AN EASY WAY OF DOING ANGULAR MOMENTUM CALCULATIONS

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A B S T R A C T

We show how the vector spherical harmonics can be expressed by the multipole transition matrices (MTM). This, together with previous work on the MTM formalism, allows for easy angular momentum calculations. We show as an example how it may be used to find the hyperfine splitting operators in a quite straightforward way for glueball calculations in the MIT bag model. A quite remarkable detail in these calculations is that one doesn't need any Clebsch-Gordan coefficients or Fierz transformations. The MTM's do the job automatically.

1 INTRODUCTION

In two previous papers^[1,2] - hereafter called paper I and II - general formulas for manifest rotation symmetric expressions for angular momentum wave functions for spin s , orbital angular momentum ℓ and total angular momentum j written in terms of Barut et al's^[3] multipole transition matrices (MTM) were presented. It should be stressed, as was also stressed in I and II, that nothing can be done with the MTM formalism that cannot be done with ordinary angular momentum formalism. However, one finds that explicit angular momentum calculations in central field problems often turns out to be easier in this formalism than by standard methods. In addition, the very compact structure of the MTM formalism gives a very 'transparent' result, displaying the spin structure automatically.

In I and II it was shown how the spherical harmonics $Y_\ell^m(\Omega)$ and the spinor spherical harmonics $\phi(s\ell j; m)$ can generally be expressed by MTM's. In this paper we complete this picture by showing how the vector spherical harmonics $\underline{Y}_{j\ell m}(\Omega)$ can also easily be expressed by MTM's. Then we give a rather extensive example to illustrate the power of the formalism. In an appendix we give a formula for the most general invariant product of two MTM's.

The use of the MTM formalism has already proved itself very convenient for calculations involving quarks in a bag model^[4], and the new formulas presented in this note could profitably be used for calculations involving gluons in a bag model, or any other model where one uses the $\underline{Y}_{j\ell m}(\Omega)$'s. As an example we show how to calculate the four-gluon vertex in "bagged" QCD.

2 ANGULAR MOMENTUM EIGENFUNCTIONS IN TERMS OF MTM'S

The angular momentum eigenfunction for intrinsic spin $\underline{S}$, orbital angular momentum $\underline{L}$ and total angular momentum $\underline{J} = \underline{L} + \underline{S}$ is given by the well-known formula

$$\phi(s\ell j; m) = \sum_{\substack{m_\ell + m_s = m \\ \ell \quad s}} \langle \ell m_\ell s m_s | j m \rangle Y_\ell^{m_\ell}(\Omega) \chi_{m_s}^{[s]} \quad , \quad (2.1)$$

where the $\chi_{m_s}^{[s]}$ is a $(2s+1)$ -component spinor (i.e. $\chi_{m_s}^{[1/2]}$ is a Pauli spinor).

In paper I we showed that $\phi(slj;m)$ can be expressed by the $(2s+1) \times (2j+1)$ dimensional MTM for 2^l -pole transition $T_{k_1 \dots k_l}^{[s,j]}$ in the following manifest rotation symmetric way:

$$\phi(slj;m) = c(slj) \hat{r}_{k_1} \dots \hat{r}_{k_l} T_{k_1 \dots k_l}^{[s,j]} \chi_m^{[j]}, \quad (2.2)$$

where $\hat{r} = \hat{r}(\Omega)$ is the unit vector in the $\underline{r}$ -direction. The MTM above is $(2s+1) \times (2j+1)$ dimensional. This is always given by the pair of upper indices within square brackets, i.e. as $[s,j]$. If the MTM is quadratic we only give one of the indices. The symbol T is used to symbolize a general MTM. For the few lowest MTM we use as before the symbol S (or σ if $s=1/2$) to indicate the dipole matrix, K to indicate quadrupole and σ again for the octupole matrices. General formulas for $c(slj)$ are given in paper II. For $s=0$ we get the spherical harmonics by using eqs. (2.1) and (2.2):

$$Y_{\ell}^{m_{\ell}}(\Omega) = c(0\ell\ell) \hat{r}_{k_1} \dots \hat{r}_{k_{\ell}} T_{k_1 \dots k_{\ell}}^{[0,\ell]} \chi_{m_{\ell}}^{[\ell]} \quad (2.3)$$

The different MTM's satisfy different algebraic relations, and we refer to papers I and II and to the literature [5,6]. Some examples of algebraic relations between MTM's are given in chapter 3 and appendix A.

The vector spherical harmonics are defined by (see for instance Edmonds^[7]):

$$Y_{\sim j\ell m}(\Omega) = \sum_{\substack{m_{\ell} + m_s = m \\ \ell + m_s = m}} \langle \ell m_{\ell} | m_s | j m \rangle Y_{\ell}^{m_{\ell}}(\Omega) \varepsilon_{\sim m_s}, \quad (2.4)$$

where $\varepsilon_{\sim m_s}$ are the spherical unit vectors. These are given by the cartesian unit vectors $\underline{e}_i$ by:

$$\underline{\xi}_{\pm} = \mp \frac{1}{\sqrt{2}} (\underline{e}_x \pm i \underline{e}_y) , \quad \underline{\xi}_0 = \underline{e}_z \quad (2.5)$$

The spherical unit vectors can also be expressed by the three MTM's $\underline{\sigma}^{[0,1]}$ given by

$$\underline{\sigma}_x^{[0,1]} = \frac{1}{\sqrt{2}} (-1, 0, 1) , \quad \underline{\sigma}_y^{[0,1]} = -\frac{i}{\sqrt{2}} (1, 0, 1) , \quad \underline{\sigma}_z^{[0,1]} = (0, 1, 0) , \quad (2.6)$$

and we see that

$$\underline{\xi}_{m_s} = \underline{\sigma}^{[0,1]} \chi_m^{[1]} \quad (2.7)$$

Inserting now eq. (2.7) into eq. (2.4) and comparing with eq. (2.1) we see that

$$\underline{Y}_{j\ell m}(\Omega) = \underline{\sigma}^{[0,1]} \phi(s=1, \ell, j; m) , \quad (2.8)$$

i.e. the vector spherical harmonics $\underline{Y}_{j\ell m}(\Omega)$ are nothing but the spinor spherical harmonics $\phi(s\ell j; m)$ multiplied from the left by the MTM $\underline{\sigma}^{[0,1]}$. Since it is already shown in paper I and II how to express $\phi(s\ell j; m)$ by the MTM's (see eq. (2.2)), we find that

$$\underline{Y}_{j\ell m}(\Omega) = c(s=1, \ell, j) \hat{r}_{k_1} \dots \hat{r}_{k_\ell} \underline{\sigma}^{[0,1]} T_{k_1 \dots k_\ell}^{[1,j]} \chi_m^{[j]} , \quad (2.9)$$

with the only possible ℓ -values given by $\ell=j\pm 1, j$. In appendix A we give some explicit expressions for $\underline{Y}_{j\ell m}(\Omega)$'s for some of the lowest values of ℓ .

Eqs. (2.8) and (2.9) constitute the main new results of this note, and we now proceed to comment on a few convenient areas of application of these formulas.

3 APPLICATIONS

In paper I it was shown how calculations involving integrals of Y_{ℓ}^m 's and $\phi(s\ell j; m)$'s are essentially all trivial integrals when the Y_{ℓ}^m 's and $\phi(s\ell j; m)$'s are expressed as in eqs. (2.2) and (2.3). The same discussion will of course also apply to integrals involving $Y_{j\ell m}$'s in addition to the above-mentioned functions. An additional advantage of using the MTM formalism is that the spin-structure of different quantities come out automatically and trivially simply by using the different multiplication rules which the different MTM's satisfy^[1,2,5,6]. That this really leads to simple calculations with the spin-content clearly displayed was demonstrated for instance in the bag-model calculations involving $P_{1/2}$ and $P_{3/2}$ quarks in ref. [4].

That this formalism is also convenient to use in calculations involving $Y_{j\ell m}$'s will now be shown in a rather detailed and extensive example. We pick an old example from Barnes, Close and Monaghan^[8]. They show how to calculate all necessary vertices in "bagged" QCD to calculate the hyperfine splitting of glueballs, using the lowest confined gluon-modes. To show the power of the present formalism we consider the 4-gluon vertex with the most complicated spin-structure considered by Barnes et al.^[8], i.e. the crossed TM-TE vertex (see eq. (3.22)). We show (see below) how we can easily get the correct spin-structure of this vertex by simply : (i) Using the MTM expressions for the $Y_{j\ell m}$'s. (ii) Using a few multiplication rules for MTM's. (iii) Doing a few trivial angular integrations.

The calculation sketched above goes as follows: Start with the Hamiltonian of the vertex of eq. (3.22) written as by Barnes et al^[8]:

$$H_I(gg \rightarrow gg) = \frac{1}{2} g^2 \int \int^{x_{ac} x_{bd}} M_{AB,CD} a_c^{\dagger} a_D^{\dagger} a_A a_B \quad (3.1)$$

with

$$M_{AB,CD} = \int_{Bag} d\vec{r} \left(\vec{A}_C^{\dagger} \times \vec{A}_A \right) \cdot \left(\vec{A}_D^{\dagger} \times \vec{A}_B \right) \quad (3.2)$$

(when neglecting a term which gives zero for colour-less s-channel glueball states). We consider only the lowest TE and TM modes with $j=1$. All matrices in the rest of this chapter are quadratic matrices for spin=1 if not explicitly stated otherwise, and all spinors χ_m are thus three-dimensional. Then the gluon wave functions are given by:

$$A_{\sim TE}^m = -i \sqrt{\frac{3}{8\pi}} \alpha_1 j_1(\omega r) \hat{r} \times \hat{z} \chi_m \quad (3.3)$$

$$A_{\sim TM}^m = \sqrt{\frac{1}{6\pi}} \alpha_1' \hat{z} \left\{ j_0(\omega' r) - \frac{3}{2} j_2(\omega' r) \hat{r}_k \hat{r}_n K_{kn} \right\} \chi_m, \quad (3.4)$$

where we have used eqs. (A.1), (A.3) and (A.5) to write the gluon wave functions in terms of MTM's. The quadrupole matrices for spin=1 are defined as^[9]:

$$K_{kn} = \frac{1}{2} (S_k S_n + S_n S_k) - \frac{2}{3} \delta_{kn} \quad (3.5)$$

For typographical reasons we suppress the unit-matrix where it is obvious that it should be, for instance multiplied by the j_0 within the parenthesis in eq. (3.4). The gluon modes are $\omega=2.74/R$ and $\omega'=4.49/R$ for the TE and TM modes respectively, where R is the bag-radius, and α_1 and α_1' are normalization constants.

For completeness we now give the necessary multiplication rules for the present calculation. These (and more) can be found in refs. [1,2,5,6,9]:

$$S_k S_n = \frac{2}{3} \delta_{kn} + \frac{i}{2} \epsilon_{knp} S_p + K_{kn} \quad (3.6)$$

$$S_k K_{np} = -\frac{1}{6} \delta_{np} S_k + \frac{1}{4} (\delta_{kn} S_p + \delta_{kp} S_n) + \frac{i}{2} (\epsilon_{knt} K_{tp} + \epsilon_{kpt} K_{tn}) \quad (3.7)$$

The above properties of the MTM's, plus eq. (A.14) and the general properties

$$\left(T_{k_1 \dots k_\ell}^{[s,j]} \right)^\dagger = T_{k_1 \dots k_\ell}^{[j,s]} \quad (3.8)$$

$$T_{kkk_3 \dots k_\ell}^{[s,j]} = 0 \quad (3.9)$$

and the fact that the MTM's are completely symmetric in all vector indices is all the MTM properties that are needed to do the present calculation.

In addition we need the following trivial angular integrals:

$$\int_{4\pi} d\Omega = 4\pi, \quad \int_{4\pi} d\Omega \hat{r}_k \hat{r}_n = \frac{4\pi}{3} \delta_{kn}$$

$$\int_{4\pi} d\Omega \hat{r}_k \hat{r}_n \hat{r}_p \hat{r}_t = \frac{4\pi}{15} (\delta_{kn} \delta_{pt} + \delta_{kp} \delta_{nt} + \delta_{kt} \delta_{np}) \quad (3.10)$$

(generally one needs a formula for the integral of an arbitrary number of $\hat{r}_i$, see ref. [1]).

Using the multiplication rules it is a simple and straightforward task to show that

$$A_{\sim TM}^{\dagger} \times A_{\sim TE}^A = -\frac{i}{18} \chi_c^{\dagger} \left\{ 4(3f_0 + f_2) \hat{r}_c + 3i(3f_0 - 2f_2) \hat{r}_c \times \underline{z} \right. \\ \left. + 6(3f_0 - 2f_2) \hat{r}_k \hat{r}_n K_{kn} + 18f_2 \hat{r}_c \hat{r}_k \hat{r}_n K_{kn} \right\} \chi_A, \quad (3.11)$$

where f_0 and f_2 are functions of ωr and $\omega' r$ only:

$$f_0(r) = \frac{\alpha_1 \alpha'_1}{4\pi} j_1(\omega r) j_0(\omega' r), \quad f_2(r) = -\frac{3}{2} \frac{\alpha_1 \alpha'_1}{4\pi} j_1(\omega r) j_2(\omega' r) \quad (3.12)$$

In eq. (3.11), the gluon "spinors" $\chi_C^{[1]}$ and $\chi_A^{[1]}$ can of course be suppressed, and eq. (3.11) can be taken as an operator equation for operators working on particle 1 (A-C).

The operator working on particle 2 (B-D) can be obtained from eq. (3.11) by noting that

$$A_{\sim TE}^{\dagger} \times A_{\sim TM} = - \left(A_{\sim TM}^{\dagger} \times A_{\sim TE} \right)^{\dagger} \quad (3.13)$$

Using now eqs. (3.11) and (3.13) in (3.2) together with the trivial angular integrations of eq. (3.10), we immediately find the vertex function $M_{AB,CD}$ expressed as a sum of terms with explicit multipole structure. In addition, this operator is of course independent of the

external spin polarizations of the gluons, and therefore we suppress the spinors $\chi^{[1]}$ and write

$$M \left(\begin{array}{c} \text{TE} \\ \text{TM} \end{array} \otimes \begin{array}{c} \text{TM} \\ \text{TE} \end{array} \right) = f_3 \mathbf{1} \otimes \mathbf{1} + f_4 \underline{S} \otimes \underline{S} + f_5 K_{kn} \otimes K_{kn}, \quad (3.14)$$

where the first operator always works on particle 1 and the second operator works on particle 2. The numbers f_3, f_4 and f_5 are given by

$$\begin{bmatrix} f_3 \\ f_4 \\ f_5 \end{bmatrix} = -\frac{1}{4\pi} (d_1 d_1')^2 \int_0^R dr r^2 j_1^2 \begin{bmatrix} \frac{1}{9} (2j_0 - j_2)^2 \\ \frac{1}{6} (j_0 + j_2)^2 \\ \frac{1}{30} (10j_0^2 + 8j_0 j_2 + 7j_2^2) \end{bmatrix}, \quad (3.15)$$

where $j_1 \equiv j_1(\omega r)$, $j_0 \equiv j_0(\omega' r)$ and $j_2 \equiv j_2(\omega' r)$ in the equation above. This can be shown to be the same as the result of Barnes et al.^[8]. The matrices $\mathbf{1}$, $\underline{S}$ and K_{kn} are the unit matrix, spin-matrices and quadrupole-matrices for spin=1 respectively. Eq. (3.14) displays clearly the general spin-structure of a Hamiltonian for any theory with rotational invariance, i.e. of the form

monopole $\times$ monopole + dipole $\times$ dipole + quadrupole $\times$ quadrupole + ...,
where of course some of the terms may be vanishing.

Using the MTM formalism in recalculating the different vertices and diagrams considered by Barnes et al.^[8], one easily gets the results below. In these calculations there is no need to carry out Fierz transformations to display the spin-structure, and we don't see a single Clebsch-Gordan coefficient or n-j symbol! The last steps are normally unavoidable in calculations of this kind^[8]. The bag-integrals (t_i , f_i and c_i) are given by expressions similar to eq. (3.15) and can be found in ref. [8]. For all the vertices, the operators (in spin space) work in a two-particle space. For the three-gluon vertex particle 1 is A (created in the vertex), and particle 2 is C-B. The spin-structure is simply the dipole-dipole term $\underline{\sigma}^{[1,0]} \times \underline{S}$. To find the expectation-value we simply insert $\chi^{[1]}$'s where possible, i.e.

$$\chi_A^\dagger \underline{\sigma}^{[1,0]} \otimes \chi_B^\dagger \underline{S} \chi_C \quad (3.16)$$

For the four-gluon and Coulomb vertices, particle 1 is A-C and 2 is B-D. The labels TE and TM refer as always to the lowest bag-model gluon modes with $\omega R=2.74$ and $\omega' R=4.49$ respectively. Our results are:

$$\begin{array}{c} \text{A, TE} \\ \text{C, TE} \end{array} \begin{array}{c} \text{B, TE} \end{array} = -i \frac{g}{2} \frac{t_1}{R} \left\{ \begin{array}{c} abc \\ \sim \end{array} \begin{array}{c} [1,0] \\ \otimes \end{array} \begin{array}{c} \sim \end{array} \right. \quad (3.17)$$

$$\begin{array}{c} \text{A, TM} \\ \text{C, TM} \end{array} \begin{array}{c} \text{B, TE} \end{array} = -i \frac{g}{2} \frac{t_2}{R} \left\{ \begin{array}{c} abc \\ \sim \end{array} \begin{array}{c} [1,0] \\ \otimes \end{array} \begin{array}{c} \sim \end{array} \right. \quad (3.18)$$

$$\begin{array}{c} \text{A, TE} \\ \text{C, TM} \end{array} \begin{array}{c} \text{B, TE} \end{array} = 0 \quad (3.19)$$

$$\begin{array}{c} \text{A, TE} \\ \text{B, TE} \end{array} \begin{array}{c} \text{C, TE} \\ \text{D, TE} \end{array} = -\frac{d_1}{R} \left\{ \begin{array}{c} xac \quad xbd \\ f_1 \end{array} \right\} \begin{array}{c} \sim \otimes \sim \end{array} \quad (3.20)$$

$$\begin{array}{c} \text{A, TE} \\ \text{B, TM} \end{array} \begin{array}{c} \text{C, TE} \\ \text{D, TM} \end{array} = -\frac{d_1}{R} \left\{ \begin{array}{c} xac \quad xbd \\ f_2 \end{array} \right\} \begin{array}{c} \sim \otimes \sim \end{array} \quad (3.21)$$

$$\begin{array}{c} \text{A, TE} \\ \text{B, TM} \end{array} \begin{array}{c} \text{C, TM} \\ \text{D, TE} \end{array} = -\frac{d_1}{R} \left\{ \begin{array}{c} xac \quad xbd \\ f \end{array} \right\} \left[\begin{array}{c} f_3 \mathbb{1} \otimes \mathbb{1} + f_4 \sim \otimes \sim + f_5 K_{kn} \otimes K_{kn} \end{array} \right] \quad (3.22)$$

$$\begin{array}{c} \text{A, TE} \\ \text{B, TE} \end{array} \begin{array}{c} \text{C, TE} \\ \text{D, TE} \end{array} = -\frac{d_1}{R} \left\{ \begin{array}{c} xac \quad xbd \\ f \end{array} \right\} \left[\begin{array}{c} c_1 \mathbb{1} \otimes \mathbb{1} + c_2 K_{kn} \otimes K_{kn} \end{array} \right] \quad (3.23)$$

$$\begin{array}{c}
 \text{A, TE} \quad \text{C, TE} \\
 \text{B, TM} \quad \text{D, TM}
 \end{array}
 \begin{array}{c}
 \text{---} \\
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 \text{---} \\
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 \end{array}
 = -\frac{\alpha_s}{R} \int^{x_{ac}} \int^{x_{bd}} \left[c_3 \mathbb{1} \otimes \mathbb{1} + c_4 K_{kn} \otimes K_{kn} \right] \quad (3.24)$$

$$\begin{array}{c}
 \text{A, TE} \quad \text{C, TM} \\
 \text{B, TM} \quad \text{D, TE}
 \end{array}
 \begin{array}{c}
 \text{---} \\
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 \text{---}
 \end{array}
 = -\frac{\alpha_s}{R} \int^{x_{ac}} \int^{x_{bd}} c_5 \tilde{S} \otimes \tilde{S} \quad (3.25)$$

It should also be noted here that the vertices are all of the general form monopole $\times$ monopole+dipole $\times$ dipole+quadrupole $\times$ quadrupole (with some of the terms absent in some vertices).

The spin "operators" Φ, S and T used by Barnes et al. [8] are actually expectation values of spin operators. It is a simple exercise using the multiplication relations for the MTM's above and eq. (2.7) to verify that

$$\begin{aligned}
 \Phi_1 &\equiv \epsilon_{\lambda_A} \cdot \epsilon_{\lambda_C}^+ = \chi_C^+ \mathbb{1} \chi_A, \quad S_1 \equiv i \epsilon_{\lambda_A} \times \epsilon_{\lambda_C}^+ = \chi_C^+ \tilde{S} \chi_A, \\
 T_1^{ij} &\equiv \frac{1}{2} (\epsilon_A^i \epsilon_C^{j+} + \epsilon_A^j \epsilon_C^{i+}) - \frac{1}{3} \delta^{ij} \epsilon_{\lambda_A} \cdot \epsilon_{\lambda_C}^+ = -\chi_C^+ K_{ij} \chi_A, \quad (3.26)
 \end{aligned}$$

thus showing that Barnes et al. [8] use the conventional definitions of the spin operators, except for the minus sign in T^{ij} .

4 CONCLUSIONS

We have shown how the vector spherical harmonics can be expressed by the multipole transition matrices (MTM). This leads to simple angular momentum calculations with the spin structure automatically displayed in problems involving vector spherical harmonics.

In an appendix (app.B), we also give the formula for the most general invariant matrix product of two MTM's.

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Appendix A

We here use the formula of eq. (2.9) and the formulas for $c(s\ell j)$ given in paper II to write explicit expressions for a few $Y_{j\ell m}$'s.

$\ell=0$

$$j=1 \quad Y_{\sim 10m}(\omega) = \sqrt{\frac{1}{4\pi}} \sigma_{\sim}^{[0,1]} \chi_m^{[1]} \quad (A.1)$$

$\ell=1$

$$j=0 \quad Y_{\sim 01m}(\omega) = -\sqrt{\frac{1}{4\pi}} \hat{r}_{\sim} \quad (A.2)$$

$$j=1 \quad Y_{\sim 11m}(\omega) = -i\sqrt{\frac{3}{8\pi}} \hat{r}_{\sim} \times \sigma_{\sim}^{[0,1]} \chi_m^{[1]} \quad (A.3)$$

$$j=2 \quad Y_{\sim 21m}(\omega) = \sqrt{\frac{3}{4\pi}} \hat{r}_k \hat{r}_{\sim} \sigma_{\sim}^{[0,1]} \sigma_k^{[1,2]} \chi_m^{[2]} \quad (A.4)$$

$\ell=2$

$$j=1 \quad Y_{\sim 12m}(\omega) = \sqrt{\frac{9}{8\pi}} \hat{r}_k \hat{r}_n \sigma_{\sim}^{[0,1]} K_{kn}^{[1]} \chi_m^{[1]} \quad (A.5)$$

$$j=2 \quad Y_{\sim 22m}(\omega) = -\sqrt{\frac{5}{4\pi}} \hat{r}_k \hat{r}_n \sigma_{\sim}^{[0,1]} K_{kn}^{[1,2]} \chi_m^{[2]} \quad (A.6)$$

$$j=3 \quad Y_{\sim 32m}(\omega) = \sqrt{\frac{15}{8\pi}} \hat{r}_k \hat{r}_n \sigma_{\sim}^{[0,1]} K_{kn}^{[1,3]} \chi_m^{[3]} \quad (A.7)$$

In practical applications it may sometimes be convenient to perform the matrix multiplications in eqs. (A.4)-(A.7). The multiplication rule

$$\sigma_k^{[0,12]} \sigma_n^{[1,22]} = K_{kn}^{[0,22]} \quad (A.8)$$

may be used to write eq. (A.4) as

$$Y_{\sim 21m}(\Omega) = \sqrt{\frac{3}{4\pi}} e_{\sim k} \hat{r}_n K_{kn}^{[0,22]} \chi_m^{[22]} \quad (A.9)$$

for instance. Furthermore, the multiplication rule

$$\sigma_k^{[0,12]} K_{np}^{[12]} = \frac{1}{3} \delta_{np} \sigma_k^{[0,12]} - \frac{1}{2} \left(\delta_{kn} \sigma_p^{[0,12]} + \delta_{kp} \sigma_n^{[0,12]} \right) \quad (A.10)$$

may be used to write eq. (A.5) as

$$Y_{\sim 12m}(\Omega) = \sqrt{\frac{1}{8\pi}} \left(\sigma_{\sim}^{[0,12]} - 3 \hat{r}_{\sim} (\hat{r}_{\sim} \cdot \sigma_{\sim}^{[0,12]}) \right) \chi_m^{[12]} \quad (A.11)$$

which is the same as given by ref. [8] by using eq.(2.7).

Another relation which is useful in manipulations of the $Y_{j\ell m}$'s is

$$\sigma_k^{[0,12]} K_{np}^{[1,22]} = \frac{i}{2} \left(\varepsilon_{knt} K_{tp}^{[0,22]} + \varepsilon_{kpt} K_{tn}^{[0,22]} \right) \quad (A.12)$$

The following relations are also useful^[1,2,5]:

$$\sigma_k^{[0,12]} \sigma_n^{[1,0]} = \delta_{kn} \quad (A.13)$$

$$\sigma_k^{[1,0]} \sigma_n^{[0,12]} = \frac{1}{3} \delta_{kn} + \frac{i}{2} \varepsilon_{knp} S_p^{[12]} - K_{kn}^{[12]} \quad (A.14)$$

$$\sigma_k^{[0,12]} S_n^{[12]} = i \varepsilon_{knp} \sigma_p^{[0,12]} \quad (A.15)$$

A lot of similar relations can be found in refs. [1,2,5,6,9].

A general formula useful in the context of vector-spherical harmonics calculations is

$$\sigma_{k_1}^{[0,12]} T_{k_2 \dots k_j}^{[1,j]} = T_{k_1 \dots k_j}^{[0,j]} \quad (A.16)$$

which is the generalization of eq. (A.8). That the left-hand side of eq. (A.16) is proportional to the right-hand side is evident. The proportionality factor has to be 1 to be consistent with our earlier conventions.

Appendix B

INVARIANT MATRIX PRODUCTS

In calculations one often encounters the problem of determining the right-hand side of the invariant matrix product equation

$$\overline{T}_{k_1 \dots k_\ell}^{[j,s]} \overline{T}_{k_1 \dots k_\ell}^{[s,j]} = A(s\ell j) \mathbb{1}^{[j]} \quad (\text{B.1})$$

where $\mathbb{1}^{[j]}$ is a $(2j+1)$ -dimensional unit matrix. This can most easily be found for all possible values of s , ℓ and j by using the invariant scalar product

$$\int_{4\pi} d\Omega \left| \phi(s\ell j; m) \right|^2 = 1 \quad (\text{B.2})$$

and then inserting the MTM representation eq. (2.2) for $\phi(s\ell j; m)$. After a simple angular integration one then finds the general formula

$$A(s\ell j) = \left| \frac{c(0\ell\ell)}{c(s\ell j)} \right|^2 \quad (\text{B.3})$$

All values of $A(s\ell j)$ can then be found by using the formulas for $c(s\ell j)$ from paper II. One finds for instance for $j \geq s \geq 1$:

$$A(s\ell j) = 2^{j-s-\ell} \frac{(s+\ell+j+1)! (\ell-s+j)! (\ell+s-j)! (2s)!}{(2j+1)! (s-\ell+j)! (2\ell)!} \quad (\text{B.4})$$

For $s \geq j \geq 1$ one can use the reflection formula for the $c(s\ell j)$'s to obtain

$$(2j+1) A(s\ell j) = (2s+1) A(j\ell s) \quad (\text{B.5})$$

and then apply eq. (B.4) for $A(s\ell j)$.

Some special cases of eq. (B.4) are of particular interest. For $s=j \geq 1$ we get quadratic spin= s matrices. For $\ell=1$ we then of course rediscover the well-known quadratic spin Casimir operator:

$$A(s1s) = s(s+1) \quad (\text{B.6})$$

For $\ell=2$ we get similarly the quadratic quadrupole Casimir operator:

$$A(s2s) = \frac{1}{6} s(s+1)(2s-1)(2s+3) \quad (B.7)$$

For $s=0$ and $\ell=j$ we find from eq. (B.3) that $A(0\ell\ell)=1$, and $A(\ell\ell 0)$ can then be found by using eq. (B.5) (although really not fulfilling the requirements for using eq. (B.5), it can be shown to be valid also here).

When writing down general formulas for different MTM expressions like eq. (B.1), the formula for $c(s\ell j)$ in paper II etc., one should always keep in mind that these general formulas are valid for the MTM's defined by the construction procedure of refs. [1,2,3,6]. If one wants to construct new MTM's, one can in the end always assure oneself that they agree with these conventions by checking if the newly constructed MTM satisfy eq. (2.2) with the $c(s\ell j)$ -coefficients calculated according to the formulas of paper II. - One could of course also choose to define the MTM's by conventionally setting $c(s\ell j) \equiv 1$, but this would be a new definition, and we don't do this.

ERRATUM TO PAPER I:

Eq. (C.19) of ref. [1] is wrong. It should read:

$$\sigma_k^{[1,2]} \sigma_n^{[2,1]} = \frac{5}{9} \delta_{kn} - \frac{5}{12} i \varepsilon_{knp} S_p^{[1]} - \frac{1}{6} K_{kn}^{[1]}$$

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