CERN-TH.4262/85

STRINGS AT FINITE TEMPERATURE

E. Alvarez^{*)}

CERN -- Geneva

ABSTRACT

A lowest order computation of the partition function of strings in thermal equilibrium is done, by first computing the world sheet σ -model at finite temperature (with topology of a torus) and then using the light cone gauge to obtain string quantities. Some cosmological applications are studied, and particular attention is paid to the phase transition signaled by the existence of a critical temperature.

^{*)}On leave from U. Autonoma, 28049 Madrid, Spain.

CERN-TH.4262/85
September 1985

1. - INTRODUCTION

One of the most important unsolved problems of contemporary physics is the nature of the initial state of the Universe itself. Classical general relativity predicts the existence of a singularity at the big bang, which is, of course, unacceptable from the physical point of view, leading in fact to a breakdown in the ability to predict physical observables. It is reasonable to think that quantum gravity effects could modify this picture, but the well-known difficulties of this theory have prevented up to now real progress in this direction.

The preceding situation has dramatically changed with the appearance of "superstring theories"¹⁾ which are candidates for higher dimensional unified theories of the four known fundamental interactions. Quantum gravity effects are computable, and the issue of the initial state can, in principle, be resolved. This does not mean that there are no difficulties: strings are formulated in a flat background in such a way that the classical gravitational field appears as a "string condensate" only; it would be much more satisfactory to formulate the theory of strings in a self-consistent dynamical space-time ab initio. A related difficulty is that most likely, perturbation theory is not enough (this happens already in classical general relativity) and a clever calculational scheme must be devised to progress further.

In spite of all these problems, we believe that the edification of a consistent "string cosmology" is one of the main topics to be studied in the near future in this particular field.

The aim of this paper is to study a thermal state of strings. Although it is perhaps not evident that the initial state will be dominated by this simplest density matrix, it is the most reasonable starting point, and is at any rate necessary in order to examine non-equilibrium regimes.

The calculation shall be done first for the Veneziano model in lowest order approximation, working in the light cone gauge, and clarifying in passing some issues of thermodynamics in non-timelike frames (such as the light cone itself). The rôle of the constraints, in particular, shall be examined in detail. In the second half of the paper an "analogue model" (i.e., a gas of particles with the same mass spectrum as the one of the free string) will be studied, and its physical consequences examined, stressing in particular some promising theories, such as the heterotic string²⁾. The paper ends with some preliminary remarks on cosmology³⁾.

Let us elaborate a bit further on the "analogue model". The spectrum of states of a free string, be it supersymmetric or not, is always of the general dual form, first introduced by Hagedorn:

$$\omega(m) = \omega_0 (m\sqrt{\alpha'})^{-a} e^{bm\sqrt{\alpha'}} \quad (1)$$

where $\omega(m)dm$ is the density of states, a and b are positive constants which are model dependent (cf. Table 1) and $m\sqrt{\alpha'}$ is a dimensionless variable, α' being the Regge slope parameter.

It has been known for some time that a physical gas with mass spectrum given by (1) cannot be at thermodynamical equilibrium at temperatures greater than a critical one, T_c [cf. work in Ref. 3) and references therein]. It is instructive to rederive this result in a "light cone" frame of reference, where the time variable has been identified with $x^+ \equiv (x^0 + x^{d-1})/\sqrt{2}$. The corresponding Hamiltonian for a free particle of mass m is the canonically conjugated variable, namely

$$p^- \equiv \frac{p_T^2 + m^2}{2p^+} \quad (2)$$

where $p_T^2 = \sum_{i=1}^{d-1} p_i^2$ is the transverse momentum squared and $p^\pm \equiv (p^0 \pm p^{d-1})/\sqrt{2}$ are positive semi-definite quantities on shell. The first thing worth noticing is that thermodynamics is not Lorentz invariant, unless we accept to transform the temperature in such a way that βH is a scalar. The partition function in this frame, i.e., $\text{tr } e^{-\beta p^-}$ does not converge, unless $W = \infty$, because there are directions with infinite momentum but finite p^- -energy. The finite quantity of interest is

$$Z(\beta) = \text{tr } e^{-\beta p^0} \quad (3)$$

which using the fact that

$$E \equiv p^0 = \frac{1}{\sqrt{2}} (p^+ + p^-) = \frac{1}{\sqrt{2}} \left(p^+ + \frac{p_T^2 + m^2}{2p^+} \right)$$

can be expressed as

$$Z(\beta) = \text{tr}_{p^+} e^{-\frac{\beta}{\sqrt{2}} p^+} \mathcal{Z}\left(\frac{\beta}{2\sqrt{2}p^+}\right) \quad (4)$$

where a transverse partition function has been defined

$$\mathcal{Z}_m(\lambda) \equiv \text{tr}_{p_T} e^{-\lambda(p_T^2 + m^2)} \quad (5)$$

[It is easy to check that from (4) and (5) the correct partition function for a Jüttner-Synge gas is obtained.] When a mass spectrum of the form $\omega(m)$ is considered, the trace in (4) diverges for $\beta \rightarrow 0$, because the transverse partition function $\zeta(\lambda)$ exponentially diverges when $\lambda \rightarrow 0$

$$\zeta(\lambda) = \int dm \omega(m) \mathcal{Z}_m(\lambda) \underset{\lambda \rightarrow 0}{\sim} e^{c/\lambda}$$

due to the fact that the exponential in the trace cannot then overcome the exponential in the mass spectrum. This is the origin of the critical temperature derived in a somewhat unconventional manner, which should nevertheless prove itself useful when studying the string. Indeed, in this case, unlike the analogue model, the light cone is not a matter of convenience, but of necessity - at least for superstrings.

Our starting point in the string computation will be formula (4); the first thing to be done is to study the world sheet σ -model at finite temperature, which is exactly analogous to $\zeta(\lambda)$ in (5) and then use that formula to compute string quantities. Physical questions of interest are the modifications of the critical temperature due to the presence of the interactions and the possibility of having a new phase for $T > T_c$ [cf. Eq. (4)] in much the same way as the deconfining temperature indicates a transition between hadron matter and quark matter in quantum chromodynamics.

2. - STRINGS AT FINITE TEMPERATURE

2.1 Introduction

In this section a one-loop string computation of the relevant partition function will be made. This means that the world sheet of the string has the topology of a torus, i.e., Euler characteristic $\chi = 0$. Corrections to the

present results would imply more complicated topologies, with smaller χ , which means higher genus and/or presence of boundaries.

The Veneziano string model, and in particular its closed string sector, will be worked out in detail, although to the order considered generalizations to physically more appealing models are not difficult. We refrain from doing this, though, because the relevant physics is studied anyway in the third section by a simpler method.

Having in mind supersymmetric extensions, a light cone formalism has been developed, although for the particular model considered, more powerful covariant methods are known [cf. Ref. 5)]. In this gauge the string action is simply

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \partial_a x^i \partial_a x^i \quad (6)$$

where $\xi^a \equiv (\sigma, \tau)$ are co-ordinates on the world sheet, $\partial_a = \partial/\partial\xi^a$, and $x^i = \chi^i(\sigma, \tau)$ $i = 1, \dots, d-2$ are the transverse co-ordinates of the string. The naive Hamiltonian corresponding to (6) is

$$h = \frac{1}{2} \int d\sigma \left\{ \frac{\pi_i^2}{T} + T (\partial_\sigma x^i)^2 \right\}$$

where $\pi_i \equiv T \partial_\tau x^i$, and the string tension, $T \equiv 1/2\pi\alpha'$. The relation between the string Hamiltonian, p^- , and the Hamiltonian of the σ -model, h , is analogous to the one in (2):

$$p^- = \frac{h}{p^+} \quad (7)$$

A similar procedure to the one used there is then in order and, in particular, the starting point is the computation of the σ -model partition function

$$Z(\lambda) \equiv \text{tr} e^{-\lambda h} \quad (8)$$

A delicate point is the presence of constraints: as is by now well known, in getting the light cone gauge the x^- variables are dependent ones, to be obtained from the equations

$$\begin{aligned} p^+ \partial_\sigma x^- &= \partial_z x^i \partial_\sigma x^i \\ p^+ \partial_z x^- &= \partial_z x^i \partial_z x^i - \partial_\sigma x^i \partial_\sigma x^i \end{aligned} \quad (9)$$

(The minus sign in the latter equation appears when an analytic continuation to Euclidean signature is made on the world sheet only; note that Lorentzian signature must always be kept for the usual space-time.)

The compatibility condition of the constraints (9) is precisely the equation of motion

$$\partial_a \partial^a x^i = 0 \quad (10)$$

and the general solution of (9) is

$$x^- = \hat{x}^- + \frac{1}{p^+} \int_0^\sigma d\sigma' \partial_z x^i(\tau, \sigma') \partial_\sigma x^i(\tau, \sigma') \quad (11)$$

where the only arbitrariness allowed is a constant (and not a function of τ).

When considering closed strings, so that

$$x^i(\sigma, z) = x^i(\sigma + v, z)$$

the integral over σ of the first constraint in (9) gives

$$\int_0^v d\sigma \partial_z x^i \partial_\sigma x^i = 0 \quad (12)$$

(whose physical meaning is the equality between the occupation numbers for the two sets of oscillators $N = \tilde{N}$). And when working at finite temperature, so that

$$x^i(\sigma, z + \lambda) = x^i(\sigma, z)$$

the τ -integral of the second constraint in (9) gives

$$\int_0^\lambda d\tau (\partial_z x^i \partial_z x^i - \partial_\sigma x^i \partial_\sigma x^i) = 0 \quad (13)$$

(This second constraint is the one that, when integrated over σ gives the mass shell condition, and subsequently, the mass spectrum.)

In order to simplify the exposition, the σ -model partition function will first be computed without constraints; and then, in the following paragraph, the constraints shall be introduced in an effective potential approach.

2.2 The world sheet σ -model at finite temperature

The whole system shall be considered in a finite space-time box of length L , so that $0 < x^i < L$; and periodic boundary conditions assumed on the walls. Given the fact that the Hamiltonian of our σ -model is quadratic, standard arguments relate the partition function to a particular functional integral:

$$Z(\lambda) = \int \mathcal{D}x^i \exp - S[x^i] \quad (14)$$

One way to define the functional integral is by reference to a normal mode decomposition. The most general off-shell expansion compatible with double periodicity (both in σ and in τ) is:

$$x^i = \hat{x}^i + \sum' x_{m,n}^i e^{i \frac{2\pi m_2 \tau}{\lambda}} e^{i \frac{2\pi n_1 \sigma}{\nu}} + \frac{N_1^i L \sigma}{\nu} + \frac{N_2^i L \tau}{\lambda} \quad (15)$$

where the sum runs over all the integers but $(0,0)$ which has been explicitly indicated; and the two further sets of integers, N_1^i and N_2^i allow for the possibility that, because the whole system is enclosed in a cubic box of length L , the physically relevant boundary condition is

$$\begin{aligned} x^i(\sigma + \nu, \tau) &= x^i(\sigma, \tau) + N_1^i L \\ x^i(\sigma, \tau + \lambda) &= x^i(\sigma, \tau) + N_2^i L \end{aligned} \quad (16)$$

or, in mathematical language, they arise because there is a non-trivial homotopy in the problem $\Pi_1(S^1) = \mathbb{Z}$. (Note however, that the derivatives $\partial_a x^i$ are ordinary periodic functions.) Their physical meaning stems from (16) in a transparent way: N_1^i is the number of times x^i winds around itself when σ runs once around

its own circle; and N_2^1 is the same thing with respect to τ . Using this decomposition, the string centre-of-mass position, defined as

$$X_{CM}^i \equiv \frac{1}{v} \int_0^v x^i(\sigma', \tau) d\sigma'$$

is written as:

$$X_{CM}^i = \hat{x}^i + \sum' x_{0m_2}^i e^{i \frac{2\pi m_2 \tau}{\lambda}} + \frac{1}{2} N_1^i L + \frac{N_2^i L \tau}{\lambda}$$

When the equations of motion $\partial^2 x^i = 0$ are used, this is found to be a linear function of τ

$$X_{CM}^i = X_{CM}^i(0) + \dot{P}^i \tau$$

but off-shell this is not the case, and the general expression must be used.

The fact that the string co-ordinates are assumed to be real, $x^i = x^{i*}$, implies a constraint in the Fourier modes

$$X_{m_1, m_2}^i = X_{-m_1, -m_2}^i$$

The path integral is now expressed in terms of an infinite product of ordinary integrals:

$$\begin{aligned} \mathcal{Z}(\lambda) = \sum_{N_1^i, N_2^i} \int d\hat{x}^i \prod_{i, m_1, m_2} \mu dx_{m_1, m_2}^i \cdot \exp \left\{ -\frac{v}{4\pi\alpha'} \frac{N_2^{i2} L^2}{\lambda} \right. \\ \left. - \frac{N_1^{i2} L^2 \lambda}{4\pi\alpha' v} - \sum' (x_{m_1, m_2}^i)^2 \frac{\pi}{\alpha'} \left(m_1^2 \frac{\lambda}{v} + m_2^2 \frac{v}{\lambda} \right) \right\} \end{aligned} \quad (17)$$

where μ is an arbitrary mass scale introduced in order to make the integration variable dimensionless. The integrals over the string zero modes, $\hat{x}^i$, give a factor proportional to the transverse volume, L^{d_T} . On the other hand, in the infinite volume (thermodynamical) limit, only the zero modes $N_1^i = N_2^i = 0$ contribute, and the partition function reads,

$$\mathfrak{z}(\lambda) = L^{d_T} \prod_{i, m_1, m_2} \mu \sqrt{\frac{d_i}{2}} \left(m_1^2 \frac{\lambda}{v} + m_2^2 \frac{v}{\lambda} \right)^{-1/2} \quad (18)$$

This infinite product can be regulated by using, for example, Epstein's Z_2 -function (see Appendix 1) with the result

$$\mathfrak{z}(\lambda) = L^{d_T} \left(\mu \sqrt{\frac{d_i}{2}} \right)^{d_T} e^{d_T Z_2'(0)} \quad (19)$$

In the Appendix, an explicit representation of the Z_2 function in terms of Jacobi's θ -functions can be found. Introducing the asymptotic expressions for these, the behaviour of the partition functions is proved to be

$$\mathfrak{z}(\lambda) \underset{\frac{\lambda}{v} \rightarrow 0}{\sim} L^{d_T} \left(\mu \sqrt{\frac{d_i}{2}} \right)^{d_T} \left(\frac{\lambda}{v} \right)^{d_T} e^{\frac{\pi d_T v}{3 \lambda}} \quad (20)$$

which coincides with the one obtained for the analogue model in the Introduction.

As the Z -function is symmetric on the exchange of λ/v by v/λ , it is plain that $\zeta(\lambda)$ should also diverge in exactly the same way in the opposite limit ($v/\lambda \rightarrow 0$). This latter divergence is nothing but a manifestation of the well-known infra-red problems of a two-dimensional massless scalar field theory [cf. for example Ref. 6)]. We shall defer any further physical discussion until the system of constraints has been introduced, which is done next.

2.3 The constraints

If the normal mode decomposition (15) is plugged into (12) and (13), and a further Fourier decomposition is made, the constraints are seen to be equivalent to

$$\begin{aligned} \delta_{m0} \left(\frac{N_2^{i2} L^2}{\lambda} - \frac{N_1^{i2} L^2 \lambda}{v^2} \right) - 4\pi i \frac{N_1^{i2} L \lambda}{v^2} m x_{m0}^i + \\ + \frac{4\pi^2}{v} \sum'_{m_1, m_2} \left[\frac{m_2^2 v}{\lambda} + m_1(m-m_1) \frac{\lambda}{v} \right] x_{m_1 m_2}^i x_{m-m_1, m_2}^i = 0 \end{aligned} \quad (21)$$

and

$$\frac{N_1^i N_2^i L^2}{\lambda} \delta_{m0} + i \frac{2\pi m N_1^i L}{\lambda} X_{0m}^i + \sum_{m_1, m_2} \frac{4\pi^2 m_1 m_2}{\lambda} X_{m_1 m_2}^i X_{-m_1, m-m_2}^i = 0 \quad (22)$$

which must be imposed for each integer value of m . (The resulting functions are real in spite of the fact that some "i" appear in those expressions.) It is evident that it is exceedingly complicated to introduce the constraints in this form in the functional integral (17). An alternative method is to define some auxiliary fields in (14) and compute

$$\begin{aligned} \mathcal{Z}(\lambda) = \int \mathcal{D}x \mathcal{D}T \mathcal{D}\xi \cdot \exp \left\{ -\frac{1}{4\pi\alpha'} \int d^2\xi \cdot \right. \\ \left. \partial_a x^i \partial_a x^i + T (\sqrt{2\alpha'} p^+ \partial_\sigma x^- - \partial_z x^i \partial_\sigma x^i) \right. \\ \left. + \xi (\sqrt{2\alpha'} p^+ \partial_z x^- - \partial_z x^i \partial_z x^i + \partial_\sigma x^i \partial_\sigma x^i) \right\} \end{aligned} \quad (23)$$

The fields S and T do not have any dynamical role; their functional integral can be explicitly done, giving rise to the constraints.

In order to perform an integration by parts, an explicit expression for x^- is needed:

$$\begin{aligned} x^-(\sigma, z) = \hat{x}^- + \frac{N_1^i N_2^i L^2}{\sqrt{2\alpha'} p^+ \lambda v} \sigma + \frac{N_2^i L}{\sqrt{2\alpha'} p^+ \lambda} (x^i(\sigma, z) - \\ x^i(0, z)) + \sum_{\substack{m_1, m_2 \\ m}} \frac{2\pi i m_2 (m - m_1)}{\sqrt{2\alpha'} p^+ \lambda m} \cdot X_{m_1 m_2}^i \cdot X_{m-m_1, m-m_2}^i \cdot \\ \cdot e^{\frac{i 2\pi m z}{\lambda}} (e^{\frac{i 2\pi m \sigma}{v}} - 1) \end{aligned}$$

where the important thing is that

$$X^-(\sigma+\nu, z) = X^-(\sigma, z) + \sum_i \frac{N_1^i N_2^i L^2}{\sqrt{2\alpha'} p^+ \lambda} \quad (24)$$

The partition function can then be expressed as:

$$\begin{aligned} Z(\lambda) = & \int dX dT d\xi \exp \left[-\frac{1}{4\pi\alpha'} \int d^2\xi \cdot \left\{ \right. \right. \\ & -\sqrt{2\alpha'} p^+ X^-(\partial_\sigma T + \partial_z \xi) + (1-\xi) \partial_z x^i \partial_z x^i + \\ & \left. \left. + (1+\xi) \partial_\sigma x^i \partial_\sigma x^i - T \partial_z x^i \partial_\sigma x^i \right\} + \right. \\ & \left. + \sum_i \frac{N_1^i N_2^i L^2}{\lambda} \int_0^\lambda dz T \right] \end{aligned} \quad (25)$$

Although this integral is still too complicated to be explicitly done, a natural thing to do is to expand on derivatives of the auxiliary fields. The first term in this expansion (the analogue of the effective potential) is

$$\begin{aligned} Z(\lambda) = & \int d\xi dT \int dX \exp \left[-\frac{1}{4\pi\alpha'} \int d^2\xi \left\{ (1-\xi) \partial_z x^i \right. \right. \\ & \left. \left. \partial_z x^i + (1+\xi) \partial_\sigma x^i \partial_\sigma x^i - T \partial_z x^i \partial_\sigma x^i \right\} + \right. \\ & \left. + \sum_i \frac{N_1^i N_2^i L^2}{\lambda} \int_0^\lambda dz T \right] \end{aligned}$$

which, by decomposing in normal modes, is easily seen to be equivalent to

$$\begin{aligned} Z(\lambda) = & \sum_{N_1^i N_2^i} \int d\xi dT L^{d_T} \prod_{i, m_1, m_2} dx_{m_1, m_2}^i \exp \\ & - \frac{(1-\xi)}{4\pi\alpha'} \frac{N_2^i L^2}{\lambda} - \frac{(1+\xi)}{4\pi\alpha'} \frac{N_1^i L^2}{\lambda} - \sum \frac{x_{m_1, m_2}^i{}^2}{2} \cdot \frac{2\pi}{\alpha'} \\ & \cdot \left[m_1^2 (1+\xi)^2 / \lambda + m_1 m_2 T + m_2^2 (1-\xi)^2 / \lambda \right] \end{aligned}$$

As long as $-1 < \xi < 1$, the discrete sums are dominated by their zero mode term in the infinite volume limit. On the other hand, the integral over Dx is still a Gaussian one, which can easily be performed, leading to:

$$Z(\lambda) = L^{d_T} \int d\xi dT (\mu \sqrt{\frac{\alpha'}{2}})^{d_T} \exp \frac{1}{2} Z_2'(\|a\|, 0) \quad (26)$$

where the result is again expressed in terms of a non-diagonal Epstein's function (cf. Appendix A)

$$Z_2(\|a\|, s) \equiv \sum \left(\frac{\lambda}{\nu} (1+\xi) n_1^2 + T n_1 n_2 + \frac{\nu}{\lambda} n_2^2 (1-\xi) \right)^{-s}$$

corresponding to a matrix

$$\|a\| \equiv \begin{pmatrix} \frac{\lambda}{\nu} (1+\xi) & T/2 \\ T/2 & \frac{\nu}{\lambda} (1-\xi) \end{pmatrix}$$

with determinant $\Delta \equiv 1 - \xi^2 - T^2/4$ again independent of λ .

2.4 Strings at finite temperature

The partition function for the system of strings is to be obtained from the corresponding one for the world sheet σ -model by means of (4), which in the infinite volume limit reads

$$Z(\beta) = \frac{L}{2\pi} \int_0^\infty \frac{dp^+}{\mu} e^{-\beta p^+} \mathcal{Z}\left(\frac{\beta}{2\sqrt{\alpha'}} p^+\right) \quad (27)$$

or, by substituting (26)

$$Z(\beta) = \frac{L}{2\pi} \int_0^\infty \frac{dp^+}{\mu} (\mu \sqrt{\frac{\alpha'}{2}})^{d-1} e^{\frac{1}{2} Z_2'(\| \bar{a} \|, 0) - \beta p^+} \quad (28)$$

where the matrix $|\bar{a}|$ is defined as:

$$\| \bar{a} \| \equiv \begin{pmatrix} \frac{\beta}{2\sqrt{\alpha'} p^+ \nu} (1+\bar{\xi}) & \bar{T}/2 \\ \bar{T}/2 & \frac{2\sqrt{\alpha'} p^+ \nu}{\beta} (1-\bar{\xi}) \end{pmatrix} \quad (29)$$

and the ordinary integral over ξ and T has been made by means of a saddle point approximation, $(\bar{\xi}, \bar{T})$ being the values at the minimum.

The physical partition function (28) diverges because of the lower limit ($p^+ \rightarrow 0$); this has nothing to do with the temperature, and corresponds to the

infra-red effect explained earlier, at the end of the second paragraph of this section. The behaviour in the other limit ($p^+ \rightarrow \infty$) depends on the detailed properties of Epstein's functions, and is, as in the analogue model, the origin of the existence of a critical temperature (namely β must be greater than β_c in order for the integral (28) to converge). To be specific

$$Z(\beta) = \frac{L^{d-1}}{2\pi} \left(\mu \sqrt{\frac{d'}{2}} \right)^{d-2} \int_0^\infty \frac{d\rho^+}{\mu} e^{-\beta \rho^+} \frac{\sqrt{\beta(1+\xi)}}{2\pi} \cdot$$

$$\cdot (2\sqrt{d'} \rho^+ \nu)^{-1/2} \cdot \eta \left[\frac{2\sqrt{d'} \rho^+ \nu}{\beta(1+\xi)} \left(-\frac{\bar{T}}{2} + i\sqrt{1-\bar{\xi}^2 - \bar{T}^2/4} \right) \right]^{-1} \cdot$$

$$\cdot \eta \left[\frac{2\sqrt{d'} \rho^+ \nu}{\beta(1+\xi)} \left(\frac{\bar{T}}{2} + i\sqrt{1-\bar{\xi}^2 - \bar{T}^2/4} \right) \right]^{-1}$$

When the minimum of the effective potential sits in the point $\bar{\xi} = \bar{T} = 0$, the integral behaves when $p^+ \rightarrow \infty$ as:

$$e^{-\beta \rho^+} e^{\frac{\pi 2\sqrt{d'} d_T}{3\beta} \rho^+}$$

leading to a maximum temperature in the absence of constraints of

$$T_c = \frac{1}{\pi \sqrt{d'}} \sqrt{\frac{3}{2d_T}} = \frac{1}{4\pi \hat{v} \sqrt{d'}}$$

where $d_T = 24$ and $v = \hat{v}^2 \pi \sqrt{d'}$, $\hat{v}$ being a dimensionless number.

3. - THE ANALOGUE MODEL

In this section the analogue model [cf. Ref. 3)], consisting of a gas of particles at finite temperature with a mass spectrum $\omega(m)dm$ of the form (1), will be analyzed, first in the canonical and then in the microcanonical ensembles.

3.1 The partition function

Assuming that there is an equal number of fermionic and bosonic degrees of freedom (which is the only remnant of supersymmetry at finite temperature), the canonical partition function can be written as:

$$\log Z_d = \frac{V_{d-1}}{(2\pi)^{d-1}} \int_{\eta_0}^{\infty} W(m) dm \int d^{d-1}k \log \frac{1 + e^{-\beta \sqrt{k^2 + m^2}}}{1 - e^{-\beta \sqrt{k^2 + m^2}}} \quad (30)$$

Here η_0 is an infra-red cut-off, defined in such a way that the asymptotic form of the spectrum (1) is valid; the figure "d" is the number of non-compact dimensions of space-time [if there is in addition a further set of dimensions compactified into a manifold of radius LR_2 , there are corrections to (30) of $O(R_2 L \eta_0)$]. V_{d-1} is a (d-1) finite volume, in which equilibrium conditions have been attained.

Expanding the logarithm in its power series, and performing some elementary integrations, (30) reduces to

$$\log Z_d = K \cdot \beta^{1-d/2} I_{d,a}(\beta) \quad (31)$$

where the constant K is defined by

$$K \equiv 2^{2-d/2} \pi^{-d/2} V_{d-1} d^{1-\frac{1-a}{2}} W_0$$

and the function $I_{d,a}(\beta)$

$$I_{d,a}(\beta) \equiv \int_{\eta_0}^{\infty} m^{-a+d/2} dm e^{b m \sqrt{a'}} \sum_n \frac{K_{d/2}((2n+1)\beta m)}{(2n+1)^{d/2}} \quad (32)$$

$K_n(z)$ being the modified Bessel functions of the third kind. Indeed, by using the asymptotic behaviour of those when $|z| \rightarrow \infty$

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}$$

it is easily proved that $I_{d,a}$ certainly diverges if $\beta < \beta_c \equiv b\sqrt{a'}$. The physical properties of the system when the critical temperature is approached from below, $\beta \rightarrow \beta_c^+$, depend on the specific values of the parameters a and b, as will be seen in the sequel.

From the knowledge of the partition function all thermodynamical quantities can be easily obtained by performing some derivatives; for instance, the mean value of the energy is

$$\langle E \rangle = - \frac{\partial}{\partial \beta} \log Z_d \quad (33)$$

and the specific heat at constant volume

$$C_V = \beta^2 \frac{\partial^2}{\partial \beta^2} \log Z_d \quad (34)$$

while the value of the energy fluctuations is

$$\sigma(E) \equiv \frac{\langle E^2 \rangle - \langle E \rangle^2}{\langle E \rangle^2} = \frac{\frac{\partial^2}{\partial \beta^2} \log Z_d}{\left(\frac{\partial}{\partial \beta} \log Z_d \right)^2} \quad (35)$$

Using standard expressions for derivatives of Bessel functions, a formula for derivatives of the partition function is obtained

$$\frac{\partial}{\partial \beta} \log Z_d = K \left[(1-d) \beta^{-d/2} I_{d,a} - \beta^{1-d/2} I_{d-2,a-2} \right] \quad (36)$$

$$\begin{aligned} \frac{\partial^2}{\partial \beta^2} \log Z_d = K \left[d(d-1) \beta^{-d/2-1} I_{d,a} + (2d-3) \beta^{-d/2} I_{d-2,a-2} \right. \\ \left. + \beta^{1-d/2} I_{d-4,a-4} \right] \end{aligned} \quad (37)$$

In Appendix B, it is shown how to compute the integrals $I_{d,a}$ in certain limits. Using those formulas, it is straightforward to check that the thermodynamical quantities converge to a finite value when $\beta \rightarrow \beta_c^+$, only when the number of non-compact dimensions of space-time is smaller than or equal to a maximum value given in Table 2.

It is perhaps worth stressing that without any dimension compactified (i.e., in their critical dimensions) HS and GSII have all the relevant thermodynamical quantities finite when $\beta \rightarrow \beta_c^+$; this is not the case though for both VDM and GSI,

where even the partition function diverges in this limit. In the latter case, the specific heat diverges even in four non-compact dimensions, which implies that if the full compactification of space-time did take place in such a string era, and the temperature was close to the critical one, fluctuations in energy grew arbitrarily large.

When the convergence condition for c_V is fulfilled, i.e., when $a > d+5/2$, a closed formula for the dispersion in energy is obtained:

$$\sigma(E) = \frac{1}{K} \sqrt{\frac{2}{\pi}} \beta_c^{\frac{d-1}{2}} \eta_0^{\xi+2} \frac{(\xi+1)^2}{\xi}$$

where $\xi \equiv a - (d+5)/2 > 0$. This can be rewritten in the form

$$\sigma(E) = \frac{1+\xi}{\xi} \frac{\eta_0}{\langle E \rangle} \quad (38)$$

which implies that $\sigma(E)$ is always small at very high mean energies.

On the other hand, as has been already remarked, when $\xi < 0$, fluctuations grow without limit when $\beta \rightarrow \beta_c^+$, and in fact the system gets out of equilibrium before reaching the limiting situation.

The most important bulk properties of this "string gas" and, in particular, its coupling to an external gravitational field are determined from the knowledge of the equation of state, which at equilibrium is equivalent to the energy momentum tensor. In the physical situation we are considering ($T \rightarrow T_c^-$), this equation is always extremely "soft" with almost zero pressure; this is the kind of matter called "dust" by cosmologists. Actually, the relation between pressure and energy density is:

$$p = \hat{\xi} \rho \quad (39)$$

with the constant $\hat{\xi}$ given by

$$\begin{aligned} \hat{\xi} &= \frac{2a-d-3}{2a-d-1} (\beta_c \eta_0)^{-1} & (2a > d+1) \\ \hat{\xi} &= d^{-1} & (2a = d+1) \\ \hat{\xi} &= \frac{2\varepsilon}{d+1-a} & (2a < d+1) \end{aligned}$$

where $\varepsilon \equiv \beta/\beta_c - 1$. Some physical consequences of (39) shall be explored in the

next section of this paper.

3.2 The microcanonical ensemble

A natural question to ask is what happens "on the other side" of the critical temperature. When the free energy remains finite in the limit $\beta \rightarrow \beta_c^+$, the situation is entirely analogous to hadron matter for which a critical temperature also exists which indicates the existence of a phase transition between a "confined phase" (hadron matter) and a "deconfined phase" (quark matter). [This was first suggested in Ref. 4).] On the nature of the new phase (the analogous of free quarks and gluons in QCD) which could exist when $T > T_c$, nothing is known at present. If the usual line of reasoning that symmetries are restored at high temperature is followed, it is likely that this phase will possess in a manifest way some symmetries broken at lower energies, and so the knowledge of this phase could be of fundamental importance.

On the other hand, when the free energy diverges when $\beta \rightarrow \beta_c^+$, the critical temperature does not indicate a phase transition, but a true maximum temperature, and it is simply not possible to have matter in the infinite volume (thermodynamic) limit, at equilibrium with $T > T_c$. This does not mean, of course, that it is not possible to have matter out of equilibrium.

The microcanonical ensemble, in particular, which is physically appropriate for a closed system can be a better representation of the matter content of the Universe, at least in the case in which all dimensions are compact. This ensemble can be formally obtained [cf. Ref. 7)] by means of a Laplace transform of the analytically continued partition function:

$$\begin{aligned} \Omega(E) &= \text{tr}^N \delta(E-H) = \text{tr}^N \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{\epsilon_0 d\beta}{2\pi i} e^{\beta(E-H)} \\ &= \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \epsilon_0 \frac{d\beta}{2\pi i} e^{\beta E} Z(\beta) \end{aligned} \quad (40)$$

where ϵ_0 is an arbitrary parameter with dimensions of energy.

The results of the preceding paragraph show that the canonical partition function diverges for $\text{Re } \beta < \beta_c$, so that the integral (40) must be done on a

contour $\text{Re } \beta = \beta_0 > \beta_c$. It turns out to be convenient to impose $|\beta - \beta_c| \gg 1$, so that the asymptotic form of the incomplete gamma function may be used to get an expression for the microcanonical density of states in terms of a single quadrature:

$$\Omega(E) = \frac{\epsilon_0}{2\pi i} \int d\beta \exp \left\{ E\beta + \sqrt{\frac{\pi}{2}} K \eta_0^{\frac{d+1}{2}-a} \beta^{-\frac{d+1}{2}} e^{-\eta_0/\beta} \right\}$$

which, approaching the integral by saddle point, leads to:

$$\Omega(E) \sim E^{-1/2} \left(\frac{E_c}{E} \right)^{E/\eta_0} \quad (41)$$

where the quantity $E_c \equiv \alpha' \frac{1-a}{2} \eta_0^{d+1-a} v_{d-1}$.

If the corresponding entropy $S(E) \sim \log \Omega(E)$ is used to formally define the temperature, one gets

$$\frac{1}{T} \equiv \frac{\partial S}{\partial E} = \frac{1}{\eta_0} \log \frac{E_c}{E} - \frac{1}{\eta_0} - \frac{1}{2E} \quad (42)$$

which is positive and well-behaved as long as $\eta_0/2 < E < E_\infty$, where E_∞ is the value at which $T = \infty$, namely

$$\frac{1}{\eta_0} \log \frac{E_c}{E_\infty} = \frac{1}{\eta_0} + \frac{1}{2E_\infty}$$

and, in particular, $E_\infty \sim E_c/e$ if $E_c \gg \eta_0$. The specific heat can also be defined as

$$C_V = - \frac{\left(\frac{\partial S}{\partial E} \right)^2}{\frac{\partial^2 S}{\partial E^2}} = \frac{2E^2 \eta_0}{T^2 (2E - \eta_0)}$$

which is negative when $E < \eta_0/2$. This property - reminiscent of black hole thermodynamics - has already been noticed by some of the authors of Ref. 3).

The microcanonical equation state is a highly non-linear relation, which can nevertheless be written in the form

$$p = \bar{\xi} s = s T \quad (43)$$

with a function $\bar{\xi}$ given by the relation

$$\left(\bar{\xi}\right)^{-1} = \log \frac{s_c}{s} - 1 - \frac{r}{2s}$$

and $r = \eta_0 / V_{d-1}$. It is worth remarking that the quantity $\bar{\xi}$ is positive in exactly the same range of parameters in which T is also positive, namely $r/2 < \rho < \rho_\infty$. The physical consequences of this approach shall be discussed next.

4. - SOME COSMOLOGICAL IMPLICATIONS

As was already said in the Introduction, the most interesting and challenging cosmological problem is related to the issue of the initial state, and its solution undoubtedly needs a full (super)string cosmology, with a self-consistent gravitational field obtained as a string condensate. A simplification of this is to assume an external gravitational field, and study the behaviour of a string thermal state in this background. The present section is actually devoted to this latter question, by assuming a specially simple cosmological form for the space-time geometry, and postulating Einstein's equations with an energy-momentum tensor given by the strings themselves. For pedagogical reasons, the case in which there are no compact dimensions is treated separately from the case in which there are some of those in spite of it being a trivial particular instance of the latter.

4.1 No compact dimensions

A particularly simple ansatz for the space-time metric is

$$ds^2 = dt^2 - B(t)^2 \delta_{ij} dx^i dx^j \quad (44)$$

where $i = 1, \dots, d-1$. Local conservation of entropy (adiabaticity) and of the energy momentum tensor (i.e., Bianchi identities) then fix the dependence with cosmic time of the entropy density $s(t)$ and energy density $\rho(t)$ in the form:

$$\begin{aligned} S &\sim B^{-(d+1)} \\ \rho &\sim B^{-(d-1)(\xi+1)} \end{aligned} \quad (45)$$

where this $\xi \equiv P/\rho$. This means that local equilibrium can be maintained by the expansion, with a time-dependent temperature

$$T \sim B^{(d-1)\xi} \quad (46)$$

In the particular situation when $2a > d+1$, so that the energy density reaches a finite limit as $T \rightarrow T_c^-$, the preceding equations imply that the only way of maintaining equilibrium is that the scale factor B tends to a non-zero limit B_c [cf. Ref. 8)].

On the other hand, in the opposite situation, when $2a < d+1$, so that the relevant thermodynamical quantities diverge as $T \rightarrow T_c^-$, equilibrium is maintained if

$$T_c - T \sim B^{2 \frac{(d-1)(\xi+1)}{d+3-2a}} \quad (47)$$

in such a way that $B \rightarrow 0$ as $T \rightarrow T_c^-$, and the matter content of the Universe would remain at finite temperature (although still with an infinite energy density) when reaching the initial singularity.

In order to reach a definite conclusion about the behaviour of string matter in cosmological backgrounds, non-equilibrium effects should, however, be studied.

4.2 Compact dimensions

The simplest ansatz allowing for the presence of some compact dimensions, y_a , which will range over a torus of length L , is:

$$ds^2 = dt^2 - R_1^2(t) \delta_{ij} dx^i dx^j - R_2^2(t) \delta_{ab} dy^a dy^b \quad (48)$$

where now x^i , $i = 1, \dots, d-1$ represent non-compact co-ordinates and y^a , $a = 1, \dots, n$ compact ones, $0 < y^a < L$. The compactification scale will be assumed to be the highest scale in the problem, so that $\eta_0 L \ll 1$. Proceeding as before, the dependence of the thermodynamical quantities with the scale factors is obtained:

$$\begin{aligned}
 S &\sim R_1^{-(d-1)} R_2^{-m} \\
 S &\sim R_1^{-(d-1)(\xi+1)} R_2^{-m(\xi+1)} \\
 \beta &\sim R_1^{\xi(d-1)} R_2^{m\xi}
 \end{aligned}
 \tag{49}$$

in such a way that when $2a > d+1$ there are now two possibilities: either both R_1 and R_2 reach a finite limit, or else there is a functional relation of the type

$$R_1 \sim R_2^{-\frac{m}{d-1}}$$

When $2a < d+1$ the evolution of the temperature with cosmic time is more complicated and given by the formula

$$T_c - T \sim R_1^{\frac{2(d-1)(\xi+1)}{d+3-2a}} \cdot R_2^{\frac{2m(\xi+1)}{d+3-2a}}
 \tag{50}$$

which clearly predicts a space-time singularity as $T \rightarrow T_c^-$.

4.3 Non-equilibrium regimes

Let us consider the matter in the Universe described by a microcanonical ensemble. This really only makes sense if the space as such is a closed manifold, but an ansatz of the general form (48) can always be assumed, as long as the compactification scale of the x^1 is much greater than the corresponding one for the y^a .

The microcanonical equation of state (43) is a non-linear one, which means that the energy momentum conservation

$$\dot{S} + (S+p) \left[(d-1) \frac{\dot{R}_1}{R_1} + m \frac{\dot{R}_2}{R_2} \right] = 0
 \tag{51}$$

cannot be analytically integrated. The logarithmic term in ξ^{-1} does, nevertheless, show a much slower dependence on ρ than the other terms in the same expression, which means that a natural first approximation is $\log \rho_c / \rho \sim \text{const.}$, giving:

$$\left(1 - \frac{\pi}{2S}\right) p = S \quad (52)$$

where $\Lambda \equiv \log \rho_c / \rho - 1$. The equation (51) can then be integrated, leading to:

$$S \left[2S(\Lambda+1) - \pi \right]^{-\left(\frac{1}{1+\Lambda}\right)} \sim R_1^{-(d-1)} R_2^{-n}$$

so that when the energy density is the dominant scale, $\rho \gg r/2(\Lambda+1)$, the explicit dependence is found:

$$S \sim R_1^{-(d-1) \frac{1+\Lambda}{\Lambda}} R_2^{-n \frac{1+\Lambda}{\Lambda}} \quad (53)$$

The physical meaning of the preceding equation (53) is that the total energy contained in an unit of comoving volume is not constant, but depends on time instead, in the form:

$$E_{\text{com.vol.}} \sim R_1^{\frac{d-1}{\Lambda}} R_2^{n/\Lambda} \quad (54)$$

It is perhaps worth remarking that, as has been stressed in the first reference of Ref. 3), it is possible to obtain a finite limit for the energy density when $T \rightarrow T_c^-$, even in the case $a < (d+1)/2$ if a time-dependent effective string tension $\alpha'(t)$ is introduced. The dependence of $\alpha'(t)$ on cosmic time is through the temperature only, not unlike the dependence of the effective coupling constant with cosmic time in inflationary models of cosmology, and is fixed by the physical requirement of finiteness when $T \rightarrow T_c^-$. To be explicit, the correct dependence when $T = T_c(1-\epsilon)$, $\epsilon \rightarrow 0^+$ is:

$$\alpha' \sim \epsilon^{\frac{d+3-2a}{1-a}}$$

which by using the equation of state $p = 0$, is equivalent to

$$\alpha' \sim B^{-\frac{2(d-1)}{1-a}} \quad (55)$$

When $1 < a < (d+1)/2$, the sign of $\dot{\alpha}'$ coincides with that of $\dot{B}$, so that when $B \rightarrow 0$, $\alpha' \rightarrow 0$ in which limit strings are known to reduce to ordinary field theories. This phenomenon was called in Ref. 3) "dynamical localization", and gives rise to interesting cosmologies.

5. - CONCLUSIONS

In the first part of this work, the relation between string thermodynamics and the partition function of the world sheet σ -model has been clarified, and a systematic one-loop calculation performed. The extension of the present results to higher loops (i.e., more complicated topologies for the string world sheet) does not present difficulties of principle.

In the second part an "analogue model", containing the dominant physical features of a thermal state of strings has been studied, and its behaviour when the temperature gets close to the critical one, $T \rightarrow T_c^-$ carefully examined. This behaviour is of particular importance, because it determines whether or not it is possible for the system to make a transition to a new phase (of "elementary constituents") when $T > T_c$.

Several interesting cosmological scenarios are studied also, in particular those which begin with a finite value for the scale parameter when $T = T_c$ (so that the initial state possesses a finite volume) and the ones based in the microcanonical ensemble.

The limitations of the present work lie, for the models for which the free energy tends to a finite limit as $T \rightarrow T_c^-$, mainly in our present lack of knowledge about the new string phase at $T > T_c$ (which would be the one which dominates the "initial state") and, for the other models, when T_c is actually a maximum temperature, in the difficulty of performing out of equilibrium studies, which seem nevertheless to dominate at small enough cosmic time. Some string two-loop computations in a thermal state are nevertheless necessary in order to study the fate of the critical temperature when non-trivial interactions are considered. Work on these matters is in progress, and we hope to be able to report on it in the near future.

ACKNOWLEDGEMENTS

It is a pleasure to acknowledge very stimulating discussions with Drs. M.3. Gavela, P. Hasenfratz and S. Randjbar-Daemi. I am also grateful to Prof. P. Minkowski and the other members of the I.T.P. of the University of Bern, where this work was begun, for the hospitality extended to me there. Finally, I would like to thank D. Amati for a careful reading of the manuscript.

APPENDIX A

Given a non-singular symmetric $p \times p$ matrix a_{ij} , Epstein's Z_p function is defined, when $\text{Re } s > 1$ as [Ref. 9)]

$$Z_p(\|a\|, s) = \sum'_{m_1, \dots, m_p = -\infty}^{\infty} \left(\sum_{i,j=1}^p a_{ij} m_i m_j \right)^{-\frac{ps}{2}} \quad (\text{A.1})$$

where the prime on the sum means that the term $\{m_p\} = \{0\}$ has to be omitted. They satisfy a "functional equation", which generalizes the usual reflexion principle of Riemann's function

$$\pi^{-\frac{ps}{2}} \Gamma\left(\frac{ps}{2}\right) Z_p(\|a\|, s) = \Delta^{-1/2} \pi^{-p/2(1-s)} \cdot \Gamma\left(\frac{p}{2}(1-s)\right) \cdot Z_p(\|a^{-1}\|, 1-s) \quad (\text{A.2})$$

where $\Delta \equiv \det(a_{ij})$. The corresponding analytic continuation is a meromorphic function, with a pole in $s = 1$ with residue $\pi^{p/2} \Delta^{-1/2} / \Gamma(p/2 + 1)$. When $s = 0$ its value is -1 and moreover, it has a set of zeroes when $s = -2k/p$, $k \in \mathbb{Z}^+$. In order to compute the determinants which appear in the present paper, the function Z_2 is needed, both in the diagonal and in the non-diagonal case, which will be now studied in turn:

i) The diagonal Z_2

It is not difficult to obtain an integral representation for the function

$$Z_2(a_1^2, a_2^2; s) \equiv \sum' (a_1^2 m_1^2 + a_2^2 m_2^2)^{-s} \quad (\text{A.3})$$

in the form

$$\Gamma(s) Z_2(a_1^2, a_2^2; s) = \frac{\pi \delta^{s-1} \Delta^{-1/2}}{s-1} - \frac{\delta^s}{s} +$$

$$\begin{aligned}
 & + \int_{\delta}^{\infty} dx \cdot x^{s-1} \left[\theta_3(0, \frac{ix a_1^2}{\pi}) \theta_3(0, \frac{ix a_2^2}{\pi}) - 1 \right] \\
 & + \pi \Delta^{-\frac{1}{2}} \int_{\delta^{-1}}^{\infty} dy \cdot y^{-1-(s-1)} \left[\theta_3(0, \frac{iy \pi}{a_1^2}) \theta_3(0, \frac{iy \pi}{a_2^2}) - 1 \right] \quad (A.4)
 \end{aligned}$$

where δ is an arbitrary positive number (it can indeed be checked that the second member does not depend on δ), and θ_3 is Jacobi's modular function, which transforms under the modular group as¹⁰⁾

$$\theta_3\left(\frac{\nu}{\tau}, -\frac{1}{\tau}\right) = (-i\tau)^{1/2} e^{\frac{i\pi \nu^2}{\tau}} \theta_3(\nu, \tau)$$

leading to the limiting behaviour

$$\begin{aligned}
 \theta_3(0, \tau) & \underset{\tau \rightarrow i\infty}{\sim} 1 + 2 e^{-\pi |\tau|} \\
 \theta_3(0, \tau) & \underset{\tau \rightarrow i0}{\sim} (\tau/i)^{-1/2} \quad (A.5)
 \end{aligned}$$

In order to compute the infinite product of physical interest

$$\det M = \prod'_{m_1, m_2} (a_1^2 m_1^2 + a_2^2 m_2^2)$$

the following relation is used

$$\log \det M = \sum'_{m_1, m_2} \log (a_1^2 m_1^2 + a_2^2 m_2^2) = -Z'_2(a_1^2, a_2^2; 0) \quad (A.6)$$

An integral representation for Z'_2 is easily found to be

$$Z'_2 = -\gamma - \log \delta - \pi \delta^{-1} \Delta^{-1/2} + \int \frac{dx}{x} \left[\theta_3(0, \frac{ia_1^2 x}{\pi}) \cdot \theta_3(0, \frac{ia_2^2 x}{\pi}) - 1 \right] + \pi \Delta^{-1/2} \int \frac{dy}{y} \left[\theta_3(0, \frac{iy\pi}{a_1^2}) \cdot \theta_3(0, \frac{iy\pi}{a_2^2}) - 1 \right] \delta^{-1} \quad (A.7)$$

Our formula (A.6) scales correctly under the transformation $\|a\| \rightarrow \|\xi a\|$, namely

$$Z'_2(\xi a_1^2, \xi a_2^2; 0) = \log \xi + Z'_2(a_1^2, a_2^2; 0) \quad (A.8)$$

leading thus, to a consistent definition of the determinant of a constant times the unit operator:

$$\det \xi \cdot \mathbb{1} = \xi \quad (A.9)$$

When computing the σ -model partition function without constraints, the function to be studied is $Z_2(8\pi^2\lambda/v, 8\pi^2v/\lambda; 0)$. The limiting behaviour of Z_2' when $\lambda/v \rightarrow 0$ can be best studied by using the non-diagonal representation of Epstein's functions to be introduced next.

ii) Non-diagonal Z_2

When studying the σ -model with constraints, the ϕ -function of interest is the non-diagonal one

$$\begin{aligned} Z_2(\|a\|, S) &\equiv \sum_{m_1, m_2 = -\infty}^{\infty} (a_{11} m_1^2 + 2a_{12} m_1 m_2 + a_{22} m_2^2)^{-S} \\ &= \sum_{m_1, m_2 = -\infty}^{\infty} a(m_1, m_2)^{-S} \end{aligned} \quad (A.10)$$

An integral representation of Z_2 is (9)

$$\pi^{-s} \Gamma(s) Z_2 = \frac{\Delta^{-1/2}}{s-1} - \frac{1}{s} + \int_1^\infty dz z^{s-1} (\theta_a(z) - 1) + \Delta^{-1/2} \int_1^\infty dz z^{-s} (\theta_{a^{-1}}(z) - 1) \quad (A.11)$$

where $\Delta \equiv \det \|a_{ij}\|$ and the θ -function is defined as

$$\theta_a(z) \equiv \sum'_{m_1, m_2} e^{-\pi z a(m_1, m_2)}$$

under a modular transformation

$$\theta_a(z) = \frac{1}{z\sqrt{\Delta}} \theta_{a^{-1}}(1/z) \quad (A.12)$$

Using the Laurent expansion of Z_2

$$Z_2(s) = \frac{c}{s-1} + c_0 + c_1(s-1) + \dots$$

and the particular expression for c_0 found by Epstein in his 1907 paper, it is not difficult to show that

$$Z_2'(0) = \log \frac{a_{11}}{4\pi^2} - 2 \log \eta(w_1) \eta(-w_2) \quad (A.13)$$

where Dedekind's function $\eta(w)$ is defined as:

$$\eta = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}) = \left(\frac{\theta_1'(0, q)}{2\pi} \right)^{1/3}$$

where

$$w_{1,2} \equiv \frac{-q_{12} \pm i\sqrt{\Delta}}{a_{11}} ; q_{1,2} \equiv e^{\pm i\pi w_{1,2}}$$

and $\theta_1(0, q)$ is the first Jacobi θ -function. By using the well-known representation of Jacobi's θ_1 -function

$$\theta'_1(0, \tau) = \pi \sum_{-\infty}^{\infty} (-1)^{n+1} (2n-1) e^{i\pi(n-\frac{1}{2})^2 \tau}$$

the limiting behaviour when $\omega \rightarrow i\infty$ of Dedekind's function is easily gotten:

$$\eta(i\tau) \underset{\tau \rightarrow \infty}{\sim} 2^{-1/3} e^{-\frac{\pi\tau}{12}}$$

in such a way that $Z'_2(0)$ in the particular diagonal case $a_{11} \equiv 8\pi^2(\lambda/v)$, $a_{12} = 0$ and $\Delta = (8\pi^2)^2$ behaves as:

$$Z'_2(0) \sim \log \frac{\lambda}{v} + \frac{\pi v}{3\lambda} \quad (\text{A.14})$$

and the limit of the determinant itself when $p^+ \rightarrow \infty$ is the one quoted in the main text. It should be obvious that both the definition and the integral representation (A.12) are easily generalized to an arbitrary number of dimensions, p , but in order to compute the determinants in this paper, it is not necessary to do so.

APPENDIX B

Let us represent by S_d the sum appearing in the integral $I_{d,a}$

$$S_d(\beta m) \equiv \sum K_{d/2} \frac{[(2m+1)\beta m]}{(2m+1)^{d/2}} \quad (B.1)$$

It is not difficult to obtain the infinite set of differential equations for the functions $S_d(z)$:

$$\frac{d^2}{dz^2} S_d(z) = \frac{d^2}{4z^2} S_d + \frac{d}{z} S_{d-2} + S_{d-4}$$

which in the $d \rightarrow \infty$ limit reduces to

$$S_d'' = \frac{d^2}{4z^2} S_d$$

namely, Bessel's equation, implying

$$S_d(\beta m) \sim K_{d/2}(\beta m) \quad (B.2)$$

(This result can be confirmed by studying both the small and asymptotic βm expansions of S_d .)

The integral $I_{d,a}(\beta)$ can be easily studied in the physical regime where $\eta_0 \gg T_c$, so that $\eta_0 \beta \gg 1$ (because $\beta > \beta_c$), and then necessarily also $\beta m \gg 1$ in the whole range of integration, leading to

$$I_{d,a} = \sqrt{\frac{\pi}{2\beta}} (\beta - \beta_c)^{a - \frac{d+1}{2}} \Gamma\left(\frac{d+1}{2} - a, \eta_0 (\beta - \beta_c)\right) \quad (B.3)$$

where $\Gamma(\alpha, z)$ is the incomplete gamma function. Using the asymptotic expansion of Γ , the behaviour of $I_{d,a}$ when $\beta \rightarrow \beta_c^+$ is easily gotten:

$$I_{d,a} \sim \sqrt{\frac{\pi}{2}} \beta_c^{-\frac{1}{2}} \frac{1}{a - \frac{d+1}{2}} \eta_0^{-a + \frac{d+1}{2}} \quad \left(a > \frac{d+1}{2}\right)$$

$$I_{d,a} \sim -\sqrt{\frac{\pi}{2}} \beta_c^{a-1-\frac{d}{2}} \log(\eta_0 \beta_c \epsilon) \quad (a = \frac{d+1}{2}) \quad (B.4)$$

+ less singular terms

$$I_{d,a} \sim \sqrt{\frac{\pi}{2}} \beta_c^{a-1-d/2} \Gamma\left(\frac{d+1}{2} - a\right) \epsilon^{a-\frac{d+1}{2}} \quad (a < \frac{d+1}{2})$$

+ less singular terms

where the parameter $\epsilon \equiv (\beta - \beta_c)/\beta_c \rightarrow 0^+$. In the particular case $a > (d+5)/2$, the limiting behaviour of the thermodynamic potential is (assuming always $\eta_0 \beta_c \gg 1$)

$$\log Z_d = \sqrt{\frac{\pi}{2}} K \eta_0^{\frac{d+1}{2}-a} \beta_c^{\frac{1-d}{2}} \left[\frac{2}{2a-d-1} - \epsilon \frac{2\eta_0 \beta_c}{2a-d-3} + \epsilon^2 \frac{(\eta_0 \beta_c)^2}{2a-d-5} + \dots \right]$$

The thermodynamical formula for the mean value of the energy

$$\langle E \rangle = -\frac{1}{\beta_c} \frac{\partial}{\partial \epsilon} \log Z_d$$

now leads to the limiting formula for the energy density when $T \rightarrow T_c^-$ (see main text).

	a	b	d _c
Veneziano Dual Model (VDM)	$\frac{d+1}{2}$	$2\pi\sqrt{\frac{d-2}{6}}$	26
Green and Schwarz I (GSI)	9/2	$\pi/8$	10
Green and Schwarz II (GSII)	10	$\pi/8$	10
Heterotic string (HS)	10	$(2+\sqrt{2})\pi$	10

Table 1: Values of the constants in the mass spectrum $\omega(m) = (m/\alpha')^{-a} e^{bm/\alpha'}$ for different string theories. The critical space-time dimension, d_c has also been explicitly indicated.

	$\Omega \equiv \log Z_d$	$\rho \equiv \frac{\langle E \rangle}{V_{d-1}}$	C_V	Thermodynamical Quantity
VDM	23	21	19	
GSI	7	5	3	
GSII	18	16	14	
HS	18	16	14	

Type of
string

Table 2: Maximum number of non-compact dimensions allowed so that the corresponding thermodynamical quantity remains finite in the limit $\beta \rightarrow \beta_c^+$.

REFERENCES

- 1) J. Schwarz, "Superstring theory", Phys. Rep. 89 (1982) 223.
- 2) D.J. Gross et al., Nucl. Phys. B256 (1985) 253.
- 3) Some preceding related works are:
E. Alvarez, Phys. Rev. D31 (1985) 418;
B. Sundborg, Nucl. Phys. B254 (1985) 583;
M.J. Bowick and L.C.R. Wijewardhana, Phys. Rev. Lett. 54 (1985) 2485;
S.H. Tye, Phys. Lett. 158B (1985) 388;
J. Polchinski, Texas preprint (1985).
M. Gleiser and J.G. Taylor, King's College preprint (1985).
- 4) N. Cabibbo and G. Parisi, Phys. Lett. 59B (1975) 67.
- 5) A.M. Polyakov, Phys. Lett. 103B (1981) 207;
O. Alvarez, Nucl. Phys. B216 (1983) 125.
- 6) S. Coleman, Comm. Math. Phys. 31 (1973) 259.
- 7) G. Horowitz, Comm. Math. Phys. 89 (1983) 117.
- 8) K. Huang and S. Weinberg, Phys. Rev. Lett. 25 (1970) 895.
- 9) P. Epstein, Math. Ann. 56 (1903) 615, 63 (1907) 205.
- 10) A. Erdelyi et al., "Higher transcendental functions", Volume 3 (Mc Graw Hill) (1953).