametrization of the fermionic curvatures $\rho_A, \nabla \chi_{\otimes}, \nabla \lambda_{iA}, \nabla \lambda_i$, it is convenient to use the following notation. Given the space-time components F_{Λ}^{ab} of F_{Λ} , we introduce new complex tensors G_A^{ab} and G_i^{ab} which have a definite duality:

$$\epsilon^{abcd} G_A^{cd} = -2i G_A^{ab} ; G_A^A = (G_A^{ab})^* \tag{3.25a}$$

$$\epsilon^{abcd} G_i^{cd} = +2i G_i^{ab} ; G_i^i = (G_i^{ab})^* \tag{3.25b}$$

and are related to F_{Λ}^{ab} by the following equation:

$$\begin{aligned}
 F_{\Lambda}^{ab} = & L_{\Lambda}^A (G_A^{ab} + a_{\perp} \bar{\lambda}^i \gamma^{ab} \lambda_{Ai}) \\
 & + L_{\Lambda}^i (G_i^{ab} + a_2 \bar{\chi}^{\otimes} \gamma^{ab} \lambda_i) \\
 & + (L^{-1})_A^{\pi} J_{\Lambda\pi} (G_A^A + a_{\perp}^* \bar{\lambda}_i \gamma^{ab} \lambda^{Ai}) \\
 & - (L^{-1})_i^{\pi} J_{\Lambda\pi} (G_i^i + a_2^* \bar{\chi}_{\otimes} \gamma^{ab} \lambda^i)
 \end{aligned} \tag{3.26}$$

involving two new coefficients to be determined by Bianchi identities.

Eq. (3.26) is invertible and one can solve it for G_A^{ab} or G_i^{ab} in terms of F_A^{ab} , $\tilde{F}_A^{ab}$ and the spinor currents. We do not need this explicit inversion: we just need to know that it exists. With the above notation we can write the remaining parametrizations as:

$$\begin{aligned}
 \rho_A = & \rho_{A|0b} V^a \wedge V^b + a_3 G_{ab}^B \gamma^a \psi^C V^b \epsilon_{ABC} \\
 & + a_4 \psi_A \wedge \bar{\chi}^\otimes \gamma^a \chi_\otimes V_a + a_5 \gamma_{ab} \psi_A \bar{\chi}^\otimes \gamma^a \chi_\otimes V^b \\
 & + a_6 \gamma_a \psi^B V^a \bar{\chi}^{Ci} \lambda_i \epsilon_{ABC} + a_7 \gamma_a \psi^B V_b \bar{\chi}^{Ci} \gamma^{ab} \lambda_i \epsilon_{ABC} \\
 & + a_8 \psi_A \bar{\lambda}^i \gamma^a \lambda_i V_a + a_9 \gamma_{ab} \psi_A \bar{\lambda}^i \gamma^a \lambda_i V^b \\
 & + a_{10} \psi_A \bar{\lambda}^{iB} \gamma^a \lambda_{iB} V_a + a_{11} \gamma_{ab} \psi_A \bar{\lambda}^{iB} \gamma^a \lambda_{iB} V^b \\
 & + a_{12} \psi_B \bar{\lambda}^{iB} \gamma^a \lambda_{iA} V_a \\
 & + a_{13} \gamma_{ab} \psi_B \bar{\lambda}^{iB} \gamma^a \lambda_{Ai} V^b \\
 & + a_{14} \chi_\otimes \bar{\psi}^B \psi^C \epsilon_{ABC} \\
 & + g a_{15} C_A^{PD} \gamma_m \psi^R V^m \epsilon_{PDR} \\
 & + g a_{16} C_M^{MC} \gamma_m \psi^C V^m \epsilon_{ABC}
 \end{aligned} \tag{3.27}$$

$$\begin{aligned}
 \nabla \chi_\otimes = & \nabla_a \chi_\otimes V^a + a_{17} G_{ab}^A \gamma^{ab} \psi_A \\
 & + a_{18} \psi_A \bar{\lambda}^{Ai} \lambda_i + g a_{19} C_M^{NB} \psi_B
 \end{aligned} \tag{3.28}$$

$$\begin{aligned} \nabla \lambda_{iA} = & \nabla_a \lambda_{iA} V^a + b_2 P_i^B \gamma^a \psi^C \epsilon_{ABC} \\ & + b_3 G_i^{ab} \gamma_{ab} \psi_A + a_{20} \psi_A \bar{\lambda}_i \chi^\otimes + \\ & + g a_{21} C_{iA}^B \psi_B + g a_{22} C_{iM}^M \psi_A \end{aligned} \quad (3.29)$$

$$\begin{aligned} \nabla \lambda_i = & \nabla_a \lambda_i V^a + b_4 P_i^A \gamma^a \psi_A + \\ & + a_{23} \psi^A \bar{\lambda}_{iA} \chi_\otimes + g a_{24} C_i^{AB} \psi^C \epsilon_{ABC} \end{aligned} \quad (3.30)$$

where we have distinguished between the b-coefficients already existing in the global limit (see eqs. (2.10)) and the a-coefficients introduced by the supergravity coupling; moreover we have explicitly written the gauge coupling constant in front of the terms involving the C - functions since they disappear at $g=0$. Inserting eqs. (3.27) - (3.30) into the Bianchi identities (3.12) fixes all the coefficients. The result is given in eqs. (3.38). Comparing with eq. (2.13), we see that the coefficients existing before the coupling to supergravity remain unchanged. On the other hand, comparing eqs. (2.10c) and (2.10d) with (3.29) and (3.30), we see that the shifts of the fermion transformation rules, forbidden at the global level by the absence of "auxiliary fields", show up in the local theory when $g \neq 0$. Indeed we can identify the functions $\mathcal{M}_{iA}^B$ and $\mathcal{N}_{iA}$:

$$\mathcal{M}_{iA}^B = g(a_{21} C_{iA}^B + a_{22} \delta_A^B C_{iM}^M) \quad (3.31a)$$

$$\mathcal{N}_{iA} = g a_{24} C_i^{PQ} \epsilon_{APQ} \quad (3.31b)$$

and we see that the supersymmetry transformations of λ_{iA} and λ_i in the presence of gauging differ from the non-gauged ones only by the addition of the terms (3.33). We find:

$$(\delta_\epsilon \lambda_{iA})_{\text{gauged}} = (\delta_\epsilon \lambda_{iA})_{\text{ungauged}} + M_{iA}{}^B \epsilon_B \quad (3.32)$$

$$(\delta_\epsilon \lambda_i)_{\text{gauged}} = (\delta_\epsilon \lambda_i)_{\text{ungauged}} + N_{iA} \epsilon^A \quad (3.33)$$

This follows from the usual identification of the supersymmetry transformation with the Lie-derivative in the fermionic directions and the use of eqs. (3.29) and (3.30).

With a similar argument we deduce:

$$(\delta_\epsilon \psi_A)_{\text{gauged}} = (\delta_\epsilon \psi_A)_{\text{ungauged}} + S_{AB} \gamma_m \epsilon^B V^m \quad (3.34)$$

$$(\delta_\epsilon \chi_\otimes)_{\text{gauged}} = (\delta_\epsilon \chi_\otimes)_{\text{ungauged}} + \mathcal{U}^A \epsilon_A \quad (3.35)$$

where

$$S_{AB} = g (a_{15} C_A{}^{PP} \epsilon_{BPQ} + a_{16} C_N{}^{MC} \epsilon_{ABC}) \quad (3.36)$$

$$\mathcal{U}^A = g a_{19} C_M{}^{MA} \quad (3.37)$$

In this way, we have determined all the transformation rules from the pure algebraic structure of the theory. Using a general theorem¹¹⁾ we are now able to calculate the potential without reference to the Lagrangian. For completeness, however, in the next section we derive the complete action.

Here we give the a- and b- coefficients appearing in the fermionic curvatures (3.27) - (3.30):

$$\begin{aligned}
 b_1 &= \frac{i}{2} & b_2 &= -i & b_3 &= 1 & b_4 &= i \\
 a_1 &= -\frac{1}{8} & a_2 &= -\frac{1}{4} & a_3 &= 2i & a_4 &= i \\
 a_5 &= \frac{i}{2} & a_6 &= 0 & a_7 &= \frac{i}{4} & a_8 &= 0 \\
 a_9 &= \frac{i}{8} & a_{10} &= 0 & a_{11} &= -\frac{i}{8} & a_{12} &= \frac{i}{4} \\
 a_{13} &= \frac{i}{4} & a_{14} &= -1 & a_{15} &= -\frac{i}{4} & a_{16} &= -\frac{i}{4} \\
 a_{17} &= \frac{1}{2} & a_{18} &= \frac{1}{4} & a_{19} &= -\frac{1}{4} & a_{20} &= 1 \\
 a_{21} &= -1 & a_{22} &= \frac{1}{2} & a_{23} &= -2 & a_{24} &= -\frac{1}{2}
 \end{aligned} \tag{3.38}$$

4) The Lagrangian

In this section we turn to the construction of the Lagrangian which, for consistency with our previous treatment of the supersymmetry transformations, must also be written in the formalism of the "group-manifold" approach. In this framework the Lagrangian must be constructed in terms of differential forms and of the two diffeomorphic invariant operations of exterior algebra (the exterior derivative d and the exterior product $\wedge$), requiring that the field equations, regarded as holding on whole superspace, admit, as a general solution, the rheonomic parametrizations of "curvatures" already determined from Bianchi identities. This is equivalent to demanding supersymmetry covariance of the x-space equations of motion, a necessary but not always sufficient condition for the supersymmetry invariance of the action. Indeed there is, in general, a difference between the supersymmetry transformations which leave the action invariant and those under which the field equations are invariant: this difference is proportional to the x-space field equations¹⁹⁾. To say it more rigorously, the supersymmetry transformations which we have determined through Bianchi identities

and which map one solutions of the x-space field equations into another, once applied to configurations, rather than solutions, become ambiguous, the ambiguity being any term which vanishes on-shell. This ambiguity has to be solved by requiring invariance of the action exactly as in the Noether approach.

Correspondingly, once we have obtained an action which yields, upon variation, the correct rheonomic parametrizations of the curvatures, in order to ensure its supersymmetry invariance, we may still have to introduce modifications, both in the Lagrangian and in the transformation rules. These modifications, however, are limited to the following two:

- i) In the transformation rules we may have to add terms proportional to x-space field equations, namely to the constraints on the inner curvature components imposed by the projection of the variational equations on the maximum number of vielbeins permissible in each equation ($V_{\Lambda} \dots \wedge V$ -projection) permissible in each equation.
- ii) In the action we may have to add terms proportional to the maximal number of vielbeins $V^{a_1} \wedge \dots \wedge V^{a_D} \epsilon_{a_1 \dots a_D}$.

Both kind of additions do not affect the previous results since the extra terms in the SUSY rules vanish on-shell and do not modify the solution of the Bianchi identities, while the extra terms in the Lagrangian contribute only to the space-time field equations.

In our specific case there are no modifications needed in the SUSY rules, but there is a $V^{a_1} \wedge \dots \wedge V^{a_D} \epsilon_{a_1 \dots a_D}$ sector which must be added to the Lagrangian: actually it is a very important sector since it includes the potential term of the scalar fields, to be discussed in the next section. According to these principles we shall, therefore, determine first all other sectors of the Lagrangian, utilizing the variational equations in superspace and then, separately the $V^{a_1} \wedge \dots \wedge V^{a_D} \epsilon_{a_1 \dots a_D}$ - sectors, by implementing supersymmetry invariance as in the Noether approach.

Before coming to the actual construction we must, however, discuss the structure of the spin-1 kinetic terms.

As we have already pointed out, the only allowed operations in the group-manifold framework are those of the exterior algebra of forms. This

rules out the usual kinetic Lagrangian

$$F_{\mu\nu} F^{\mu\nu} = F \wedge *F \quad (4.1)$$

which involves the notion of Hodge-duality (*-operation) whose extension to superspace would, anyhow, be problematic. This difficulty is circumvented by the use of first-order formalism. Instead of (4.1) one writes

$$\rho_{\nu(SPIN 1)}^{(KIN)} = F^{ab} F_{ab} \epsilon_{c_1 \dots c_4} V_{\Lambda}^{c_1} \dots V_{\Lambda}^{c_4} - 12 F^{ab} F V_{\Lambda}^b V^{\Lambda} \epsilon_{abpq} \quad (4.2)$$

where $F^{ab} = -F^{ba}$ is a 0-form regarded as an independent variable and

$$F = dA + A \wedge A \quad (4.3)$$

is the field strength (2-form) of the vector fields. The variation of $\rho_{\nu(SPIN 1)}^{(KIN)}$ in δF^{ab} yields an equation whose general solution is:

$$F = F_{ab} V^a \wedge V^b \quad (4.4)$$

Eq. (4.4) identifies F_{ab} with the $V^a \wedge V^b$ -components of the 2-form F and provides the necessary transition from first- to second-order formalism. Differently from eq. (4.1), the Lagrangian (4.2) has a natural extension to the case of supersymmetric theories. Suppose that, solving the Bianchi identities, we have found the following rheonomic parametrization of F :

$$F = F_{ab} V^a \wedge V^b + \text{const } \bar{\lambda} \gamma^a \psi \wedge V_a \quad (4.5)$$

where λ is a convenient spin 1/2 field (gaugino) and ψ denotes the gravitino. Then we can replace eq. (4.2) by ^{14),15)}:

$$\rho_{\nu(SPIN 1)}^{(KIN)} = F^{ab} F_{ab} \epsilon_{c_1 \dots c_4} V_{\Lambda}^{c_1} \dots V_{\Lambda}^{c_4} - 12 F^{ab} (F - \text{const } \bar{\lambda} \gamma^m \psi \wedge V_m)_{\Lambda} \wedge V^{\Lambda} V^b V^{\Lambda} \epsilon_{abpq} \quad (4.6)$$

and upon variation in δF_{ab} , we not only identify F_{ab} with the $V^a \wedge V^b$ -components of F but we also enforce the rheonomic parametrization (4.5) as a variational equation. We expect therefore that the spin-1 kinetic Lagrangian will have the structure (4.6).

In the case of theories including scalar fields, there is, however, an extra complication due to the "duality rotations". According to the results of ¹⁸⁾, already recalled in section 2, we know that the isometry group G of the scalar σ -model acts on the m -vector fields of the theory as a group of duality transformations. This means the following.

i) There is an inclusion mapping:

$$G \hookrightarrow U_{Sp}(m, m) = U(m, m) \cap Sp(2m, \mathbb{C}) \quad (4.7)$$

which to each element $g \in G$ associates a $2m \times 2m$ complex matrix $S(g)$ with the following block structure:

$$S(g) = \begin{pmatrix} \phi_0(g) & \phi_1^*(g) \\ \phi_1(g) & \phi_0^*(g) \end{pmatrix} \quad (4.8)$$

where the $m \times m$ matrices ϕ_0 and ϕ_1 fulfill the conditions:

$$\phi_0^+ \phi_0 - \phi_1^+ \phi_1 = \mathbb{1} \quad (4.9a)$$

$$\phi_0^+ \phi_1^* - \phi_1^+ \phi_0^* = 0 \quad (4.9b)$$

Equations (4.9) guarantee that $S(g)$ is simultaneously pseudo-unitary and symplectic as required by eq. (4.7).

ii) The equations of motion and Bianchi identities of the m -vector fields

$\hat{A}_\mu^{(\pm)}(x)$ can be written as a system of $2m$ equations of the following type
($\hat{F}_{\mu\nu}^{(\pm)}$ denotes the self-dual part of $F_{\mu\nu}$):

$$\partial_\mu \begin{pmatrix} \hat{F}_{\mu\nu}^{(+)} \\ \hat{F}_{\mu\nu}^{(-)} \end{pmatrix} = 0 \quad (4.10)$$

which are left invariant by the transformations of G defined through the action of the matrix $S(g)$:

$$\begin{pmatrix} \hat{F}_{\mu\nu}^{(+)} \\ \hat{F}_{\mu\nu}^{(-)} \end{pmatrix}' = S(g) \begin{pmatrix} \hat{F}_{\mu\nu}^{(+)} \\ \hat{F}_{\mu\nu}^{(-)} \end{pmatrix} \quad (4.10)$$

Now given a parametrization $L(z)$ of the scalar coset manifold G/H , we can construct $S(L(z))$ and the Lagrangian which yields equations of motion with the invariance property (4.10) has, in terms of $\phi_0(L(z)) = \phi_0$ and $\phi_1(L(z)) = \phi_1$, a universal structure. Indeed, according to ¹⁷⁾ we must have

$$\mathcal{L}_{(SPIN 4)}^{(kin)} = \overset{(+)}{F}_{\lambda\mu\nu} \overset{(+)}{F}_{\Sigma\mu\nu} \mathcal{A}^{\lambda\Sigma} + \overset{(-)}{F}_{\lambda\mu\nu} \overset{(-)}{F}_{\Sigma\mu\nu} \bar{\mathcal{A}}^{\lambda\Sigma} \quad (4.11)$$

where the $m \times m$ matrix $\mathcal{A}$ is symmetric ($\mathcal{A}^T = \mathcal{A}$) and is given by

$$\mathcal{A} = (\phi_0^+ + \phi_1^+)^{-1} (\phi_0^+ - \phi_1^+) \quad (4.12)$$

In the Noether approach, therefore, one first chooses an embedding of G into $Usp(m, m)$ and, having obtained ϕ_0 and ϕ_1 , one is able to write the kinetic Lagrangian (4.11).

The supersymmetry transformation rules are then engineered in such a way as to suit the Lagrangian (4.11).

In our approach, however, the supersymmetry transformation rules have already been fixed in Bianchi identities and we have to see how the harmonics appearing in the rheonomic curvatures are related to the $\mathcal{A}_{\lambda\Sigma}$ function. To this purpose consider eq. (3.26) which relates the $V^a_{\lambda} V^b_{\Sigma}$ components of $F_{\lambda\Sigma}$ to the tensors G_A^{ab} , G_i^{ab} appearing in the ρ_A and $\nabla\lambda_{Ai}$ parametrizations. If we introduce the splitting of $F_{\lambda\Sigma}^{ab}$ into self-dual and antiself-dual parts

$$F_{\lambda\Sigma}^{ab} = \overset{(+)}{F}_{\lambda\Sigma}^{ab} + \overset{(-)}{F}_{\lambda\Sigma}^{ab} \quad (4.13a)$$

$$\epsilon_{abcd} \overset{(\pm)}{F}_{\lambda\Sigma}^{cd} = \pm 2i \overset{(\pm)}{F}_{\lambda\Sigma}^{ab} ; \quad \overset{(\pm)}{F}_{\lambda\Sigma}^{ab} = (\overset{(\mp)}{F}_{\lambda\Sigma}^{ab})^* \quad (4.13b)$$

we can rewrite eq. (3.26) as follows:

$$\overset{(+)}{F}_\Lambda{}^{ab} = \left[(u^{-1})_{\Lambda A}, (u^{-1})_\Lambda{}^i \right] \begin{bmatrix} G_{ab}^A \\ G_{iab} \end{bmatrix} + \quad (4.14)$$

$$+ \left[(u^{-1})_{\Lambda A}, (u^{-1})_\Lambda{}^i \right]^* \begin{bmatrix} -\frac{1}{8} \bar{\lambda}^j \gamma_{ab} \lambda_{Aj} \\ -\frac{1}{4} \bar{\chi}_\otimes \gamma_{ab} \lambda^i \end{bmatrix}$$

where we have introduced a $(3+m) \times (3+m)$ matrix u^{-1} with the following components:

$$(u^{-1})_{\Lambda A} = (L^{-1})_A{}^\pi J_{\Lambda\pi} = (L_\Lambda{}^A)^* \quad (4.15a)$$

$$(u^{-1})_\Lambda{}^i = L_\Lambda{}^i \quad (4.15b)$$

Let us now consider the part of the Lagrangian which contains the kinetic terms of the spin 3/2-field, the gauginos and the vector field strengths. According to our previous discussion we shall write it in the following way:

$$\begin{aligned} \mathcal{L} = & \dots - 4 (\bar{\Psi}^A \gamma^a \rho_A + \bar{\rho}^A \gamma^a \Psi_A)_\Lambda V_a + \\ & + i [\delta_1 (\bar{\chi}_\otimes \gamma_a \nabla \chi_\otimes - \nabla \bar{\chi}_\otimes \gamma^a \chi_\otimes) + \delta_2 (\bar{\lambda}^i \gamma^a \nabla \lambda_i - \nabla \bar{\lambda}^i \gamma_a \lambda_i) \\ & + \delta_3 (\bar{\lambda}^{Ai} \gamma^a \nabla \lambda_{Ai} - \nabla \bar{\lambda}^{Ai} \gamma^a \lambda_{Ai})] V^b V^c V^d \epsilon_{abcd} + \dots + \\ & + \delta_4 (\Omega^{\Lambda Z} \overset{(+)}{F}_\Lambda{}^{ab} \overset{(+)}{F}_Z{}^{ab} + \bar{\Omega}^{\Lambda Z} \overset{(-)}{F}_\Lambda{}^{ab} \overset{(-)}{F}_Z{}^{ab}) \epsilon_{c_1 \dots c_4} V^{c_1} - V^{c_4} + \dots \end{aligned} \quad (4.16)$$

$$\begin{aligned}
 & -24i\delta_4 (\Omega_{\Lambda\Sigma}^{(+)ab} - \bar{\Omega}_{\Lambda\Sigma}^{(+)} F_{\Lambda}^{ab})_{\Lambda} (F_{\Sigma} - iL_{\Sigma}^A \bar{\chi}^{\otimes} \gamma_c \psi_A \wedge V^c \\
 & - \frac{i}{2} L_{\Sigma}^i \bar{\chi}_{iA} \gamma^c \psi^A V_c - i(L_{\Sigma}^A)^* \bar{\chi}_{\otimes} \gamma_c \psi^A \wedge V^c \\
 & - \frac{i}{2} (L_{\Sigma}^i)^* \bar{\chi}^{iA} \gamma_c \psi_A \wedge V^c)_{\Lambda} V_a V_b \\
 & + F_{\Lambda} \{ i\delta_5 (B_{\Lambda\Sigma} L_{\Sigma}^A \epsilon_{ABC} \bar{\psi}^B \wedge \psi^C - \bar{B}_{\Lambda\Sigma} (L_{\Sigma}^A)^* \epsilon^{ABC} \bar{\psi}_B \wedge \psi_C) \\
 & + \delta_6 (\bar{B}_{\Lambda\Sigma} L_{\Sigma}^i \bar{\chi}_{iA} \gamma^a \psi^A - B_{\Lambda\Sigma} (L_{\Sigma}^i)^* \bar{\chi}^{iA} \gamma_a \psi_A)_{\Lambda} V_a + \\
 & + \dots
 \end{aligned} \tag{4.16}$$

continued)

where $\Omega_{\Lambda\Sigma}$ is the kinetic vector metric which should have the form (4.12) in terms of the symplectic matrix $S(L(z))$ representing the $SU(3, n)/SU(3) \otimes SU(n) \otimes U(1)$ coset element $L_{\Lambda}^{\Sigma}(z)$, appearing in the Pauli term is a function of the scalars yet to be determined and $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ are coefficients to be determined as well.

To this effect we consider the $\delta \bar{\psi}^A$ - variation and the $\delta \bar{\chi}^{ai}$ - variation and we look at the cancellation of the $G^{ab} \psi_A V \dots V$ terms in the corresponding equations.

From $\delta \bar{\psi}^A$ we obtain:

$$\begin{aligned}
 & -8\gamma^a \rho_A \wedge V_a + 24i\delta_4 (\Omega_{\Lambda\Sigma}^{(+)} F_{\Lambda}^{ab} - \bar{\Omega}_{\Lambda\Sigma}^{(+)} F_{\Lambda}^{ab}) L_{\Sigma}^B \epsilon_{BAC} \cdot \\
 & \cdot \psi^C V_a V_b + 2i\delta_5 F_{\Lambda} B_{\Lambda\Sigma} L_{\Sigma}^B \epsilon_{BAC} \psi^C + \dots = 0
 \end{aligned} \tag{4.17}$$

Recalling the rheonomic parametrization (3.27) of ρ_A and projecting eq. (4.17) on $\psi^c \wedge V^a \wedge V^b$ we obtain

$$16i G_{ab}^B \epsilon_{ABC} - 24i \delta_4 (\Omega_{\Lambda\Sigma} \overset{(+)}{F}_\Lambda^{ab} - \bar{\Omega}_{\Lambda\Sigma} \overset{(-)}{F}_\Lambda^{ab}) L_\Sigma^B \epsilon_{ABC} \\ - 2i \delta_5 (B_{\Lambda\Sigma} \overset{(+)}{F}_\Lambda^{ab} + B_{\Lambda\Sigma} \overset{(-)}{F}_\Lambda^{ab}) L_\Sigma^B \epsilon_{ABC} + \dots = 0 \quad (4.18)$$

In this equation the self-dual and antiself-dual field strengths must cancel separately. Since G_{ab}^B is self-dual the cancellation of the $\overset{(-)}{F}_\Lambda^{ab}$ - terms implies

$$+ 24i \delta_4 \bar{\Omega}_{\Lambda\Sigma} \overset{(-)}{F}_\Lambda^{ab} - 2i \delta_5 B_{\Lambda\Sigma} \overset{(-)}{F}_\Lambda^{ab} = 0 \quad (4.19)$$

which can be achieved if we set

$$\delta_5 = 12 \delta_4 \quad ; \quad B_{\Lambda\Sigma} = \bar{\Omega}_{\Lambda\Sigma} \quad (4.20)$$

Utilizing eq. (4.20) and looking at the cancellation of the $\overset{(+)}{F}_\Lambda^{ab}$ - terms we get

$$16i G_{ab}^B - 48i \delta_4 (\text{Re } \Omega_{\Lambda\Sigma}) \overset{(+)}{F}_\Lambda^{ab} L_\Sigma^B = 0 \quad (4.21)$$

Recalling eqs. (4.14) and (4.15), we see that eq. (4.21) is fulfilled if $\text{Re } \Omega_{\Lambda\Sigma}$ fulfils the following properties (we use $\Omega = \Omega^T$):

$$[(u^{-1})^+]^B_\Sigma (\text{Re } \Omega_{\Sigma\Lambda}) (u^{-1})_{\Lambda A} = \delta_A^B \quad (4.22a)$$

$$[(u^{-1})^+]^B_\Sigma (\text{Re } \Omega_{\Sigma\Lambda}) (u^{-1})_{\Lambda}^i = 0 \quad (4.22b)$$

and if

$$\delta_4 = \frac{1}{3} \quad (4.23)$$

Considering next the $\delta \bar{\lambda}^{Ai}$ - variation we get

$$\begin{aligned}
 & 2i \delta_3 \gamma^a \nabla \lambda_{Ai} V^b V^c V^d \epsilon_{abcd} \\
 & - 12 \delta_4 (\Omega^{\Lambda \Sigma} \overset{(+)}{F}_\Lambda^{ab} - \bar{\Omega}^{\Lambda \Sigma} \overset{(-)}{F}_\Lambda^{ab}) (L_\Sigma^i)^* \gamma^c \psi_A V_c V_a V_b \quad (4.24) \\
 & - \delta_6 B_{\Lambda \Sigma} F_\Lambda (L_\Sigma^i)^* \gamma^c \psi_A V_c + \dots = 0
 \end{aligned}$$

Recalling the rheonomic parametrization (3.29) of $\nabla \lambda_{Ai}$ and projecting eq. (4.24) into $\psi_A \wedge V_p V_q V_r \in \epsilon^{pqrv}$, we obtain:

$$\begin{aligned}
 & - 48i \delta_3 G_i^{ta} \gamma_a - 24i \delta_4 (\Omega^{\Lambda \Sigma} \overset{(+)}{F}_\Lambda^{ta} + \bar{\Omega}^{\Lambda \Sigma} \overset{(-)}{F}_\Lambda^{ta}) (L_\Sigma^i)^* \gamma_a \\
 & - 2i \delta_6 \bar{\Omega}_{\Lambda \Sigma} (\overset{(+)}{F}_\Lambda^{ta} - \overset{(-)}{F}_\Lambda^{ta}) (L_\Sigma^i)^* \gamma_a = 0 \quad (4.25)
 \end{aligned}$$

where eq. (4.20) has already been used. The cancellation of the $\overset{(-)}{F}_\Lambda^{ta}$ terms yields

$$\delta_6 = 12 \delta_4 = 4 \quad (4.26)$$

while the cancellation of the $\overset{(+)}{F}_\Lambda^{ta}$ terms is achieved if

$$\delta_3 = -\frac{1}{3} \quad (4.27)$$

and if

$$[(\mathcal{U}^{-1})^+]_{i \Sigma} (R_\epsilon \Omega_{\Sigma \Lambda}) [\mathcal{U}^{-1}]_\Lambda^j = \delta_{i \Sigma}^j \quad (4.28a)$$

$$[(\mathcal{U}^{-1})^+]^\beta_\Sigma (R_\epsilon \Omega_{\Sigma \Lambda}) [\mathcal{U}^{-1}]_\Lambda^j = 0 \quad (4.28b)$$

Putting together eqs. (4.28) and (4.22) we obtain the matrix equation

$$(U^{-1})^\dagger \operatorname{Re} \Omega U^{-1} = \mathbb{1} \Rightarrow \operatorname{Re} \Omega = U^\dagger U \quad (4.29)$$

Equation (4.29) tells us that the matrix Ω , which must have the form (4.12) in order to guarantee $SU(3, n)$ duality invariance of the field equations, must also fulfil eq. (4.29) in order to be consistent with rheonomy, namely in order to yield a supersymmetric Lagrangian.

Hence equation (4.29) provides a very powerful consistency check on the whole formalism. In Appendix C we study the explicit embedding of $SU(3, n)$ into $USp(3+n, 3+n)$ and in this way we arrive at the determination of $\Omega_{\Lambda Z}$ which is given by equations (C21), (C22) and (C23). We then consider the real part of this matrix and we show that indeed it satisfies eq. (4.29).

This concludes our preliminary discussion of the vector field kinetic metric $\Omega_{\Lambda Z}$. The Lagrangian can now be constructed using $\Omega_{\Lambda Z}$ and the following auxiliary matrices which are used to saturate indices in a $SU(3) \otimes SU(n)$ invariant manner:

$$W^{AB} = L_\Lambda^A L_Z^B \bar{\Omega}^{\Lambda Z} ; \quad W_{AB} = (W^{AB})^* \quad (4.30a)$$

$$W^{ij} = L_\Lambda^i L_Z^j \Omega_{\Lambda Z} ; \quad W_{ij} = (W^{ij})^* \quad (4.30b)$$

The procedure is that of writing the most general ansatz consistent with the following symmetries and then determining the coefficients from the variational equations. The symmetries to be respected are:

- a) Lorentz and $SU(3) \otimes SU(n)$ manifest invariance.
- b) Parity. All terms must have the same parity $P = -1$ as the Einstein term.
- c) Global scale invariance. We must respect the invariance under the scale transformations:

$$\begin{aligned}
 V^a &\mapsto e V^a & A_\Lambda &\mapsto e A_\Lambda \\
 \psi^A &\mapsto e^{1/2} \psi^A & \lambda_{Ai} &\mapsto e^{-1/2} \lambda_{Ai} \\
 \omega^{ab} &\mapsto \omega^{ab} & \lambda_i &\mapsto e^{-1/2} \lambda_i \\
 \chi &\mapsto e^{-1/2} \chi & z_A^i &\mapsto z_A^i
 \end{aligned} \tag{4.31}$$

where e is the scaling parameter. This is a symmetry of the Bianchi identities (3.12) and a Lagrangian consistent with them must have the same symmetry.

As a matter of convenience the Lagrangian can be split as follows

$$\mathcal{L} = \mathcal{L}(\text{ungauged}) + \Delta\mathcal{L}(\text{gauging}) \tag{4.32}$$

where $\mathcal{L}_{(\text{ungauged})}$ is the Lagrangian at $g=0$. The full Lagrangian is obtained by introducing G-covariant derivatives inside the definition of F_A^i and of the Q-connections and by adding $\Delta\mathcal{L}_{(\text{gauging})}$ to be determined in the next section.

$\mathcal{L}_{(\text{ungauged})}$ can be written as follows:

$$\begin{aligned}
 \mathcal{L}(\text{ungauged}) = & \mathcal{L}_{\text{KIN}} + \mathcal{L}_{\text{PAULI}} + \mathcal{L}_{\text{TORSION}} + \mathcal{L}_{(4\text{FERMI})} + \\
 & + \mathcal{L}_{(4\text{FERMI})}^{(4V)}
 \end{aligned} \tag{4.33}$$

where we have separated $\mathcal{L}_{(4\text{FERMI})}^{(4V)}$ in the 4-Fermi sector. This part, proportional to $V^a V^b V^c V^d \epsilon_{abcd}$ we determine via the Noether method.

Our result is:

$$\mathcal{L}_{\text{KIN}} = \epsilon_{abcd} R^{ab} \wedge V^c \wedge V^d - 4 (\bar{\psi}^A \gamma_a \rho_A + \bar{\rho}^A \gamma_a \psi_A) V^a \tag{4.34a}$$

$$\begin{aligned}
 & -\frac{i}{3} \left[4 (\bar{\chi}^{\otimes} \gamma_a \nabla \chi_{\otimes} - \nabla \bar{\chi}^{\otimes} \gamma_a \chi_{\otimes}) + (\bar{\lambda}^i \gamma_a \nabla \lambda_i - \nabla \bar{\lambda}^i \gamma_a \lambda_i) \right. \\
 & + \left. (\bar{\lambda}^{Ai} \gamma_a \nabla \lambda_{Ai} - \nabla \bar{\lambda}^{Ai} \gamma_a \lambda_{Ai}) \right] V_b \wedge V_c \wedge V_d \epsilon^{abcd} + \\
 & + \frac{1}{3} \left[P_{A|a}^i (P_i^A - \bar{\lambda}^{iB} \psi_c \epsilon^{ABC} - \bar{\lambda}^i \psi^A) + \right. \\
 & + P_i^A{}_{|a} (P_i^A - \bar{\lambda}^{iB} \psi^c \epsilon_{ABC} - \bar{\lambda}^i \psi_A) \left. \right] V_b \wedge V_c \wedge V_d \epsilon^{abcd} \\
 & + \frac{1}{3} \left(\Omega^{\Lambda\Sigma} \overset{(+)}{F}_{\Lambda}^{ab} \overset{(+)}{F}_{\Sigma}^{ab} + \bar{\Omega}^{\Lambda\Sigma} \overset{(-)}{F}_{\Lambda}^{ab} \overset{(-)}{F}_{\Sigma}^{ab} - \frac{1}{2} P_{A|a}^i P_{i|a}^A \right) \cdot \\
 & \cdot \epsilon_{c_1 \dots c_4} V^{c_1} \wedge \dots \wedge V^{c_4} - 8i \left(\Omega^{\Lambda\Sigma} \overset{(+)}{F}_{\Lambda}^{ab} - \bar{\Omega}^{\Lambda\Sigma} \overset{(-)}{F}_{\Lambda}^{ab} \right) \cdot \\
 & \cdot \left(F_{\Sigma} - i L_{\Sigma}^A \bar{\chi}^{\otimes} \gamma_c \psi_A \wedge V^c - \frac{i}{2} L_{\Sigma}^i \bar{\lambda}_{iA} \gamma^c \psi^A \wedge V^c - \right. \\
 & \left. - i (L_{\Sigma}^A)^* \bar{\chi}_{\otimes} \gamma_c \psi^A \wedge V^c - \frac{i}{2} (L_{\Sigma}^i)^* \bar{\lambda}_{iA} \gamma_c \psi^A \wedge V^c \right) \wedge V_a \wedge V_b
 \end{aligned}$$

(4.34a
continued)

$$\begin{aligned}
 \rho_{(PAULI)} = F_{\Lambda} \{ & 4i \left(\bar{\Omega}_{\Lambda\Sigma} L_{\Sigma}^A \epsilon_{ABC} \bar{\psi}^B \psi^C - \Omega_{\Lambda\Sigma} (L_{\Sigma}^A)^* \epsilon^{ABC} \right. \\
 & \cdot \bar{\psi}^B \wedge \psi_C \left. \right) + 4 \left(\Omega_{\Lambda\Sigma} L_{\Sigma}^i \bar{\lambda}_{iA} \gamma^a \psi^A - \bar{\Omega}_{\Lambda\Sigma} (L_{\Sigma}^i)^* \bar{\lambda}^{iA} \gamma^a \psi_A \right) V^a \\
 & - 8 \left(\Omega_{\Lambda\Sigma} L_{\Sigma}^A \bar{\chi}^{\otimes} \gamma^a \psi_A - \bar{\Omega}_{\Lambda\Sigma} (L_{\Sigma}^A)^* \bar{\chi}_{\otimes} \gamma^a \psi^A \right) V_a \\
 & + 2 \left(\bar{\Omega}_{\Lambda\Sigma} L_{\Sigma}^A \bar{\lambda}^i \gamma_{ab} \lambda_{Ai} - \Omega_{\Lambda\Sigma} (L_{\Sigma}^A)^* \bar{\lambda}_i \gamma_{ab} \lambda^{Ai} \right) \wedge V_a \wedge V_b \\
 & + 4i \left(\Omega_{\Lambda\Sigma} L_{\Sigma}^i \bar{\chi}^{\otimes} \gamma_{ab} \lambda_i - \bar{\Omega}_{\Lambda\Sigma} (L_{\Sigma}^i)^* \bar{\chi}_{\otimes} \gamma_{ab} \lambda^i \right) V^a \wedge V^b \\
 & - 2i \left(P_i^A \bar{\lambda}^{iB} \gamma_{ab} \psi^c \epsilon_{ABC} - P_A^i \bar{\lambda}_{iB} \gamma_{ab} \psi_c \epsilon^{ABC} \right) \wedge V^a \wedge V^b \\
 & + 2i \left(P_i^A \bar{\lambda}^i \gamma_{ab} \psi_A - P_A^i \bar{\lambda}_i \gamma_{ab} \psi^A \right) \wedge V^a \wedge V^b \}
 \end{aligned}$$

(4.34b)

$$\begin{aligned}
 \mathcal{L}_{(TORSION)} &= 4 R_a \wedge V^a \wedge [\bar{\chi}^\otimes \gamma_b \chi_\otimes - \bar{\chi}^i \gamma_b \chi_i + \bar{\chi}^{\Delta i} \gamma_b \chi_{\Delta i}] \wedge V^b \\
 \mathcal{L}_{(4FERMI)} &= i [\bar{\Psi}^A \wedge \Psi^B \wedge \bar{\Psi}^P \wedge \Psi^Q \epsilon_{ABC} \epsilon_{PQR} W^{CR} - \bar{\Psi}_A \wedge \Psi_B \wedge \bar{\Psi}_P \wedge \Psi_Q \cdot \\
 &\quad \cdot \epsilon^{ABC} \epsilon^{PQR} W_{CR} + 2i (\bar{\chi}^{iA} \gamma_a \chi_{iB} \bar{\Psi}_A \wedge \gamma_b \Psi^B - \\
 &\quad - \bar{\chi}^{iA} \gamma_a \chi_{iA} \bar{\Psi}_B \wedge \gamma_b \Psi^B + \bar{\chi}^i \gamma_a \chi_i \bar{\Psi}_A \wedge \gamma^b \Psi^A) \wedge V_a \wedge V_b \\
 &\quad - \frac{i}{2} (\bar{\Psi}_A \wedge \Psi_B \bar{\chi}^{\Delta i} \gamma_{ab} \chi^{\beta j} W_{ij} - \bar{\Psi}^A \wedge \Psi^B \bar{\chi}_{\Delta i} \gamma_{ab} \chi_{\beta j} W^{ij}) \wedge V^a \wedge V^b \\
 &\quad - \frac{i}{2} (\bar{\Psi}_A \wedge \gamma_{ab} \Psi_B \bar{\chi}^{\Delta i} \chi^{\beta j} W_{ij} - \bar{\Psi}^A \wedge \gamma_{ab} \Psi^B \bar{\chi}_{\Delta i} \chi_{\beta j} W^{ij}) \wedge V^a \wedge V^b \\
 &\quad - \frac{i}{3} (\bar{\chi}^i \chi_{iA} \bar{\chi}_\otimes \gamma_a \Psi^A + \bar{\chi}_i \chi^{iA} \bar{\chi}^\otimes \gamma_a \Psi_A) V^b V^c V^d \epsilon_{abcd} \\
 &\quad + (\bar{\chi}_i \gamma_{ab} \chi^{iA} \bar{\chi}^\otimes \gamma_c \Psi_A - \bar{\chi}^i \gamma_{ab} \chi_{iA} \bar{\chi}_\otimes \gamma_c \Psi^A) V^a V^b V^c \\
 &\quad - i (\bar{\chi}^i \gamma_a \chi^{jA} \bar{\chi}_\otimes \Psi_A W_{ij} + \bar{\chi}_i \gamma_a \chi_{jA} \bar{\chi}^\otimes \Psi^A W^{ij}) V_b V_c V_d \epsilon^{abcd} \\
 &\quad - (\bar{\chi}^i \gamma_a \chi^{jA} \bar{\chi}_\otimes \gamma_{bc} \Psi_A W_{ij} + \bar{\chi}_i \gamma_a \chi_{jA} \bar{\chi}^\otimes \gamma_{bc} \Psi^A W^{ij}) V^a V^b V^c \\
 &\quad - i (\bar{\Psi}_A \gamma_{ab} \Psi_B \bar{\chi}^\otimes \chi^\otimes W^{AB} - \bar{\Psi}^A \gamma_{ab} \Psi^B \bar{\chi}_\otimes \chi_\otimes W_{AB}) V^a V^b
 \end{aligned} \tag{4.34 c}$$

$$\begin{aligned}
 \mathcal{L}_{(4V)}^{(4V)} &= \frac{1}{96} \epsilon_{abcd} V^a V^b V^c V^d \left\{ -4 \bar{\chi}_i \chi^\otimes \bar{\chi}^i \chi_\otimes + \right. \\
 &\quad + (4 \bar{\chi}^\otimes \gamma^m \chi_\otimes + \bar{\chi}^i \gamma^m \chi_i - \bar{\chi}^{\Delta i} \gamma^m \chi_{\Delta i})^2 + \\
 &\quad \left. + \bar{\chi}_{\Delta i} \gamma^{ab} \chi^i (3 W^{AB} \bar{\chi}^i \gamma_{ab} \chi_{\beta i} - \bar{\chi}_i \gamma^{ab} \chi^{\Delta i}) \right\} + h.c.
 \end{aligned} \tag{4.34 e}$$

As a final remark we remind the reader that the action is in first-order formalism: hence $\tilde{F}_\Lambda^{ab}$, $\tilde{F}_\Lambda^{ab}$ and $P^i_{\Lambda ia}$ must be varied as independent dynamical variables. $SU(3) \otimes SU(n)$ local invariance is manifest due to the presence of the scalar fields providing the metrics W^{AB} and W^{ij} by means of which we contract the "wrong" set of indices. In the limit where the scalar fields are zero, however, $W^{AB} \rightarrow S^{AB}$, $W^{ij} \rightarrow S^{ij}$ and global invariance under $SU_3 \otimes SU_n$ is not present.

5) The Scalar Field Potential

In section 4 we calculated $\mathcal{L}_{(ungauged)}$. In this section we study $\Delta\mathcal{L}_{(gauging)}$ which is particularly interesting because it contains the scalar field potential.

In full generality we can write:

$$\Delta\mathcal{L}_{(gauging)} = \mathcal{L}_{(potential)} + \mathcal{L}_{(2\text{ Fermi})} + \mathcal{L}_{(4V)}^{(4V)} \quad (5.1)$$

where $\mathcal{L}_{(2\text{ Fermi})}$ is a sum of bilinear terms in the fermions containing at least one gravitino ψ_A . It can be obtained straightforwardly from the equations of motion and reads:

$$\begin{aligned} \mathcal{L}_{(2\text{ Fermi})} = & -2i \left[\delta_1 \bar{\chi}^\alpha \gamma^\alpha \psi_A u^A + \delta_1 \bar{\chi}^\alpha \gamma^\alpha \psi^A u_A \right. \\ & + \delta_2 \bar{\chi}^i \gamma^\alpha \psi^A \mathcal{M}_{iA} + \delta_2 \bar{\chi}^i \gamma^\alpha \psi_A \mathcal{M}^{iA} + \delta_3 \bar{\chi}_{\alpha i} \gamma^\alpha \psi^B \mathcal{M}^{iA}_B \\ & + \delta_3 \bar{\chi}^{\alpha i} \gamma^\alpha \psi_B \mathcal{M}_{iA}^B \left. \right] V^b V^c V^d \epsilon_{abcd} \\ & - 4 (S_{AB} \bar{\psi}^A \gamma^{ab} \psi^B - S^{AB} \bar{\psi}_A \gamma_{ab} \psi_B) V_a V_b \end{aligned} \quad (5.2)$$

where the coefficients $\delta_1, \delta_2, \delta_3$ are given in eq. (4.10a) and the "shifts" of the fermion SUSY rules are defined by eqs. (3.31), (3.36) and (3.37).

$\mathcal{L}_{(2\text{Fermi})}$ and $\mathcal{L}_{(2\text{Fermi})}$, on the other hand, being proportional to four vierbeins, must be obtained by the Noether method. As far as $\mathcal{L}_{(2\text{Fermi})}^{(4V)}$ is concerned one finds:

$$\begin{aligned} \mathcal{L}_{(2\text{Fermi})}^{(4V)} = & -\frac{1}{6} \left[-C^i \bar{\chi}^\otimes \lambda_i - C_i{}^{PD} \bar{\chi}^\otimes \lambda^{iR} \epsilon^{PPR} \right. \\ & + \frac{1}{2} C^A \bar{\lambda}^i \lambda_{Ai} + C_i{}^{Aj} \bar{\lambda}^i \lambda_{Aj} \\ & + \frac{1}{2} C_A{}^{ij} \bar{\lambda}_{Bi} \lambda_{Cj} \epsilon^{ABC} \\ & + C_i \bar{\chi}^\otimes \lambda^i + C_i{}^{PD} \bar{\chi}^\otimes \lambda^{iR} \epsilon^{PPR} \\ & + \frac{1}{2} C_A \bar{\lambda}_i \lambda^{Ai} + C_i{}^{Aj} \bar{\lambda}_i \lambda^{Aj} \\ & \left. + \frac{1}{2} C_{ij}{}^A \bar{\lambda}^{Bi} \lambda^{Cj} \epsilon_{ABC} \right] \end{aligned} \quad (5.3)$$

The calculation of the potential is a more delicate matter. We write:

$$\mathcal{L}_{(Potential)} = -g^2 W(z) \epsilon_{abcd} V^a V^b V^c V^d \quad (5.4)$$

where $W(z)$ is the scalar potential. Supersymmetry invariance of the action requires a cancellation between the contributions of $\mathcal{L}_{(Potential)}$ and those of $\mathcal{L}_{(2\text{Fermi})}$ (see eq. (5.2)). This occurs if and only if the following equation holds true:

$$\begin{aligned} -g^2 W(z) \delta_A^B = & 2 S_{AM} S^{MB} - \frac{2}{3} M_A M^B - \frac{1}{6} N_{iA} N^{iB} \\ & - \frac{1}{6} M^{iH}{}_A M_{iH}{}^B \end{aligned} \quad (5.5)$$

Eq. (5.5) requires that the traceless part of the r.h.s. cancel identically. This non-trivial condition provides an excellent check on all the previous computations. We refer to Appendix B for more details. Eq. (5.5) is a general result, essentially model-independent, which links the potential to the fermion shifts¹²⁾.

The potential can now be easily computed taking the trace of eq.(5.5). From the Jacoby identities for the group G and the definition of C functions one obtains:

$$|C_i|^2 = |C_P|^2 + |C_A^{iP}|^2 - |C_A^{PQ}|^2 \quad (5.6)$$

Using eq. (5.6) we arrive at the simple result:

$$12 g^2 W(x) = \frac{1}{2} |C_{iA}^B|^2 + \frac{1}{2} |C_i^{PQ}|^2 - (|C_A^{PQ}|^2 - |C_P|^2) \quad (5.7)$$

The last term in the bracket is the contribution from the gravitino mass matrix $|S_{AB}|^2 = \frac{1}{8} (|C_A^{PQ}|^2 - |C_P|^2)$.

Note that in the absence of matter multiplets $W = -1/2$ and we recover $N=3$ anti-De Sitter supergravity²²⁾.

Conclusion

In this paper we have constructed the complete $N=3$ matter coupled supergravity for an arbitrary number of vector multiplets gauging one of the two following classes of groups G :

$$G = SO(3) \times (\text{Compact gauge group})$$

$$G = SO(1,3) \times (\text{Compact gauge group})$$

The study of the potential and of the possible partial supersymmetry breakings is postponed to future work.

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APPENDIX A: MATRIX ALGEBRA IN D=4

Our conventions are the following ones:

(1)

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} \quad \eta_{ab} = \text{diag}(1, -1, -1, -1) \quad (\text{A.1})$$

(2) Charge conjugation and gamma five:

$$C\gamma^a C^{-1} = -\gamma^{aT}, \quad C^T = -C, \quad C^2 = -1 \quad (\text{A.2})$$

$$\gamma^5 = -i\gamma_1\gamma_2\gamma_3\gamma_4, \quad (\gamma^5)^2 = 1 \quad (\text{A.3})$$

$$C\gamma^5 C^{-1} = \gamma^{5T}, \quad C\gamma^5\gamma^a C^{-1} = (\gamma^5\gamma^a)^T \quad (\text{A.4})$$

(3) Hermiticity:

$$\gamma_1^\dagger = \gamma_1, \quad \begin{pmatrix} \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}^\dagger = -\begin{pmatrix} \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}, \quad \gamma^{5\dagger} = \gamma^5, \quad \gamma_1\gamma^{a\dagger}\gamma_1 = \gamma^a \quad (\text{A.5})$$

(4) Lorentz-generators:

$$\gamma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b] \quad \gamma_1\gamma_{ab}^\dagger\gamma_1 = \gamma_{ab} \quad (\text{A.6})$$

$$C\gamma^{ab}C^{-1} = -\gamma^{ab}$$

(5) Dirac-algebra identities

$$[\gamma^a, \gamma^{bc}] = 2(\eta^{ab}\gamma^c - \eta^{ac}\gamma^b) \quad (A.7)$$

$$\{\gamma^a, \gamma^{bc}\} = 2i \epsilon^{abcd} \gamma^d \quad (A.8)$$

$$\gamma_a \gamma_{bc} = (\eta_{ab}\gamma_c - \eta_{ac}\gamma_b) + i \epsilon_{abcd} \gamma^d \gamma_5 \quad (A.9)$$

$$\gamma_5 \gamma_{ab} = -\epsilon_{abcd} \gamma^{cd} \quad (A.10)$$

$$\gamma_5 \gamma_a \gamma_b = \eta_{ab} \gamma_5 - i \epsilon_{abcd} \gamma^{cd} \quad (A.11)$$

$$\text{Tr}(\gamma_a \gamma_b \gamma_c \gamma_d \gamma_5) = 4i \epsilon_{abcd} \quad (A.12)$$

$$\epsilon_{abcd} = -\epsilon^{abcd} \quad \epsilon_{1234} = +1 \quad (A.13)$$

$$\epsilon_{abcd} \epsilon^{rsjk} = (-4!) \delta_{abcd}^{rsjk} \quad (A.14)$$

(6) Properties of fermion currents:

If χ_A is a Majorana spin 1/2 0-form, and Ξ_A a Majorana spin 3/2 1-form we have

$$\lambda_A = (1 + \gamma_5) \chi_A \quad \lambda^A = (1 - \gamma_5) \chi_A \quad (A.15)$$

$$\psi_A = (1 + \gamma_5) \Xi_A \quad \psi^A = (1 - \gamma_5) \Xi_A \quad (A.16)$$

$$\bar{\lambda}^A = \bar{\chi}_A (1 - \gamma_5) \quad \bar{\lambda}_A = \bar{\chi}_A (1 + \gamma_5) \quad (A.17)$$

$$\bar{\psi}^A = \bar{\Xi}_A (1 - \gamma_5) \quad \bar{\psi}_A = \bar{\Xi}_A (1 + \gamma_5) \quad (A.18)$$

$$(\bar{\lambda}_A \lambda_B)^\dagger = \bar{\lambda}^B \lambda^A = \bar{\lambda}^A \lambda^B \quad (\text{A.19})$$

$$(\bar{\lambda}^A \gamma^a \lambda_B)^\dagger = \bar{\lambda}^B \gamma^a \lambda_A = -\bar{\lambda}_A \gamma^a \lambda^B \quad (\text{A.20})$$

$$(\bar{\lambda}^A \gamma^{ab} \lambda^B)^\dagger = -\bar{\lambda}_B \gamma^{ab} \lambda_A = \bar{\lambda}_A \gamma^{ab} \lambda_B \quad (\text{A.21})$$

$$(\bar{\psi}_A \psi_B)^\dagger = -\bar{\psi}^B \psi^A = \psi^A \psi^B \quad (\text{A.22})$$

$$(\bar{\psi}^A \gamma^a \psi_B)^\dagger = -\bar{\psi}^B \gamma^a \psi_A = -\bar{\psi}_A \gamma^a \psi^B \quad (\text{A.23})$$

$$(\bar{\psi}^A \gamma^{ab} \psi^B)^\dagger = \bar{\psi}_B \gamma^{ab} \psi_A = \bar{\psi}_A \gamma^{ab} \psi_B \quad (\text{A.24})$$

APPENDIX B: THE SCALAR POTENTIAL

Let us define

$$T_A^B = 2 S_{AH} S^{HB} - \frac{2}{3} U_A U^B - \frac{1}{6} N_{iA} N^{iB} - \frac{1}{6} M_{iH}^B M^{iH}_A \quad (B.1)$$

where we substitute the values of $\delta_1, \delta_2, \delta_3$. In what follows, the traceless part of a tensor is denoted by a circle.

We want to show that

$$\overset{\circ}{T}_A^B = 0 \quad (B.2)$$

To this effect we introduce the following definitions:

$$X_A^B = C_A^{PQ} C_{PQ}^B ; \quad x_A^B = C_A^{iQ} C_{iQ}^B \quad (B.3a)$$

$$Y_A^B = C_{AP}^B C^P + C_A^{BP} C_P ; \quad y_A^B = C_{Ai}^B C^i + C_A^{Bi} C_i \quad (B.3b)$$

$$Z_A^B = C_A C^B \quad (B.3c)$$

$$W_A^B = C_{AP}^M C_M^{BP} ; \quad w_A^B = C_i^{PB} C_{PA}^i \quad (B.3d)$$

where

$$C_P = C_{PM}^M ; \quad C^P = C_M^{MP} ; \quad C_P = (C^P)^* \quad (B.4a)$$

$$C_i = C_{iH}^H ; \quad C^i = C_M^{Hi} ; \quad C_i = -(C^i)^* \quad (B.4b)$$

By definition the matrices of eqs. (B.3) are Hermitian.

The proof of eq. (B.2) relies on the following three lemmas:

Lemma 1 :

$$\overset{\circ}{Z}_A^B = \overset{\circ}{W}_A^B + \frac{1}{2} \overset{\circ}{X}_A^B - \overset{\circ}{Y}_A^B \quad (B.5)$$

Lemma 2 :

$$C_A^{\Lambda P} C_{\Lambda R}^B = -C_R^{\Lambda P} C_{\Lambda A}^B + C_A^{BP} C_{\Lambda R}^{\Lambda} \quad (B.6)$$

Lemma 3 :

$$2X_A^B - Y_A^B - 2W_A^B = -(2x_A^B - y_A^B + 2w_A^B) \quad (B.7)$$

Proof of lemma 1 : This lemma is a consequence of the structure of the permutation group on 3-indices. From eqs. (3.38) and (3.32) one obtains

$$S_{AB} = -\frac{i}{4} (C_A^{PQ} \epsilon_{BPQ} + \epsilon_{ABM} C^M) \quad (B.8)$$

Multiplication by ϵ^{ABR} shows that S_{AB} is symmetric as it should be. Hence S_{AB} is also given by:

$$S_{AB} = -\frac{i}{8} (C_A^{PQ} \epsilon_{BPQ} + C_B^{PQ} \epsilon_{APQ}) \quad (B.9)$$

Calculating $2S_{AM} S^{MB}$ from either (B.8) or (B.9) one gets

$$2S_{AM} S^{MB} = \frac{1}{8} [2X_A^B - 2Y_A^B - Z_A^B + \delta_A^B Z_M^M] \quad (B.10a)$$

$$2S_{AM} S^{MB} = \frac{1}{32} [6X_A^B - 4W_A^B - 4Y_A^B + 2\delta_A^B X_M^M] \quad (B.10b)$$

and by equating the two results one proves lemma 1.

Corollary of Lemma 1

$$\left(2 S_{AM} S^{MB} - \frac{2}{3} U_A U^B \right)_{\substack{\text{TRACELESS} \\ \text{PART}}} = \frac{1}{12} (2 \dot{X}_A^B - \dot{Y}_A^B - 2 \dot{W}_A^B) \quad (B.11)$$

Proof of Lemma 2 : Lemma 2 is a consequence of the Jacobi identities of the gauge group G . From the definition of the C functions given in eqs. (3.23), we find

$$\begin{aligned} C_A^{\Lambda P} C_{\Lambda R}^B &= (L^{-1})_A^\Sigma (L^{-1})_R^\Omega L_A^P L_R^B \int_\Sigma^{\Lambda\Delta} \int_{\Lambda\Omega R'} J^{\Gamma\Gamma'} = \\ &= -(L^{-1})_A^\Sigma L_A^P (L^{-1})_R^\Omega L_R^B \left[\int_{\Lambda'R\Sigma} \int_\Omega^{\Lambda\Delta} + \int_{\Lambda\Sigma\Omega} \int_{R'}^{\Lambda\Delta} \right] J^{\Gamma\Gamma'} \\ &= -C_R^{\Lambda P} C_{\Lambda A}^B - C_{\Lambda A}^B C_A^{\Lambda P} \quad (B.12) \end{aligned}$$

q.e.d.

Proof of Lemma 3. Take the real part of the trace $P = R$ in eq. (B.6). The proof of eq. (B.2) now follows from eq. (B.11) and lemma 3 using the following identity (cfr (B.3)) :

$$-\frac{1}{6} [N_{iA} N^{iB} + M_{iA}^A M^{iB}] = \frac{1}{12} [2 \dot{X}_A^B - \dot{Y}_A^B - 2 \dot{W}_A^B] + \frac{1}{24} \delta_A^B C^i C_i \quad (B.13)$$

Indeed we find

$$\begin{aligned} \frac{\circ}{T}_A^B &= \left(2 S_{AM} S^{MB} - \frac{2}{3} U_A U^B \right)_{\substack{\text{TRACELESS} \\ \text{PART}}} - \\ &= -\frac{1}{6} (N_{iA} N^{iB} + M_{iA}^B M^{iB})_{\substack{\text{TRACELESS} \\ \text{PART}}} = \quad (B.14) \\ &= \frac{1}{12} (2 \dot{X}_A^B - \dot{Y}_A^B - 2 \dot{W}_A^B + 2 \dot{X}_A^B - \dot{Y}_A^B - 2 \dot{W}_A^B) = 0 \end{aligned}$$

q.e.d.

APPENDIX C: THE $\mathcal{Q}_{AZ}$ MATRIX AND THE EMBEDDING OF $SU(3,N)$ INTO THE SYMPLECTIC GROUP.

First of all we consider the embedding of the pseudo-unitary algebra $U(p,q)$ into the symplectic algebra.

Let J be the pseudo-unitary metric $J = \text{diag}(\underbrace{+, +, \dots, +}_p, \underbrace{-, -, \dots, -}_q)$
In our case $p=3$, $q=m$.

An element of the $SU(p,q)$ Lie algebra is a matrix H of dimension $(p+q) \times (p+q)$ satisfying

$$JH^\dagger = -JH \quad (C1)$$

while an element of the $Usp(p+q, p+q)$ Lie algebra is a matrix $2(p+q) \times 2(p+q)$ with the following block structure:

$$\delta S = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix} \quad (C2)$$

where

$$T = -T^\dagger ; \quad V = V^T \quad (C3)$$

Equation (C3) is the infinitesimal form of eqs. (4.9) of the text. Introducing the projection operators on the p -dimensional and q -dimensional subspaces:

$$P_p = \frac{1+J}{2} ; \quad P_q = \frac{1-J}{2} \quad (C4)$$

and the following decomposition of H

$$H = H_{pp} + H_{qq} + H_{pq} + H_{qp} \quad (C5)$$

where

$$H_{pp} = P_p H P_p ; \quad H_{qq} = P_q H P_q ; \quad H_{pq} = P_p H P_q ; \quad H_{qp} = P_q H P_p \quad (C6)$$

we claim that setting

$$T = H_{pp} + H_{qq}^* \quad ; \quad V = H_{pq} + H_{qp}^* \quad (C7)$$

equations (C3) are satisfied as a consequence of (C1). Explicit evaluation of the commutators shows that equations (C7) define a representation of the pseudo-unitary Lie algebra. Indeed, calculating the commutator $[H^1, H^2] = H^3$ we get

$$\begin{aligned} [H_{pp}^1, H_{pp}^2] + H_{pq}^1 H_{qp}^2 - H_{pq}^2 H_{qp}^1 &= H_{pp}^3 \\ H_{pp}^1 H_{pq}^2 - H_{pp}^2 H_{pq}^1 + H_{pq}^1 H_{qp}^2 - H_{pq}^2 H_{qp}^1 &= H_{pq}^3 \end{aligned} \quad (C8)$$

+ the same equations with $(p \leftrightarrow q)$.

While computing $[\delta S_1, \delta S_2] = \delta S_3$, we obtain

$$\begin{aligned} T_3 &= [T_1, T_2] + V_1^* V_2 - V_2^* V_1 = H_{pp}^3 + (H_{qq}^3)^* \\ V_3 &= V_1 T_2 - V_2 T_1 + T_1^* T_2 - T_2^* T_1 = H_{pq}^3 + (H_{qp}^3)^* \end{aligned} \quad (C9)$$

which proves our statement. Equation (C7) describes therefore the required embedding of the $U(p, q)$ Lie algebra into $U_{sp}(p+q, p+q)$

We consider now the relation between the finite group element

$L = \exp(H)$ parametrizing the coset manifold $SU(p, q)/SU_p \times SU_q \times U_1$ and the finite element $S = \exp(\delta S)$

Let us set $H_{pp} = H_{qq} = 0$ and $H_{pq} = H_{qp}^\dagger = B$, namely

$$H = \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} \quad (C10)$$

Following standard procedures ¹⁹⁾ we obtain

$$L = \exp H = \begin{pmatrix} (1 + XX^\dagger)^{1/2} & X \\ X^\dagger & (1 + X^\dagger X)^{1/2} \end{pmatrix} \quad (C11)$$

where

$$X = B (\sinh \sqrt{B^T B}) / \sqrt{B^T B} \quad (C12)$$

Correspondingly we have

$$T = 0, \quad V = \left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array} \right) \quad (C13)$$

and exponentiating we find

$$S = \exp(\delta S) = \left(\begin{array}{c|c} (1 + \phi_1^* \phi_1)^{1/2} & \phi_1^* \\ \hline \phi_1 & (1 + \phi_1 \phi_1^*)^{1/2} \end{array} \right) \quad (C14)$$

where

$$\phi_1 = \left(\begin{array}{c|c} 0 & X^* \\ \hline X^T & 0 \end{array} \right) \quad (C15)$$

$$\phi_0 = (1 + \phi_1^* \phi_1)^{1/2} = \left(\begin{array}{c|c} (1 + X X^T)^{1/2} & 0 \\ \hline 0 & (1 + X^T X^*)^{1/2} \end{array} \right) \quad (C16)$$

Equations (C15) and (C16) provide the relation between the coset representative $L(X)$ and its symplectic representation $S(X)$.

We can now apply the general result (4.12) and we obtain the following formula for the vector field kinetic metric:

$$a = (\phi_0^T + \phi_1^T)^{-1} (\phi_0^T - \phi_1^T) = [H(-X)]^{-1} H(X) \quad (C17)$$

where

$$H(X) = \left(\begin{array}{c|c} (1+XX^T)^{1/2} & -X \\ \hline -X^T & (1+X^TX^*)^{1/2} \end{array} \right) \quad (C18)$$

In order to write α in explicit form, we should calculate the inverse matrix $[H(-X)]^{-1}$. Let us set

$$[H(X)]^{-1} = \left(\begin{array}{c|c} A(X) & B(X) \\ \hline C(X) & D(X) \end{array} \right) \quad (C19)$$

Imposing the equation $[H(X)]^{-1} H(X) = \mathbb{1}$, one easily verifies that $A(X)$ and $D(X)$ are even functions of X while $B(X)$ and $C(X)$ are odd functions:

$$\begin{aligned} A(-X) &= A(X); \quad D(-X) = D(X); \quad B(-X) = -B(X) \\ C(-X) &= -C(X) \end{aligned} \quad (C20)$$

Utilizing this fact we conclude that

$$\alpha = \left(\begin{array}{c|c} \mathbb{1} + 2BX^T & -2B(1+X^TX^*)^{1/2} \\ \hline -2C(1+XX^T)^{1/2} & \mathbb{1} + 2CX \end{array} \right) \quad (C21)$$

where $B = B(X)$ and $C = C(X)$ are determined through the equations:

$$B \left[(1+X^TX^*)^{1/2} - X^T(1+XX^T)^{-1/2}X \right] = (1+XX^T)^{-1/2}X \quad (C22)$$

$$C \left[(1+XX^T)^{1/2} - X(1+X^TX^*)^{-1/2}X^T \right] = (1+X^TX^*)^{-1/2}X^T \quad (C23)$$

In the text we show that in order for the rheonomic parametrizations of the curvatures to be solutions of the field equations, the following relation must be fulfilled

$$\text{Re } a = \underset{\text{def.}}{u^\dagger u} \equiv K^{-1} \quad (\text{C24})$$

where u^{-1} is the matrix defined by eqs. (4.14) and (4.15). In the parametrization (C11) of the coset, u^{-1} reads

$$u^{-1} = \left(\begin{array}{c|c} (1 + X^* X^T)^{1/2} & X \\ \hline X^T & (1 + X^\dagger X)^{1/2} \end{array} \right) \quad (\text{C25})$$

and we get

$$K = u^{-1}(u^\dagger)^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (\text{C26})$$

where

$$\begin{aligned} \alpha &= \mathbb{1} + X^* X^T + X X^\dagger & \beta &= (1 + X^* X^T)^{1/2} X^* + X (1 + X^\dagger X)^{1/2} \\ \delta &= \mathbb{1} + X^T X^* + X^\dagger X & \gamma &= (1 + X^T X^*)^{1/2} X^T + (1 + X^\dagger X)^{1/2} X^\dagger \end{aligned} \quad (\text{C27})$$

Equation (C24) is fulfilled if $\text{Re } a K = \mathbb{1}$. On the other hand, this is true if the following equations hold:

$$\begin{aligned} (\mathbb{1} + B X^T + B^* X^\dagger) \alpha - [B (1 + X^T X^*)^{1/2} + B^* (1 + X^\dagger X)^{1/2}] \gamma &= \mathbb{1} \\ (\mathbb{1} + B X^T + B^* X^\dagger) \beta - [B (1 + X^T X^*)^{1/2} + B^* (1 + X^\dagger X)^{1/2}] \delta &= 0 \\ -[C (1 + X X^\dagger)^{1/2} + C^* (1 + X^* X^T)^{1/2}] \alpha + (\mathbb{1} + C X + C^* X^*) \beta &= 0 \\ -[C (1 + X X^\dagger)^{1/2} + C^* (1 + X^* X^T)^{1/2}] \beta + (\mathbb{1} + C X + C^* X^*) \gamma &= \mathbb{1} \end{aligned} \quad (\text{C28})$$

The verification of eqs. (C28) is very tedious. We did it by a power series expansion up to cubic order in the general case and to all orders in the special cases $X = X^*$ or $X = -X^*$. Eqs. (C28) can be checked rather easily in the case of a single vector multiplet, namely in the case $n=1$ where X becomes a column vector $X = X_A$

In this simple case we find:

$$B_A = \frac{1}{\mu} X_A \quad ; \quad C_A = \frac{1}{\mu} (\gamma_1 X_A + \gamma_2 X^2 X_A^*) \quad (C29)$$

where

$$|X|^2 = X_A \bar{X}_A \quad X^2 = X_A X_A \quad (C30)$$

$$\mu = 1 + |X|^2 - X^2, \quad \gamma_1 = \sqrt{1 + |X|^2}, \quad \gamma_2 = \frac{1}{|X|^2} (1 - \gamma_1) \quad (C31)$$

so that,

$$a = \left(\begin{array}{c|c} \delta_{AB} + \frac{2}{\mu} X_A X_B & -\frac{2\gamma_1}{\mu} X_A \\ \hline -\frac{2\gamma_1}{\mu} X_B & 1 + 2\frac{X^2}{\mu} \end{array} \right) \quad (C32)$$

while

$$K = \left(\begin{array}{c|c} \delta_{AB} + X_A X_B^* + X_A^* X_B & \gamma_1 (X_A + X_A^*) \\ \hline \gamma_1 (X_B + X_B^*) & \gamma_1 + |X|^2 \end{array} \right) \quad (C33)$$

The condition $\text{Re } aK = \mathbb{1}$, namely eqs. (C28), is easily seen to be true by straightforward matrix multiplication.

As a final remark we note that the matrix (C32) in the projective variables

$$z_A = \frac{X_A}{\sqrt{1 + |X|^2}} \quad (C34)$$

is analytic and has the simpler expression:

$$a_{\Lambda\Sigma} = \begin{pmatrix} \delta_{AB} + 2 \frac{z_A z_B}{1 - z^2} & -2 z_A \\ -2 z_B & \frac{1 + z^2}{1 - z^2} \end{pmatrix} \quad (C35)$$

where $z^2 = z_M z_M$. This is in line with that happens in other supergravity theories (notably N=1 ³⁾, N=2 (minimal coupling) ⁷⁾ and N=5) where the vector kinetic metric is also an analytic function of scalar fields.

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