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In fact, linear independence of the killing vectors on p. 4 is an immediate consequence of Eq. (24); the argument given in the text (Eqs. 25-27) is unnecessary.

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CURRENT CONSERVATION AND PARALLELISM IN NONLINEAR SIGMA MODELS*

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ABSTRACT

We examine the consequences of a strong current conservation condition, which figures in the bosonization of certain free fermion theories, when applied to a general bosonic nonlinear sigma model in two dimensions. The condition is shown to lead directly to the existence of parallelizing Killing vector fields on the internal manifold. The current algebra is found to have the expected Kac Moody form.

Bosonic nonlinear sigma models in two dimensions have some striking properties. Witten has shown that, if the internal manifold is a group, a value for the coefficient of the Wess-Zumino term can be chosen such that the theory becomes equivalent to a theory of free fermions.¹ Curtright, *et al.*,² then pointed out that this value corresponds to a parallelizing torsion on the internal manifold, and explicitly displayed ultraviolet finiteness of the bosonic and supersymmetric models to two loop order. A proof of ultraviolet finiteness to all orders has recently been given.³

An important part of Witten's proof¹ involved showing that the bose and fermi theories have matching current algebras, and in particular that the light cone currents (of both theories) satisfy

$$\partial_+ J_{a-} = \partial_- J_{a+} = 0 \quad (1)$$

(We emphasize that the choice of light cone coordinates is not arbitrary. Eq. (1), in the fermi theory, is the statement that the left and right chiral currents are separately conserved.)

The arguments of Ref. [2], on the other hand, are based entirely upon geometrical properties of the underlying manifold. In particular, it was shown that the introduction of the Wess-Zumino term corresponds to the introduction of a torsion on the internal manifold, and that the value of the coefficient which "linearizes" the theory corresponds exactly to a parallelizing torsion. The work presented here is an attempt to better understand the connection between these two seemingly very different approaches.

The kinds of internal manifold usually considered, in both nonlinear sigma models and Kaluza-Klein theories, are coset spaces⁴ or, more generally, arbitrary Riemannian manifolds. Here we consider the latter, and take as our Lagrangian

$$\mathcal{L} = g_{ab} \partial_\mu \phi^a \partial^\mu \phi^b + b \epsilon_{ab} \epsilon^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b, \quad (2)$$

where b is a constant. Our entire approach is to find what restrictions the conditions (1) put on this theory.

Now, if our Lagrangian is to give some conserved currents, it clearly must have some symmetries. So suppose

$$\delta \phi^j = \epsilon \lambda^\alpha A_\alpha^j(\phi), \quad (3)$$

is such a set of symmetries, labelled by α . Here the $\delta\lambda^\alpha$ are arbitrary constants and the A_α^i are as yet unknown functions of the fields ϕ^a . Inserting (3) into (2) and requiring that the action be invariant gives the two conditions

$$(\mathcal{L}_{A_\alpha} g)_{ab} \equiv A_\alpha^i \partial_j g_{ab} + A_{\alpha,i} g_{jb} + A_{\alpha,j} g_{ai} = 0 \quad (4)$$

$$(\mathcal{L}_{A_\alpha} \epsilon)_{ab} = \partial_a \theta_{\alpha b} - \partial_b \theta_{\alpha a} \quad (5)$$

where $\mathcal{L}_{A_\alpha}$ means the Lie derivative along the vector field A_α^i , and θ is included to allow for the fact that the Lagrangian may change by a total derivative (in fact ϵ_{ab} is only defined up to a curl). So the A_α^i are Killing vectors on the internal manifold.⁵

The currents corresponding to these symmetries are found, via Noether's method, to be

$$J_\alpha^\mu = A_\alpha^i (g_{jk} \partial^\mu \phi^k + b \epsilon_{jk} \epsilon^{\mu\nu} \partial_\nu \phi^k) - b \theta_{\alpha k} \epsilon^{\mu\nu} \partial_\nu \phi^k. \quad (6)$$

Note that Eqs. (5) and (6) are invariant under the "gauge transformation"

$$e_{ij} \rightarrow e_{ij} + \partial_i X_j - \partial_j X_i, \quad (7)$$

provided θ transforms as

$$\theta_{\alpha k} \rightarrow \theta_{\alpha k} + A_\alpha^i \partial_i X_k - A_\alpha^k \partial_k X_i. \quad (8)$$

The equations of motion are

$$\square \phi^b + \Gamma_{ac}^b \partial_\mu \phi^a \partial^\mu \phi^c - S^b_{ac} \epsilon^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^c = 0 \quad (9)$$

with

$$S_{abc} \equiv \frac{3}{2} b \epsilon_{[ab,c]}. \quad (10)$$

It is natural then to identify S as a torsion.² We then write $\mathcal{D}(D)$ for the covariant derivative with (without) torsion, so for example on a vector²

$$\mathcal{D}_a V_b \equiv D_a V_b + S^c_{ab} V_c. \quad (11)$$

Note that, since the exterior derivative commutes with the Lie derivative, Eq. (5) gives

$$(\mathcal{L}_{A_\alpha} S)_{abc} = 0. \quad (12)$$

We now examine the requirement

$$\partial_- J_{\alpha+} = 0 \quad (13)$$

(current conservation, which follows from (4) and (5)), then implies that $\partial_+ J_{\alpha-} = 0$ also. Implementing (13) on the current in (6) gives

$$\mathcal{D}_m (A_{\alpha k} + b (A_\alpha^i \epsilon_{jk} - \theta_{\alpha k})) = 0 \quad (14)$$

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The symmetric part of (14) implies that $A_\alpha^i \epsilon_{jk} - \theta_{\alpha k}$ is a Killing vector, i.e., for some constant matrix $f_{\alpha\beta}$,

$$A_\alpha^i \epsilon_{jk} - \theta_{\alpha k} = f_{\alpha\beta} A_{\beta k}. \quad (15)$$

What conditions can we find on $f_{\alpha\beta}$? Inserting (15) in (5) gives

$$b f_{\alpha\beta} D_k A_{\beta j} = A_\alpha^i S_{jki} \quad (16)$$

Then the antisymmetric part of (14), together with (16), gives

$$(\delta_{\alpha\gamma} - b^2 f_{\alpha\beta} f_{\beta\gamma}) A_\gamma^i S_{kmi} = 0 \quad (17)$$

Therefore in matrix notation

$$b^2 f^2 = \mathbf{I} \quad (18)$$

and

$$F^\pm \equiv \mathbf{I} \pm b f \quad (19)$$

are projection operators. Thus Eq. (14) may be written

$$\mathcal{D}_m F_{\alpha\beta}^+ A_{\beta k} = 0 \quad (20)$$

In fact, choosing a basis of Killing vectors which diagonalizes $b f$, Eq. (16) becomes

$$D_k A_{\alpha j} = \pm A_\alpha^i S_{jik} \quad (21)$$

and the Killing vectors fall into two classes: those which are covariantly constant when the torsion is taken to be $+S_{jik}$ (class 1), and those which are covariantly constant when the torsion is taken to be $-S_{jik}$ (class 2). This means that a submanifold of the internal manifold, of dimension at least one half of the number of Killing vectors, is parallelized.

That parallelism follows from Eq. (21) may be seen as follows. First note that for Killing vectors from a given class,

$$\begin{aligned} \partial_m (A_\alpha^i A_{\beta j}) &= A_\beta^i D_m A_{\alpha j} + A_\alpha^i D_m A_{\beta j} \\ &= \pm (A_\beta^i A_\alpha^j + A_\alpha^i A_\beta^j) S_{mj}{}^i \\ &= 0 \end{aligned} \quad (22)$$

from the antisymmetry of S . Therefore,

$$A_\alpha^i A_{\beta j} = c_{\alpha\beta} \quad (23)$$

where $c_{\alpha\beta}$ is a constant real symmetric matrix. We take Eq. (23) to stand for two separate equations, one for each class. We can then diagonalize both c 's by

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orthogonal transformations which leave Eq. (21) intact. Since the metric g_{ab} is assumed Riemannian, both c -matrices are positive definite, and we can rescale the Killing vectors so that both classes satisfy

$$A_\alpha^\rho A_{\beta\rho} = \delta_{\alpha\beta} \quad (24)$$

separately. It now follows that the Killing vectors of a given class form a linearly independent set at every point. For suppose there were functions $f_\alpha(\phi)$ such that

$$f_\alpha A_\alpha^j = 0 \quad (25)$$

for the Killing vectors from a given class. Differentiating, with a covariant derivative which includes the appropriate torsion, gives

$$(\partial_m f_\alpha) A_\alpha^j = 0 \quad (26)$$

Then from Eq. (24)

$$\partial_m f_\alpha = 0 \quad (27)$$

i.e. the f 's are constant. But the fact that the A_α^j 's are Killing vectors means that there exist no constant f 's for which Eq. (25) holds. So the A_α^j 's, from a given class, form a linearly independent set at each point.

Now the formula

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] A_{\alpha\beta} = \mathcal{R}_{\alpha\beta}^{\gamma\delta} A_{\gamma\delta} = 0 \quad (28)$$

means that the internal manifold contains a submanifold whose generalized curvature $\mathcal{R}$ (curvature including torsion) vanishes. (Note that the torsion term which usually appears on the right-hand side of Eq. (28) vanishes as a result of Eq. (21)). Since the A_α^j 's within a given class (of which let there be n in class 1, m in class 2) are linearly independent at each point, the dimension of the parallelized submanifold is at least $\max(m, n)$. In fact, since the only parallelizable manifolds are semisimple and simple groups and the seven sphere,⁶ and since S^7 has no Wess-Zumino term in two dimensions,² our parallelized submanifold is necessarily a group. But a semisimple group G of dimension $\dim(G)$ admits $2\dim(G)$ Killing vectors,⁴ of which, for a suitable choice of torsion, half may be chosen to be covariantly constant. This means that m is necessarily equal to n , and the isometry group of the metric is necessarily of the form $G \times G$. This however does not imply that one cannot bosonize a chiral fermion. The light cone currents are

$$J_{\alpha\pm} = F_{\alpha\beta}^\pm A_{\beta k} \partial_\pm \phi^k \quad (29)$$

and bosonizing chiral fermions corresponds to taking the fields ϕ^k in (29) to be functions of x^+ (or x^-) only. (Note that as a result of Eq. (15) the currents are independent of ϵ_{ij}).

We now calculate the light cone current commutators directly, and check that they satisfy a Kac Moody algebra.^{1,7,8} From (2) the canonical momentum is

$$\pi_j = g_{jk} \partial_0 \phi^k + b \epsilon_{jk} \partial_1 \phi^k \quad (30)$$

where $\partial_0 \equiv \partial/\partial t$, etc. Since g_{jk} has an inverse, Eq. (30) is solvable for $\partial_0 \phi^k$, and there are no constraints. We define the structure constants of the isometry group, $f_{\alpha\beta\gamma}$, by

$$(\mathcal{L}_{A_\alpha} A_\beta)^\gamma = f_{\alpha\beta\gamma} A_\gamma^\gamma \quad (31)$$

We then find

$$[J_{\alpha+}, J_{\beta-}]_{P.B.} = \frac{1}{\sqrt{2}} F_{\alpha\gamma}^+ F_{\beta\delta}^- \delta(x-x') f_{\gamma\epsilon} A_\epsilon \partial_+ \phi^\epsilon \quad (32)$$

In the basis described above, in which

$$\begin{aligned} F_{\alpha\beta}^+ &= 2\delta_{\alpha\beta} \text{ for } \alpha \leq n, \text{ zero otherwise} \\ F_{\alpha\beta}^- &= 2\delta_{\alpha\beta} \text{ for } \alpha > n, \text{ zero otherwise} \end{aligned} \quad (33)$$

the condition

$$F_{\alpha\gamma}^+ F_{\beta\delta}^- f_{\gamma\epsilon} = 0 \quad (34)$$

is a manifestation of the direct product nature of the isometry group, and in fact follows directly from Eqs. (31) and (21). For the other brackets we find

$$[J_{\alpha\pm}, J_{\beta\pm}]_{P.B.} = -\sqrt{2} f_{\alpha\beta\gamma} J_{\gamma\pm} \delta(x-x') \pm 4\delta_{\alpha\beta} \partial_1 \delta(x-x') \quad (35)$$

which can be cast into standard form^{1,8} by a suitable rescaling of the currents and structure constants.

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