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SO(4) REDUCTION OF THE SU(N) YANG-MILLS-DIRAC EQUATIONS*

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ABSTRACT

We reduce by the action of an $SO(4)$ subgroup of the conformal group of space-time the equations in compactified Minkowski space which describe $SU(N)$ Yang-Mills fields minimally coupled to Dirac spinor fields transforming under the lowest-dimensional representation(s) of $SU(N)$. The reduced systems can be interpreted as Hamiltonian systems with symmetry constrained by a condition on the associated moment maps.

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1. INTRODUCTION

In a previous paper¹, we considered the reduction of the $SU(n)$, $SO(n)$ and $Sp(n)$ Yang-Mills equations under the $SO(4)$ subgroup of the conformal group of space-time $C(3,1)$. This work was based on an earlier study² of the geometric formulation of invariant gauge fields under smooth group actions. These methods have also been applied to derive the dimensional reduction procedure³, to reduce and solve gauge⁴ and matter-coupled-gauge systems^{5,6}, and to determine invariant spinor fields with gauge freedom.⁷

Our purpose in the following is to extend these investigations to the case of coupled Yang-Mills-Dirac equations on conformally compactified Minkowski space with massless Dirac spinor fields transforming under the lowest dimensional representation(s) of the gauge group $SU(N)$. For $N = 2$, reductions under certain compact subgroups of $C(3,1)$, including $SO(4)$, have already been done and invariant solutions found.⁸ We here consider higher dimensional gauge groups.

In section 2, we give the characterization of $SU(N)$ gauge fields and Dirac spinors invariant under the $SO(4)$ action. Such actions are generally characterized by an homomorphism of the isotropy subgroup of $SO(4)$ into $SU(N)$, but for a large class of homomorphisms, either no non-trivial invariant Dirac spinor fields exist or, as shown in the Appendix, the field equations force the invariant spinors to be zero. Thus, only particular homomorphisms allow Yang-Mills systems coupled with one lowest dimensional multiplet of spinors, and we consider some of these in section 3. More precisely, we look at the reduction of the $SU(2n)$ Yang-Mills-Dirac equations on the manifold $S^1 \times S^3$ with Lorentzian metric (diffeomorphic to the compactified Minkowski space) for typical $SO(4)$ embeddings. In a convenient gauge, the reduced systems are seen to be interpretable as Hamiltonian systems with $U(n)$ symmetry constrained by the condition that the $SU(n)$ part of the moment map (i.e., the associated conserved quantities) vanishes. Assuming that either one of the two Weyl components equals zero, we further simplify the residual systems by use of this constraint to obtain a set of one-dimensional systems interacting via inverse square potentials. Finally, we present a non-trivial invariant solution to the $SU(4)$ coupled system, the solution on the compactified space being expressible as a solution on Minkowski space by an appropriate transformation.^{6,9}

2. $SO(4)$ INVARIANT FIELDS

First, we summarize some of the notations given in references 1 to 6 that will be used in the following. Let $\bar{M}$ be the conformally compactified Minkowski space, identified with the group $U(2)$ and for simplicity, let us work on the twofold covering $U(1) \times SU(2)$, identified as $S^1 \times S^3$, with points $p = (e^{i\psi}, v)$, $e^{i\psi} \in U(1)$, and $v \in SU(2)$.

We also consider the following natural group actions on $U(1) \times SU(2)$:

- (a) Left action of $SU(2)$: $L_g: (e^{i\psi}, v) \mapsto (e^{i\psi}, gv)$, where $g \in SU(2)$,
- (b) Right action of $SU(2)$: $R_g: (e^{i\psi}, v) \mapsto (e^{i\psi}, vg)$, where $g \in SU(2)$,
- (c) Left action of $SU(2) \times SU(2)$: $L_{(g',g)}: (e^{i\psi}, v) \mapsto (e^{i\psi}, g'vg^{-1})$, where $g, g' \in SU(2)$,

(d) Left action of the diagonal subgroup $SU(2)_D \equiv (SU(2) \times SU(2))_D$: $D_g: (e^{i\psi}, v) \mapsto (e^{i\psi}, gvg^{-1})$, where $g \in SU(2)$.

In terms of the Cartesian coordinates $\{x^\mu\}$, the injection of the Minkowski space (M) in its compactified version $\bar{M}$ is defined by:

$$e^{i\psi} = u^5 + iu^0, \quad \text{and} \quad u = u^4 - iu^3\sigma_3, \quad (2.1a)$$

($i = 1, 2, 3$), with $\{\sigma_i\}$ representing the Pauli matrices, where

$$u^\mu = \frac{x^\mu}{r}, \quad u^4 = \frac{(1 + x_\mu x^\mu)}{2r}, \quad \text{and} \quad u^5 = \frac{(1 - x_\mu x^\mu)}{2r}, \quad (2.1b)$$

and $r = [x_0^2 + \frac{1}{4}(1 - x_\mu x^\mu)^2]^{1/2}$.

Its (singular) inverse is given by:

$$(e^{i\psi}, v) \mapsto \left\{ x^\mu = \frac{u^\mu}{u^4 + u^5} \right\}. \quad (2.2)$$

The subgroup $SO(4) \subset SO_0(4, 2) \sim SU(2, 2)/Z_2$ (where the subindex "0" specifies the identity component) acts on the (u^1, u^2, u^3, u^4) subspace, and its twofold covering $SU(2) \times SU(2)$ is the corresponding subgroup of $SU(2, 2)$. The isotropy subgroup of $SU(2) \times SU(2)$ at the reference point $p_0 = (e^{i\psi}, 1_2)$ is the diagonal subgroup $SU(2)_D$ and the orbits are the S^3 corresponding to fixed ψ .

On $U(1) \times SU(2)$, there exists a natural $SO(2) \times SO(4)$ left invariant Lorentzian metric, denoted $\bar{g}$, which is conformal to the Minkowski metric g :

$$g = r^2 \bar{g} = r^2 \left(\theta^0 \otimes \theta^0 - \sum_i \theta^i \otimes \theta^i \right). \quad (2.3)$$

The 1-forms θ^0 and θ^i are defined as left invariant forms and they constitute a global set of orthonormal co-frames on $S^1 \times S^3$. Explicitly, they can be expressed as:

$$\theta^0 = d\psi, \quad \text{and} \quad \theta^i = \frac{1}{2}(v^\dagger dv)^i, \quad (v \in SU(2)) \quad (2.4)$$

We denote the associated dual frames $\{e_\mu\}$. Since these are orthonormal, the linear generators $\{\gamma^\mu\}$ of the Clifford algebra are just the ordinary Dirac matrices. As in reference 6, we use the following representation:

$$\gamma^0 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma_i \sigma_2 \\ \sigma_2 \sigma_i & 0 \end{bmatrix}, \quad \text{and} \quad (2.5)$$

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{bmatrix} 1_2 & 0 \\ 0 & -1_2 \end{bmatrix}.$$

2.1 INVARIANT $SU(N)$ YANG-MILLS FIELDS:

It follows from the result of reference 2 that each equivalence class of principal bundles $P(\bar{M} \sim S^1 \times [(SU(2) \times SU(2))/SU(2)_D], SU(N))$, admitting a lift of the group action of $SU(2) \times SU(2)$, is characterized by a conjugacy class of homomorphisms (λ) of the isotropy subgroup $SU(2)_D$ into the gauge group (structural group) $SU(N)$. Since $SU(2)$ is a simple and simply connected group, the problem of determination, up to conjugacy, of its homomorphisms into $SU(N)$ is equivalent to the classification of the $\mathfrak{su}(2)$ -subalgebras into $\mathfrak{su}(N)$. Mal'cev¹⁰ and Dynkin¹¹ have solved this for all semi-simple Lie subalgebras. In the case of $\mathfrak{su}(2) \subset \mathfrak{su}(N)$, these classes are in one-to-one correspondence with the systems of highest weights (spins) of (non-trivial) $\mathfrak{su}(2)$ irreducible representations constrained so that the sum of their associated dimensions is less than or equal to N .

Among these classes, we shall mainly be interested in homomorphisms which we shall call "homogeneous". For m, N , and $N/m \in \mathcal{N}$ (natural numbers), these correspond to $SU(2)$ subgroups formed from N/m identical irreducible $SU(2)$ representations $D^j(g)$ of (highest) weight $j = (m-1)/2$:

$$\lambda : (g, g) \in SU(2)_D \mapsto 1_{\frac{N}{m}} \otimes D^j(g) \in SU(N), \quad g \in SU(2). \quad (2.6)$$

The explicit calculation of the $SO(4)$ invariant gauge fields for these classes of bundles is given in reference 1, and we recall the results below. The invariant connections $\omega(\psi)$ obtained are:

$$\omega(\psi) = H'(\psi) \otimes D^j(\tau_i)\theta^i + \Gamma(\psi) \otimes 1_m \theta^0, \quad (2.7)$$

where $H' \in \mathcal{N}(N/m)$, the space of $N/m \times N/m$ Hermitian matrices, $\Gamma(\psi) \in \mathfrak{su}(N/m)$, $\{\tau_i \equiv \sigma_i/2i\}$, $\psi \in S^1$. The $\Gamma(\psi)$ component may be thought of as a gauge potential on S^1 with respect to the residual gauge group. This may, of course, be gauged to zero. However, for the purposes of Euler-Lagrange variations, it is preferable to retain this gauge freedom.

The "non-homogeneous" homomorphisms may in general be written as embeddings of the type:

$$\lambda : (g, g) \in SU(2)_D \mapsto \left(1_{\frac{M_a}{m_a}} \otimes D^{j_a}(g) \right) \oplus \left(1_{\frac{M_b}{m_b}} \otimes D^{j_b}(g) \right) \oplus \dots \left(1_{\frac{M_n}{m_n}} \otimes D^{j_n}(g) \right) \in SU\left(\sum_{i=1}^n M_i\right), \quad (2.8)$$

where $j_k \equiv (m_k - 1)/2$, $M_k \in \mathcal{N}$, $\forall k \in \{0, 1, \dots, n\}$. A calculation shows that the resulting invariant fields correspond to an uncoupled direct sum of each homogeneous invariant field except for every pair of weights $(j, j+1)$, which in addition

produces “off-diagonal” contributions. For example, if the embedding λ is:

$$\lambda : (g, g) \in SU(2)_D \mapsto \begin{pmatrix} 1 & M \\ 2j+3 & \end{pmatrix} \otimes D^{j+1}(g) \oplus \begin{pmatrix} 1 & M' \\ 2j+1 & \end{pmatrix} \otimes D^j(g) \in SU(M+M'), \quad (2.9)$$

the most general $SO(4)$ invariant gauge field has the simple form:

$$\omega(\psi) = \begin{bmatrix} H_M \otimes D^{j+1}(\tau_i) & K \otimes \Omega(\tau_i) \\ -K^\dagger \otimes \Omega^\dagger(\tau_i) & H_{M'} \otimes D^j(\tau_i) \end{bmatrix} \theta^i + c(\psi)\theta^0, \quad (2.10)$$

where: $H_M \in \mathcal{H}(M/2j+3)$, $H_{M'} \in \mathcal{H}(M'/2j+1)$, K is a $M/2j+3 \times M'/2j+1$ complex matrix, the $\{\Omega(\tau_i)\}$ stands for $(2j+3) \times (2j+1)$ matrices which are expressed in terms of Clebsch-Gordan coefficients for the coupling of j and $j+1$ to 1, and the element $c(\psi)$ belongs to the centralizer of the image $\lambda_*(su(2)_D)$ in the gauge Lie algebra $su(M+M')$.

2.2. INVARIANT $SU(N)$ DIRAC SPINOR FIELDS

Let us define $\Psi \in C^4 \times C^N$ as a massless Dirac spinor field transforming under the fundamental representation of $SU(N)$, denoted $\bar{D}$. With respect to our choice of Dirac matrices and orthonormal co-frames, the $SO(4)$ invariance condition reads (for details, see references 6 and 7):

$$\Psi(L(g', g)p) = \begin{bmatrix} D^{(1/2,0)}(g) & 0 \\ 0 & D^{(1/2,0)*}(g) \end{bmatrix} \otimes \bar{D}(\rho^{-1}((g', g), p)) \Psi(p), \quad (2.11)$$

$V(g', g) \in SU(2) \times SU(2)$ and $p \in U(1) \times SU(2)$, where $D^{(1/2,0)}$ and $D^{(0,1/2)}$ (equivalent to the complex conjugate of $D^{(1/2,0)}$) are basic representations of $SL(2, C)$. The “transformation function” $\rho^{-1}((g', g), p)$ which characterizes the group action on the principal fibre bundle (see ref. 2), may be chosen independent of the point p since the homomorphism λ extends smoothly to an homomorphism (Λ) of $SU(2) \times SU(2)$ into $SU(N)$. In fact, we may define:

$$\rho^{-1}((g', g), p) = \Lambda((g', g)) = \lambda((g, g)). \quad (2.12)$$

Consequently, the Dirac spinor field must satisfy the following linear isotropy condition at the reference point $p_0 = (e^\psi, 1_2)$ for any homogeneous homomorphism:

$$\Psi(p_0) = \begin{bmatrix} g & 0 \\ 0 & g^* \end{bmatrix} \Psi(p_0) \begin{pmatrix} 1 & N \\ N & \end{pmatrix} \otimes D^j(g)^T, \quad (2.13)$$

$\forall g \in SU(2)$, $\Psi \in C^{4 \times N}$. However, Schur’s lemma forbids the existence of non-trivial invariant Dirac spinor fields unless the highest weight j equals $1/2$. In that case, the corresponding invariant spinor takes the form:

$$\Psi(\psi) = \begin{bmatrix} \xi^T(\psi) \otimes \sigma_2 \\ \eta^T(\psi) \otimes 1_2 \end{bmatrix}, \quad (2.14)$$

where $\xi, \eta \in C^{n \times 1}$ are functions of $\psi \in S^1$.

We remark that spinors transforming under any representation of the gauge group will have non-trivial $SO(4)$ invariant fields if the restriction to $\lambda(SU(2)_D)$ of the representation contain at least one irreducible $j = 1/2$ component. For example, the reduction by $SO(4)$ of simple super Yang-Mills systems for any homogeneous homomorphism is equivalent to the sourceless Yang-Mills case since these embeddings do not provide any non-trivial invariant spinor field in the adjoint representation of $SU(N)$.

3. $SU(2n)$ YANG-MILLS-DIRAC EQUATIONS:

In the orthonormal basis $\{\theta^\mu\}$ defined above, the pseudo-riemannian connection of $\bar{g}$ ($\Gamma_{\nu\rho}^\mu$) is derived from the Maurer-Cartan structure equations:

$$d\theta^0, \text{ and } d\theta^i + \epsilon_{ijk}\theta^j \wedge \theta^k = 0. \quad (3.1)$$

This implies that the only non-zero components of $\Gamma_{\rho\nu}^\mu$ are:

$$\Gamma_{jk}^i = \epsilon_{ijk}, \quad (3.2)$$

for

$$d\theta^\mu + \Gamma_{\rho\nu}^\mu \theta^\rho \wedge \theta^\nu = 0. \quad (3.3)$$

Thus, we can write the canonical spin connection⁶ as:

$$\tilde{\sigma} = \tilde{\sigma}_\rho \theta^\rho = \frac{1}{8} [\gamma^\mu, \gamma^\nu] \Gamma_{\mu\nu\rho} \theta^\rho = \frac{1}{8} [\gamma^i, \gamma^j] \epsilon_{ijk} \theta^i. \quad (3.4)$$

It follows that the action $(\mathcal{A})$ on $U(1) \times SU(2)$ with orthonormal basis $\{\theta^\mu\}$ is given for the Yang-Mills-Dirac system by:

$$\mathcal{A} = \int_{\bar{M}} \left\{ -\frac{1}{2k} \text{tr} (F \wedge *F) + \frac{i}{2} [\text{tr}(\bar{\Psi} \gamma^\mu (e_\mu + \tilde{\sigma}_\mu) \Psi + \bar{\Psi} \gamma^\mu \Psi \tilde{D}(A_\mu)^T) - \text{Hermitian conjugate}] V \right\} \quad (3.5)$$

Here $V = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$ is the volume element, $F = D\omega = \frac{1}{2} F_{\mu\nu}^a T_a \theta^\mu \wedge \theta^\nu$, is the curvature associated to the gauge field $\omega = A_\mu^a T_a \theta^\mu$, and $*F$ represents its dual relatively to the metric $\bar{g}$. The $\{T_a\}$ forms a basis of the gauge Lie algebra such that: $\text{tr}(T_a T_b) = k\delta_{ab}$, and k may be any negative real constant.

The Yang-Mills-Dirac equations determined from $\mathcal{A}$ are: (i) Yang Mills:

$$*D * F = J, \quad (3.6)$$

which possess a 1-form spinor current with values in the gauge Lie algebra:

$$J = i \text{tr} (\bar{\Psi} \gamma_\mu \Psi T_a^T) \theta^\mu T_a, \quad (3.7)$$

and (ii) Dirac:

$$\gamma^\mu [(e_\mu + \bar{\sigma}_\mu)\Psi + \Psi \bar{D}(A_\mu)^T] = 0. \quad (3.8)$$

3.1 REDUCTION

Inserting the explicit forms (2.7) and (2.14) for the respective $SO(4)$ invariant gauge and spinor fields in the action (3.5), we arrive at the reduced action

$$A_R = \int_{S^1} \mathcal{L}_R \theta^0 \quad (3.9a)$$

where

$$\begin{aligned} \mathcal{L}_R = \text{tr} \left\{ \frac{1}{2} [(\mathcal{D}H)^2 - (1 - H^2)^2] - \frac{2ik}{3} [\mathcal{D}\xi\xi^\dagger + \mathcal{D}\eta\eta^\dagger - \xi(\mathcal{D}\xi)^\dagger - \eta(\mathcal{D}\eta)^\dagger] \right. \\ \left. - 2k [(\eta\eta^\dagger - \xi\xi^\dagger)H] \right\}, \end{aligned} \quad (3.9b)$$

is the reduced Lagrangian density written in terms of a residual gauge element (Γ) of zero curvature, a set of "scalar components" (H), and spinor remnants (ξ and η). The "covariant derivatives" with respect to the residual component Γ are defined by:

$$\mathcal{D}H \equiv \dot{H} + [\Gamma, H], \quad (3.10a)$$

$$\mathcal{D}\xi \equiv \dot{\xi} + \Gamma\xi, \quad (3.10b)$$

$$\mathcal{D}\eta \equiv \dot{\eta} + \Gamma\eta, \quad (3.10c)$$

where the dot indicates the derivative of the variable with respect to ψ . The variational equations consist of:

$$\mathcal{D}(\mathcal{D}H) - 2H(1 - H^2) = -2kh', \quad (3.11a)$$

$$[H, \mathcal{D}H] = \frac{4ik}{3} h_0, \quad (3.11b)$$

and

$$\mathcal{D}\xi + \frac{3i}{2} H\xi = 0, \quad (3.11c)$$

$$\mathcal{D}\eta - \frac{3i}{2} H\eta = 0, \quad (3.11d)$$

where $\Gamma \in su(n)$, $H' \equiv H + 1_n \in \mathcal{H}(n)$, h_0 is identified as the traceless part of $h \equiv \eta\eta^\dagger + \xi\xi^\dagger$, and $h' \equiv \eta\eta^\dagger - \xi\xi^\dagger$. One can check that the reduced Lagrangian density ($\mathcal{L}_R$) and the corresponding equations are left invariant by the gauge transformations $U \in SU(n)$:

$$\Gamma \rightarrow U^\dagger \Gamma U + U^\dagger \dot{U}, \quad (3.12a)$$

$$H \rightarrow U^\dagger H U, \quad (3.12b)$$

$$\xi \rightarrow U^\dagger \xi, \quad (3.12c)$$

and

$$\eta \rightarrow U^\dagger \eta. \quad (3.12d)$$

We shall now choose a gauge in which the residual component Γ vanishes, and thus the equations (3.11) simplify to:

$$\ddot{H} - 2H(1 - H^2) = 2kh', \quad (3.13a)$$

$$[H, \dot{H}] = \frac{4ik}{3} h_0, \quad (3.13b)$$

and

$$\dot{\xi} + \frac{3i}{2} H\xi = 0, \quad (3.13c)$$

$$\dot{\eta} - \frac{3i}{2} H\eta = 0. \quad (3.13d)$$

Note that if this gauge had been fixed before the Euler-Lagrange variation (in A_R), the equation (3.13b) would not be obtained. Substitution of the invariant fields in (3.6) and (3.8) in the gauge $\Gamma = 0$ also produces (3.13a - d), as the $SO(4)$ reduction derived for a multiplet of spinors transforming under the contragredient fundamental representation D^* .

From the equations (3.13a), (3.13c) and (3.13d), it follows that the anti-Hermitian matrix:

$$I = [H, \dot{H}] - \frac{4ik}{3} h, \quad (3.14)$$

is conserved. This constant is related to the invariance of the system under the $U(n)$ transformations: $H \rightarrow U H U^\dagger$, $\xi \rightarrow U\xi$, and $\eta \rightarrow U\eta$ ($U \in U(n)$). The second Yang-Mills-Dirac equation (3.13b) may be recognized as the vanishing of the traceless ($su(n)$) part. Moreover, taking the trace shows that the quantity: $|\xi|^2 + |\eta|^2$, must be a real constant. Note that I is the sum of two terms coming respectively from the Yang-Mills and Dirac spinor fields.

3.2 HAMILTONIAN SYSTEM:

In the following, we formulate the above as a Hamiltonian system with symmetry constrained by a condition on the associated moment map.

Consider $\mathcal{H}(n)$ and $\mathcal{C}^n$ as spaces with the respective Hermitian inner products $\langle H, H' \rangle = \text{tr}(HH')$, $H, H' \in \mathcal{H}(n)$, and $\langle \xi, \pi \rangle = \sum_{i=1}^n \xi^i \pi^{i*}$, $\xi, \pi \in \mathcal{C}^n$. Correspondingly, we define on $\mathcal{H} \times \mathcal{H} \times \mathcal{C}^n \times \mathcal{C}^n$ the symplectic structure:

$$\Omega(H, P, \xi, \eta) = \text{tr}(dH \wedge dP) - \frac{4ik}{3} \sum_{i=1}^n (d\xi^i \wedge d\xi^{i*} + d\eta^i \wedge d\eta^{i*}), \quad (3.15)$$

where $(H, P, \xi, \eta) \in \mathcal{H} \times \mathcal{H} \times \mathcal{C}^n \times \mathcal{C}^n$.

On this space, the Hamiltonian:

$$\mathcal{H}_R = \frac{1}{2} \text{tr} (P^2 + (1 - H^2)^2) + 2k(\langle H\eta, \eta \rangle - \langle H\xi, \xi \rangle), \quad (3.16)$$

gives rise to the following (Hamilton) equations:

$$\dot{H} = P \quad (3.17a)$$

$$\dot{P} = 2H(1 - H^2) - 2k(\eta\eta^\dagger - \xi\xi^\dagger), \quad (3.17b)$$

and

$$\dot{\xi} = -\frac{3i}{2}H\xi, \quad \xi^{\star T} = \frac{3i}{2}\xi^{\star T}H, \quad (3.17c)$$

$$\dot{\eta} = \frac{3i}{2}H\eta, \quad \eta^{\star T} = -\frac{3i}{2}\eta^{\star T}H, \quad (3.17d)$$

which are equivalent to the three reduced equations (3.13a), (3.13c), and (3.13d) of the Yang-Mills-Dirac system obtained. The $U(n)$ action defined by:

$$U : (H, P, \xi, \xi^*, \eta, \eta^*) \mapsto (UHU^\dagger, UPU^\dagger, U\xi, U\xi^*, U\eta, U\eta^*),$$

is symplectic and preserves the Hamiltonian. Its moment map¹² is:

$$I(H, P, \xi, \xi^*, \eta, \eta^*) = [H, P] - \frac{4ik}{3}(\eta\eta^\dagger + \xi\xi^\dagger) \in \mathfrak{u}^*(n) \text{ (the dual } U(n) \text{ Lie algebra)}. \quad (3.18)$$

Adding the condition that the $\mathfrak{su}(n)$ part $(I - \frac{1}{n}\text{tr } I)$ equals zero, reproduces the entire system (3.13 a-d) interpreted as a constrained Hamiltonian system. We also have the real constant: $|\xi|^2 + |\eta|^2$, which remains free.

3.3 ADDITIONAL REDUCTION:

If we suppose that one of the two Weyl components vanishes, that is either $\eta = 0$ or $\xi = 0$, we can further reduce the Yang-Mills-Dirac equations with the help of the relation (3.13b). Let us assume that $\eta = 0$, the trace of the moment map I then implies that

$$|\xi|^2 = \sum_{i=1}^n |\xi_i|^2 = \frac{Cn}{2}, \quad (3.19)$$

where C is an arbitrary real constant. However, since H is Hermitian, it can be diagonalized with a ψ -dependent transformation in $U(n)$:

$$H_D(\psi) = U(\psi)HU^\dagger(\psi) = \begin{bmatrix} \lambda_1(\psi) & 0 \\ 0 & \dots & \lambda_n(\psi) \end{bmatrix}, \quad (3.20)$$

with $U \in U(n)$. We also define:

$$\tilde{P} \equiv UPU^\dagger, \quad (3.20a)$$

$$\xi' \equiv U\xi, \quad \xi'^* \equiv U^*\xi^*. \quad (3.20b)$$

The constraint I (3.18) now reads:

$$(\lambda_i - \lambda_j)\tilde{P}_{ij} = \frac{4ik}{3}(\xi'_i\xi'_j{}^* - C\delta_{ij}), \quad (3.21)$$

where i, j take the values: $1, \dots, n$. For $C \neq 0$, it follows that $\lambda_i \neq \lambda_j$ for $i \neq j$. Substituting Hamilton's equation (3.17a) in the relation (3.21), we get an expression for the element $u(\psi) \equiv UU^\dagger$ of the Lie algebra $\mathfrak{u}(n)$ of $U(n)$: (i) if $i \neq j$:

$$u_{ij} = \frac{4ik\xi'_i\xi'_j{}^*}{3(\lambda_j - \lambda_i)^2}, \quad (3.22)$$

and (ii) if $i = j$: the terms u_{ij} are left undetermined since they correspond to the elements of the centralizer of H_D in $\mathfrak{u}(n)$, and hence can be ignored.

Upon substitution of (3.22) in the two remaining Yang-Mills-Dirac equations, we derive from the diagonal terms of (3.13a) that:

$$\tilde{\lambda}_i + \sum_{j \neq i} \frac{32k^2C^2}{9(\lambda_j - \lambda_i)^3} - 2\lambda_i(1 - \lambda_i^2) - 2kC = 0, \quad (3.23)$$

for every $i = 1, \dots, n$; while the off-diagonal contributions are automatically satisfied. Finally, equation (3.23c) reduces to:

$$\xi'_i - \frac{4ik}{3}C \sum_{i \neq j} \frac{1}{(\lambda_j - \lambda_i)^2} \xi'_j + \frac{3i}{2} \lambda_i \xi'_i = 0, \quad (3.24)$$

for every $i = 1, \dots, n$. Setting $\xi = 0$ instead of η leads to a similar set of equations.

We remark that the system (3.13) can be regarded as a set of n one-dimensional systems with quartic potentials and Calogero type interaction¹³⁻¹⁴. Once solved for the eigenvalues $\{\lambda_i\}$, ξ is determined by quadrature and the solution $U(\psi)$ from the definition of u . In the case of vanishing Fermi fields, we have $\xi = 0$, $\eta = 0$, and $C = 0$ in equation (3.23). It is then found that U is a constant and (3.23) decouples giving the general solution in terms of elliptic functions as in reference 1.

We have not been able to integrate (3.23) in general, but in the next section, a particular solution to (3.13) is presented.

3.4 $SU(4)$ SOLUTION:

As noted above, the moment map splits into the sum of two parts, corresponding to the gauge field and spinor field contributions. We shall derive a specific solution for the case where not just the sum but each term separately vanishes. That is, we assume:

$$[H, \tilde{H}] = 0, \quad (3.25a)$$

and

$$\eta\eta^\dagger + \xi\xi^\dagger = \frac{(|\xi|^2 + |\eta|^2)}{n} \mathbf{1}_n. \quad (3.25b)$$

However, the equation (3.25b) can only be satisfied by non-trivial ξ and η if $n = 1$ or 2. The $n = 1$ case corresponds to the $SU(2)$ Yang-Mills-Dirac system which has been solved in references 6 and 8. For $n = 2$, (3.25b) implies that:

$$\xi^\dagger \eta = 0, \quad (3.26a)$$

and

$$|\xi|^2 = |\eta|^2. \quad (3.26b)$$

Let us suppose for simplicity a normalization: $|\xi|^2 = |\eta|^2 = 1$, and initial conditions which respect the constraint (3.26a):

$$\xi_0 \equiv \begin{bmatrix} e^{i\phi} \\ 0 \end{bmatrix}, \text{ and } \eta_0 = \begin{bmatrix} 0 \\ e^{i\phi'} \end{bmatrix}, \quad (3.27)$$

with real numbers ϕ, ϕ' . We can thus express $\xi(\psi)$ and $\eta(\psi)$ as:

$$\xi(\psi) = U(\psi)\xi_0, \text{ and } \eta(\psi) = V(\psi)\eta_0, \quad (3.28)$$

where $U, V \in U(2)$ depend on the parameter $\psi \in S^1$. Consequently, the relation (3.25b) becomes:

$$U\sigma_3 U^\dagger - V\sigma_3 V^\dagger = 0, \quad (3.29)$$

and the first Yang-Mills-Dirac equation takes the form:

$$\tilde{H} - 2H(1 - H^2) = 2kU\sigma_3 U^\dagger, \quad (3.30)$$

However, (3.25a) requires that: $H(\psi) = h^0(\psi)1_2 + h^3(\psi)U\sigma_3 U^\dagger$, where h^0 and h^3 are real functions of ψ . From the Dirac equation, it then follows that (3.30) is solved by:

$$H(\psi) = \begin{bmatrix} \lambda(\psi) & 0 \\ 0 & \omega(\psi) \end{bmatrix}, \quad (3.31)$$

where λ and ω satisfy respectively the equations:

$$\tilde{\lambda} - 2\lambda + 2\lambda^3 - 2k = 0, \quad (3.32a)$$

and

$$\tilde{\omega} - 2\omega + 2\omega^3 + 2k = 0. \quad (3.32b)$$

The solutions to (3.32) can be expressed in terms of elliptic functions and correspondingly, the Dirac spinor solutions are given by:

$$\xi(\psi) = \begin{bmatrix} e^{i\phi} \exp\left(\frac{2}{2i} \int_{\psi_0}^{\psi} \lambda d\psi\right) \\ 0 \end{bmatrix}, \quad (3.33a)$$

and

$$\eta(\psi) = \begin{bmatrix} 0 \\ e^{i\phi'} \exp\left(\frac{3}{2i} \int_{\psi_0}^{\psi} \omega d\psi\right) \end{bmatrix}. \quad (3.33b)$$

For a different choice of initial conditions: $g\xi_0$ and $g\eta_0$, with g a constant element of $U(2)$, the solution can be written as: $g\xi(\psi)$, $g\eta(\psi)$, and $gH(\psi)g^\dagger$.

4. SUMMARY

In this work, we have examined the reduction by $SO(4)$ symmetry of $SU(N)$ Yang-Mills fields minimally coupled to massless Dirac spinor fields transforming under the lowest dimensional representation(s) of $SU(N)$. We showed that only a restricted class of homomorphisms characterizing the $SO(4)$ invariant $SU(N)$ gauge fields allows non-vanishing invariant Dirac spinor fields. These homomorphisms are specified by sets of consecutive spins (highest weights) increasing by one unit and starting with value $1/2$. We also explicitly reduced the $SU(2n)$ Yang-Mills-Dirac systems corresponding to the "homogeneous" homomorphisms with spin $1/2$. An interpretation of these systems in terms of Hamiltonian systems with symmetry constrained by a condition on the associated moment maps was formulated. This condition requires the vanishing of the traceless part of the conserved quantity, which is composed of the sum of contributions coming from the gauge and spinor fields. In the case where each contribution of the traceless part equals zero, the spinor fields is trivial for every $n > 2$ and the solution to the corresponding sourceless Yang-Mills systems is presented in reference 1. For the gauge groups $SU(2)$ ($n = 1$) and $SU(4)$ ($n = 2$), the coupled systems can be completely solved in terms of elliptic functions, with non-trivial spinor solutions. Finally, setting one of the two residual spinor components to zero, we were able to further reduce the $SU(2n)$ Yang-Mills-Dirac system to derive a set of one-dimensional systems interacting via a Calogero type potential.

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APPENDIX

We discuss here the reduction of the Yang-Mills-Dirac equations by non-homogeneous homomorphisms of $SU(2)$. Specifically, we find from the field equations that the homomorphisms which may lead to non-zero $SO(4)$ invariant spinor solutions are consisting of a sequence (2.7) with $j_0 = \frac{1}{2}$, $j_1 = \frac{3}{2}, \dots, j_n = \frac{1}{2} + n$ ($n \geq 1$). To see this, we consider the θ^0 component (or constraint component) of the Yang-Mills equations with spinor sources.

Let us write any homomorphism λ as a sum of homogeneous parts associated with disjoint sets of consecutive highest weights: $\left\{ \frac{1}{2}, \frac{3}{2}, \dots, \frac{1}{2} + n \right\} \oplus \left\{ j_0^1, j_0^1 + 1, \dots, j_0^1 + n^1 \right\} \oplus \dots \oplus \left\{ j_0^\ell, j_0^\ell + 1, \dots, j_0^\ell + n^\ell \right\}$, ($n^1, \dots, n^\ell, \ell \in NU\{0\}$):

$$\lambda : (g, g) \in SU(2)_D \mapsto \sum_{a=0}^{\ell} \sum_{i=0}^{n^a} \oplus (1_{m_i^a} \otimes D^{j_0^a + i}(g)) \in SU \left(\sum_{a=0}^{\ell} \sum_{i=0}^{n^a} M_i^a \right), \quad (A.1)$$

with $j_0^0 = \frac{1}{2}$, $M_i^a = [(2j_0^a + 1) + 2i] m_i^a$, $m_i^a, n^a \in NU\{0\}$.

In a convenient gauge, we know that the θ^0 contribution to the $SO(4)$ invariant gauge field can be made to vanish. Corresponding to the homomorphism (4.1), the invariant field ω is expressed as the direct sum of overlapping sums of contributions (2.10) for each disjoint set. For instance, if $\ell = 0$ (i.e., one set of consecutive weights with $j_0^0 = \frac{1}{2}$, $M_i = M_i^0$, $m_i = m_i^0$, and $n = n^0$):

$$\omega(\psi) = \theta^i \begin{bmatrix} H_0 \otimes \tau_i & G_0 \otimes \Omega\left(\frac{1}{2}, \frac{1}{2}\right) \\ -G_0^\dagger \otimes \Omega\left(\frac{3}{2}, \frac{1}{2}\right)^\dagger(\tau_i) & H_1 \otimes D^{\frac{3}{2}}(\tau_i) \\ 0 & \ddots \\ \vdots & \vdots \\ H_{n-2} \otimes D^{n-\frac{3}{2}}(\tau_i) & G_{n-2} \otimes \Omega\left(n-\frac{1}{2}, n-\frac{1}{2}\right)(\tau_i) \\ -G_{n-2}^\dagger \otimes \Omega\left(n-\frac{1}{2}, n-\frac{3}{2}\right)^\dagger(\tau_i) & H_{n-1} \otimes D^{n-\frac{1}{2}}(\tau_i) \\ & -G_{n-1}^\dagger \otimes \Omega\left(n+\frac{1}{2}, n-\frac{1}{2}\right)^\dagger(\tau_i) & H_n \otimes D^{n+\frac{1}{2}}(\tau_i) \end{bmatrix} \quad (A.2)$$

where $H_i \in \mathcal{H}(m_i)$, G_i are complex matrices, and the matrices $\Omega^{(j+1, j)}(\tau_i)$ are expressed in terms of Clebsch-Gordan coefficients coupling $j+1$ and j to 1. We add (direct sum) to the right hand side of (A.2) a similar matrixial expression for each supplementary disjoint sequence.

As we recall from section 2, only the $j = \frac{1}{2}$ homogeneous part of the embedding allows a non-trivial contribution to the $SO(4)$ invariant spinor field transforming under the fundamental representation of the gauge group. Explicitly, we get for $\Psi \in \mathcal{C}^{4 \times \sum_a M_i^a}$:

$$\Psi(\psi) = \begin{bmatrix} \xi^T \otimes \sigma_2 & A = 0 \\ \eta^T \otimes 1_2 & \end{bmatrix}, \quad (A.3)$$

where $A \in C^{4 \times M'}$, $M' = \sum_{i=1}^n M_i + \sum_{a=1}^l \sum_{i=0}^{n^i} M_i^a$, and $\xi, \eta \in C^{m_0 \times 1}$.

We now insert the invariant fields in the equation (3.6) and evaluate the θ^0 component of each member. The θ^0 part of the spinor current (J) is found to be:

$$J|_{\theta^0} = i\alpha(|\eta|^2 + |\xi|^2) \left(-\frac{1}{M_0} 1_{M_0} \oplus \frac{1}{M'} 1_{M'} \right) \theta^0, \quad (A.4)$$

where $\alpha = \left(\frac{-iM_0 M'}{M_0 + M'} \right)$ is a constant.

We also compute the part $*D * F|_{\theta^0}$ of the Yang-Mills-Dirac equation (3.6). However, each disjoint sequence is associated with a different element (i.e., matrix of type (A.2)) of the direct sum in the expression for the invariant ω and the field equations do not mix the components of different elements. Thus, let us consider the element corresponding to the sequence of highest weights beginning with $\frac{1}{2} : \{\frac{1}{2}, \frac{3}{2}, \dots, j_n = \frac{1}{2} + n\}$. Taking the trace on each side of the θ^0 contribution of (3.6), we derive that:

$$(1) \quad \frac{3}{(2j_n + 1)} \mathcal{G}_{n-1} = i \frac{\alpha}{M'} (|\eta|^2 + |\xi|^2) m_n,$$

$$(2 \leq i \leq n) \quad \frac{3}{(2j_{n+1-i} + 1)} (-\mathcal{G}_{n+1-i} + \mathcal{G}_{n-i}) = i \frac{\alpha}{M'} (|\eta|^2 + |\xi|^2) m_{n+1-i},$$

$$(n+1) \quad 3\mathcal{G}_0 = i\alpha(|\eta|^2 + |\xi|^2), \quad (A.5)$$

where by definition

$$\mathcal{G}_i \equiv \text{tr}(G_i \dot{G}_i^+ - \dot{G}_i G_i^+), \quad (A.6)$$

for $i = 1, \dots, n+1$.

From these $(n+1)$ relations, it follows that

$$\frac{M_i}{M'} \sum_{i=1}^n (|\eta|^2 + |\xi|^2) = (|\eta|^2 + |\xi|^2). \quad (A.7)$$

But this is only satisfied if either: (1) $m_i^a = 0$ for every $a \geq 1$ and every $i \in \{0, \dots, n^a\}$, or (2) $|\eta|^2 + |\xi|^2 = 0$, which implies that $\xi = \eta = 0$.

We thus conclude from the reduced field equations (θ^0 part) that only those "non-homogeneous" homomorphisms associated with the sequence of spins of $SU(2) : \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{1}{2} + n\}$, may lead to a coupling with a non-trivial $SO(4)$ invariant lowest dimensional multiplet of spinors. All the other non-homogeneous homomorphisms require that $\xi = \eta = 0$ and hence reduce the problem to Yang-Mills systems without sources.