EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

CERN/LEP-TH/85-26

ALGORITHMS FOR TRACKING OF CHARGED PARTICLES
IN CIRCULAR ACCELERATORS

F. Christoph Iselin
CERN, Geneva, Switzerland

Paper presented at the 1985 Joint Accelerator School
on Non-Linear Dynamics
30 January - 5 February, 1985,
Santa Margarita di Pula, Sardinia

Geneva, June 1985

ALGORITHMS FOR TRACKING OF CHARGED PARTICLES IN CIRCULAR ACCELERATORS

F. Christoph Iselin
CERN, Geneva, Switzerland

1. INTRODUCTION

An important problem in accelerator design is the determination of the largest stable betatron amplitude. This stability limit is also known as the **dynamic aperture**. The equations describing the particle motion are non-linear, and the Linear Lattice Functions cannot be used to compute the stability limits. The stability limits are therefore usually searched for by particle tracking. One selects a set of particles with different betatron amplitudes and tracks them for many turns around the machine. The particles which survive a sufficient number of turns are termed stable.

This paper concentrates on conservative systems. For this case the particle motion can be described by a Hamiltonian, i.e. tracking particles means application of **canonical transformations**. Canonical transformations are equivalent to **symplectic mappings**, which implies that there exist invariants. These invariants should not be destroyed in tracking.

2. NOTATION

We begin with the general Hamiltonian in Cartesian coordinates

$$H = \{(mc^2)^2 + (\mathbf{p} - e\mathbf{A})^2\}^{1/2} + e\phi$$

Where m is the particle rest mass and c is the velocity of light. $\mathbf{A}$ is the magnetic vector potential, ϕ the electric scalar potential, e the particle charge, and $\mathbf{p}$ its momentum vector.

We use the curvilinear reference system defined by Courant and Snyder [3]. In this system the vector potential has only one component along the reference curve. The scalar potential ϕ is zero by the Coulomb gauge. In this reference system the Hamiltonian becomes

$$H = \{(mc^2)^2 + p_x^2 + p_y^2 + (p_s - eA_s)^2\}^{1/2}$$

Note that p_s and A_s are the vector components in the s -direction, multiplied by $[1 + h(s)x]$, where $h(s)$ is the curvature of the reference curve [3]. It is convenient to use the arc length s along the reference curve instead of the time t as an independent variable. A convenient set of canonical variables is then

$$(x, p_x), (y, p_y), (-ct, E/c)$$

Transformation to these variables produces the final form for the Hamiltonian

$$H = -p_s = -(1 + hx) (s \cdot p) \\ = -(1 + hx) e(s \cdot A) - (1 + hx) \{(E/c)^2 - (mc)^2 - p_x^2 - p_y^2\}^{1/2}$$

We denote by $(s \cdot p)$ and $(s \cdot A)$ the scalar products of the unit vector in the s -direction with p and A respectively, i.e. the components of p and A in the s -direction. For subsequent considerations it is now useful to write the components of the phase space vector in the order

$$V = \begin{pmatrix} x \\ y \\ -ct \\ p_x \\ p_y \\ E/c \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}$$

Where the first three components are the positions, and the next three components are the momenta. With this convention the equations of motion become

$$\begin{aligned} \frac{dx}{ds} &= + \frac{\partial H}{\partial p_x} & \frac{dp_x}{ds} &= - \frac{\partial H}{\partial x} \\ \frac{dy}{ds} &= + \frac{\partial H}{\partial p_y} & \frac{dp_y}{ds} &= - \frac{\partial H}{\partial y} \\ \frac{d(-ct)}{ds} &= + \frac{\partial H}{\partial (E/c)} & \frac{d(E/c)}{ds} &= - \frac{\partial H}{\partial (-ct)} \end{aligned}$$

We also define the 6-dimensional gradient operator

$$\partial/\partial V = \begin{pmatrix} \partial/\partial v_1 \\ \partial/\partial v_2 \\ \partial/\partial v_3 \\ \partial/\partial v_4 \\ \partial/\partial v_5 \\ \partial/\partial v_6 \end{pmatrix}$$

Then we may write the equations of motion in the form

$$\frac{dV}{ds} = S \frac{\partial H}{\partial V}$$

where we make use of the matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

3. SYMPLECTICITY

Let us consider a particle with arbitrary initial conditions for $s = 0$

$$x_0, y_0, -t_0, p_{x0}, p_{y0}, E_0$$

$$\text{or } v_k(0) \text{ for } k = 1, \dots, 6$$

and obtain the solution for its orbit

$$v_i(s) = v_i(s; x_0, y_0, -t_0; p_{x0}, p_{y0}, E_0)$$

We may linearize the motion in the neighbourhood of this orbit as follows:

$$\Delta v_i(s) = \sum_{k=1}^6 \left\{ \frac{\partial v_i(s)}{\partial v_k(0)} \Delta v_k(0) \right\}$$

Using the Jacobian matrix

$$M_{ik}(s_2, s_1) = \frac{\partial v(s_2)}{\partial v(s_1)}$$

we may then write the linearized motion as

$$\Delta V(s) = M(s, 0) \Delta V(0)$$

The Hamiltonian character of the motion implies that the Jacobian is symplectic, i.e. that the condition

$$M^T S M = S, \quad \text{or} \quad M^{-1} = S^T M^T S$$

holds. The superscript T denotes the transpose of the matrix, and the matrix S has been defined in Section 2. A proof for the symplecticity of the Jacobian can be found in references [4,7,9].

4. THIN LENS CODES (KICK CODES)

Several computer programs for particle tracking make use of thin lens approximations. Examples are PATRICIA [14] by H. Wiedemann and RACETRACK [15] by A. Wrulich.

These computer codes model a linear portion of the machine by using the linear transfer matrix. Thus the transformation for a linear portion is

$$V(s_2) = M(s_2, s_1) V(s_1)$$

The transfer matrix M is symplectic and it can be accumulated before particle tracking begins. It should be noted that for the usual canonical variables, an exactly linear canonical transformation does not exist. Consider for example a drift space. The Hamiltonian is

$$H = - \{ (E/c)^2 - (mc)^2 - p_x^2 - p_y^2 \}^{1/2}$$

From which the equations of motion for the horizontal plane become

$$dx/ds = p_x \{ (E/c)^2 - (mc)^2 - p_x^2 - p_y^2 \}^{-1/2}$$

$$dp_x/ds = 0$$

Their solution

$$x_2 = x_1 + s p_x \{ (E/c)^2 - (mc)^2 - p_x^2 - p_y^2 \}^{-1/2}$$

$$p_{x2} = p_{x1}$$

is clearly nonlinear in p_x and p_y . It is however possible to use the linear approximation

$$x_2 = x_1 + s p_x / p_0$$

$$p_{x2} = p_{x1}$$

This preserves the symplecticity of the mapping, as long as p_0 is a constant.

In these codes non-linear elements are treated as thin lenses. The effect of a thin lens can be viewed as a kick acting on the particle, hence thin lens programs are often called "kick codes". Let the vector potential be expanded as a truncated Taylor series

$$A_s = \text{Re} \sum_{n=1}^N \frac{B_n (x - iy)^{n+1}}{(n+1)!}$$

In a right-handed coordinate system (x, y, s) the integrated magnetic field can then be approximated as

$$\int_0^L (B_x + iB_y) ds = L \sum_{n=1}^N \frac{B_n (x - iy)^n}{n!} = L \left\{ \frac{\partial A_s}{\partial y} - i \frac{\partial A_s}{\partial x} \right\}$$

Using the magnetic rigidity $B\rho$, the thin lens kicks are

$$\frac{\Delta p_x}{p_0} = - \frac{1}{B\rho} \int_0^L B_y ds = - \frac{1}{B\rho} \int_0^L \frac{\partial A_s}{\partial x} ds$$

$$\frac{\Delta p_y}{p_0} = + \frac{1}{B\rho} \int_0^L B_x ds = \frac{1}{B\rho} \int_0^L \frac{\partial A_s}{\partial y} ds$$

This approximation assumes that the magnetic field strength is constant along any particle orbit, and it is valid only when the transverse particle position does not change significantly as a function of s while traversing a magnet. These codes have the advantage of providing a very fast tracking method, especially when there are only a few non-linear elements.

Proof of symplecticity: It is easily verified that a matrix of the form

$$M = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$$

with symmetric A is symplectic. Now the Jacobian has the form

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a & b & 0 & 1 & 0 & 0 \\ b & c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$a = \frac{1}{B\rho} \int_0^L \frac{\partial^2 A_s}{\partial x^2} ds$$

$$b = \frac{1}{B\rho} \int_0^L \frac{\partial^2 A_s}{\partial x \partial y} ds$$

$$c = \frac{1}{B\rho} \int_0^L \frac{\partial^2 A_s}{\partial y^2} ds$$

are the derivatives of the force components with respect to the positions.

5. MATRIX METHODS

Second-order Matrix Methods were used originally by K. L. Brown et al. in TRANSPORT [1,2]. Later they were also applied by R. Servranckx in DIMAT [13] and by F. C. Iselin in MAD [11].

Let us consider a truncated Taylor expansion

$$\begin{aligned} v_i(s_2) &= \sum_{j=1}^6 R_{ij} v_j(s_1) \\ &+ \sum_{j=1}^6 \sum_{k=1}^6 T_{ijk} v_j(s_1) v_k(s_1) \\ &+ \sum_{j=1}^6 \sum_{k=1}^6 \sum_{l=1}^6 U_{ijkl} v_j(s_1) v_k(s_1) v_l(s_1) \\ &+ \text{terms of order 4 and higher} \end{aligned}$$

Truncation may affect symplecticity seriously. The Jacobian of the mapping is found by differentiation as

$$\begin{aligned} M_{ij} &= \frac{\partial v_i(s_2)}{\partial v_j(s_1)} \\ &= R_{ij} + 2 \sum_{k=1}^6 T_{ijk} v_k(s_1) + 3 \sum_{k=1}^6 \sum_{l=1}^6 U_{ijkl} v_k(s_1) v_l(s_1) \\ &+ \text{terms of order three and higher} \end{aligned}$$

Substitution in the symplecticity condition gives

$$\begin{aligned}
 S_{ij} &= \sum_{k=1}^6 \sum_{l=1}^6 M_{ki} S_{kl} M_{lj} \\
 &= \sum_{k=1}^6 \sum_{l=1}^6 R_{ki} S_{kl} R_{lj} \\
 &+ \sum_{k=1}^6 \sum_{l=1}^6 \sum_{m=1}^6 \{2R_{ki} S_{kl} T_{ljm} + 2T_{kim} S_{kl} R_{lj}\} v_m \\
 &+ \sum_{k=1}^6 \sum_{l=1}^6 \sum_{m=1}^6 \sum_{n=1}^6 \{4T_{kim} S_{kl} T_{ljn} + 3R_{ki} S_{kl} U_{ljmn} + 3U_{kimn} S_{kl} R_{lj}\} v_m v_n \\
 &+ \text{terms of order three and higher}
 \end{aligned}$$

Because of the symplecticity of the matrix R , the constant terms are equal on both sides of the equation. If the expansion is not truncated then the linear and higher terms cancel; but if we truncate the expansion, say by dropping terms of third and higher order, we omit the second-order terms arising from the U_{ijkm} in the four-fold sum. The remaining terms in this sum, arising from the T_{ijk} , vanish only in exceptional cases and the symplecticity condition is true to second order only.

An example where a polynomial expansion fails is shown in Figures 1 and 2. Given a simple accelerator cell with strong sextupole components in rather long dipoles, particles are tracked both with a second-order symplectic method and a non-symplectic second-order polynomial expansion. The motion is restricted to the horizontal plane. Having one degree of freedom only, one would expect that the motion follows a smooth, if distorted, curve in the phase plane. It turns out that the symplectic method indeed produces a smooth curve, while tracking with the polynomial expansion generates a rather fuzzy curve. The reason is that cubic and higher-order terms generated by cross-coupling of the strong sextupoles are neglected in the polynomial expansion, thus making it non-symplectic. The transformation can be made to approach symplecticity by breaking the long magnets into pieces and treating the pieces one at a time. The error is proportional to the square of the step size. It is also possible to find a mapping using a generating function of third order [6,10] which agrees to order two with the R and T coefficients. The results are completely equivalent to application of a Lie transformation of order three (see below).

LHC - LP=79 M - 60 DEG - K2B=0.05

PHASE SPACE PLOT FOR ALL PARTICLES
OBSERVATION POINT AT "#S"

DATE: 03/04/85

TIME: 16.54.05

Figure 1: Symplectic Tracking in an Accelerator Cell

LHC - LP=79 M - 60 DEG - K2B=0.05

PHASE SPACE PLOT FOR ALL PARTICLES
OBSERVATION POINT AT "#S"
DATE: 03/04/85 TIME: 16.54.05

Figure 2: Non-symplectic Tracking in an Accelerator Cell

6. NUMERICAL INTEGRATION METHODS

Sometimes it is attractive to use a numerical integration method, like for example the Runge-Kutta Method. However care must be taken since not all numerical integration methods preserve symplecticity. As an example consider the one-dimensional system

$$\frac{d^2x}{ds^2} + k^2x = 0$$

whose solution can be described by the transfer matrix with the components

$$m_{11} = + \cos(ks)$$

$$m_{12} = + (1/k) \sin(ks)$$

$$m_{21} = - k \sin(ks)$$

$$m_{22} = + \cos(ks)$$

This transfer matrix is symplectic, because its determinant is one:

$$m_{11} m_{22} - m_{12} m_{21} = 1$$

To solve the same system with a fourth-order Runge-Kutta method we re-write the original equations as

$$\begin{aligned} \frac{dx}{ds} &= p & \frac{dp}{ds} &= - k^2x \end{aligned}$$

Solving this system with the standard Runge-Kutta method we obtain a transfer matrix with the components

$$m_{11} = \left\{ 1 - \frac{(ks)^2}{2!} + \frac{(ks)^4}{4!} \right\}$$

$$m_{12} = \left\{ ks - \frac{(ks)^3}{3!} \right\} / k$$

$$m_{21} = \left\{ ks - \frac{(ks)^3}{3!} \right\} \cdot (-k)$$

$$m_{22} = \left\{ 1 - \frac{(ks)^2}{2!} + \frac{(ks)^4}{4!} \right\}$$

This solution is correct to order 4 in s , but the determinant of the transfer matrix is not exactly 1:

$$m_{11} m_{22} - m_{12} m_{21} = 1 - \frac{(ks)^6}{72} + \frac{(ks)^8}{576}$$

i.e. the transfer matrix is symplectic to order six. However the error can be made arbitrarily small by decreasing the integration step.

7. GENERATING FUNCTION METHODS

We have mentioned that Hamiltonian motion can be described by canonical transformations. These in turn can be derived from generating functions. Assume we know a generating function of the form

$$F = F\{q(s_1); p(s_2)\} = F(q_1; p_2)$$

This leads to the three implicit equations for $p(s_2)$

$$p_i(s_1) = \frac{\partial F\{q(s_1); p(s_2)\}}{\partial q_i(s_1)}, \quad i = 1, 2, 3$$

which can be solved e.g. by Newton iteration, and three explicit equations for $q(s_2)$

$$q_i(s_2) = \frac{\partial F\{q(s_1); p(s_2)\}}{\partial p_i(s_2)}, \quad i = 1, 2, 3$$

The Jacobian of this mapping is symplectic. To prove this fact, we introduce the four "partial Jacobians"

$$F_{ppij} = \frac{\partial^2 F}{\partial p_i \partial p_j}, \quad i, j = 1, 2, 3$$

$$F_{pqij} = \frac{\partial^2 F}{\partial p_i \partial q_j}, \quad i, j = 1, 2, 3$$

$$F_{qpji} = \frac{\partial^2 F}{\partial q_i \partial p_j}, \quad i, j = 1, 2, 3$$

$$F_{qqij} = \frac{\partial^2 F}{\partial q_i \partial q_j}, \quad i, j = 1, 2, 3$$

The total differentials for q_2 and p_1 are

$$dq_2 = F_{pq} dq_1 + F_{pp} dp_2$$

$$dp_1 = F_{qq} dq_1 + F_{qp} dp_2$$

We solve the second equation in terms of dp_2 and substitute in the first equation. The Jacobian of the generated mapping can then be written in partitioned form as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

with the submatrix definitions

$$M_{11} = F_{pq} - F_{pp} F_{qp}^{-1} F_{qq}$$

$$M_{12} = F_{pp} F_{qp}^{-1}$$

$$M_{21} = - F_{qp}^{-1} F_{qq}$$

$$M_{22} = F_{qp}^{-1}$$

The symplecticity condition can now be computed block-wise:

$$M_{11}^T M_{21} - M_{21}^T M_{11} = 0$$

$$M_{11}^T M_{22} - M_{21}^T M_{12} = + I$$

$$M_{12}^T M_{21} - M_{22}^T M_{11} = - I$$

$$M_{12}^T M_{22} - M_{22}^T M_{12} = 0$$

Using the symmetry

$$F_{pq} = F_{qp}^T$$

the proof is straightforward.

8. CANONICAL INTEGRATION METHOD

The canonical integration method was invented by R. Ruth [12]. It is based on a set of generating functions which lead to explicit equations. These functions essentially represent an alternation of drift spaces and thin lenses. The drift lengths and thin lens strengths are selected such as to minimize the error terms. Generating functions for drift spaces have the form

$$F_1 = q_1 p_2 + G_1(p_2, s)$$

$$\Rightarrow p_1 = \frac{\partial F_1}{\partial q_1} = p_2$$

$$\Rightarrow p_2 = p_1$$

$$q_2 = \frac{\partial F_1}{\partial p_2} = q_1 + \frac{\partial G_1}{\partial p_2}$$

Generating function for thin-lens transformations have the form

$$F_2 = q_2 p_1 + G_2(q_2, s)$$

$$\Rightarrow q_1 = \frac{\partial F_2}{\partial p_1} = q_2$$

$$\Rightarrow q_2 = q_1$$

$$p_2 = \frac{\partial F_2}{\partial q_2} = p_1 + \frac{\partial G_2}{\partial q_2}$$

Both types of generating functions lead to explicit equations for both positions and momenta. The functions G_1 and G_2 are determined so as to eliminate all low-order powers of s in the Hamiltonian. For sufficiently small s the Hamiltonian then becomes nearly constant, and the transformed equations of motion become

$$\frac{dq}{ds} \approx 0$$

$$\frac{dp}{ds} \approx 0$$

They are solve trivially by

$$q = \text{constant}$$

$$p = \text{constant}$$

Example: Let the Hamiltonian be given in the form

$$H = g(p) + \phi(q, s)$$

Then the force acting on the particle has the components

$$F_i = - \frac{\partial \phi}{\partial q_i}, \quad i = 1, 2, 3$$

We also define the vector with the components

$$G_i(p) = \frac{\partial g(p)}{\partial p_i}, \quad i = 1, 2, 3$$

A map of third-order in s is then e.g. given by the three-step procedure

$$p_1 = p_0 + c_1 h F(q_0, s_0)$$

$$q_1 = q_0 + d_1 h G(p_1)$$

$$p_2 = p_1 + c_2 h F(q_1, s_0 + d_1 h)$$

$$q_2 = q_1 + d_2 h G(p_2)$$

$$p_3 = p_2 + c_3 h F(q_2, s_0 + (d_1 + d_2) h)$$

$$q_3 = q_2 + d_3 h G(p_3)$$

The constants c_i and d_i are determined such as to cancel powers of s up to order 3 in the Hamiltonian. This happens for the values

$$\begin{array}{lll} c_1 = 7/24 & c_2 = 3/4 & c_3 = -1/24 \\ d_1 = 2/3 & d_2 = -2/3 & d_3 = 1 \end{array}$$

9. LIE ALGEBRAIC METHODS

Lie algebraic methods have been proposed for particle tracking by A. J. Dragt et al. [4,5,7,8,9].

9.1 Definitions

For two arbitrary phase space functions

$$f = f(q,p), \quad g = g(q,p)$$

the Poisson bracket is defined as

$$[f,g] = \sum_{i=1}^3 \left\{ \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right\}$$

and the Lie operator associated with a function f acting on a function g is defined by the Poisson bracket

$$:f:g = [f,g]$$

Iterated Lie operators are defined as

$$:f:^2g = [f, [f, g]]$$

$$:f:^3g = [f, [f, [f, g]]]$$

We then define a Lie transformation as the formal exponential series

$$\exp(:f:) V = \sum_{n=0}^{\infty} \frac{:f:^n}{n!} V$$

This is a symplectic mapping for any function f [4,7].

9.2 Application for Tracking

The Hamiltonian equations of motion can be written using a Poisson bracket or a Lie operator:

$$dV/ds = - [H, V] = - :H: V$$

Comparing with an ordinary linear differential equation a formal solution to this equation can be obtained as

$$V(s_2) = \exp(- : \int H ds :) V(s_1)$$

This solution is valid, as long the condition

$$[H(s_1), H(s_2)] = 0$$

holds for all s_1, s_2 [4,7]. The solution can be factored as

$$V(s_2) = \exp(:F:) V(s_1) = \exp(:F_2:) \exp(:F_3:) \exp(:F_4:) \dots V(s_1)$$

where the F_k are **homogeneous polynomials** of order k in the phase space variables. The part of the Lie transformation for $k = 2$ can also be written as

$$\exp(:F_2:) V(s_1) = M V(s_1)$$

where M is the standard linear transfer matrix. This approach has the following advantages:

- In the factorized form transformations can be concatenated by simple polynomial manipulation. This speeds up tracking considerably if the concatenated transformations are precalculated.
- Truncation is possible without destroying symplecticity. It may however cause a loss of precision.

Application of Lie transformations has the following drawbacks:

- Exact concatenation generates very high order terms and thus large sets of coefficients requiring long computing times. Truncation with a loss of precision cannot be avoided.
- Application of a Lie transformation means evaluation of an infinite series

$$V(s_2) = \sum_{n=0}^{\infty} \frac{:F:^n}{n!} V(s_1)$$

Truncation of this series **destroys symplecticity**.

9.3 How to Find a Lie Transformation

Let the Hamiltonian be expanded as a sum of homogeneous polynomials, H_n , of degree n ,

$$H(V) = H_2(V) + H_3(V) + H_4(V) + \dots$$

The first step is to find the linear transfer matrix using the polynomial H_2 only:

$$V = M V(s_1)$$

This linear transformation is obviously canonical. It transforms the Hamiltonian to the interaction picture

$$\begin{aligned} H'(V) &= H_3(M^{-1}V) + H_4(M^{-1}V) \\ &= H'_3(V,s) + H'_4(V,s) \end{aligned}$$

In a second step, find the third-order part of the Lie transformation:

$$F_3 = F_3(V,L), \text{ where}$$

$$F_3(V,s) = - \int_0^s H'_3(V,s) ds$$

This canonical transformation eliminates the third order part of the Hamiltonian. The fourth order part becomes

$$H''_4(V,s) = H'_4(V,s) - (1/2) [F_3(V,s), H'_3(V,s)]$$

In a third step, we find the fourth-order part of the Lie transformation

$$F_4(V) = - \int_0^L H''_4(V,s) ds$$

9.4 Concatenation of Lie Transformations

Assume we know two Lie transformations F and G characterized by the following:

F: matrix M_f , polynomials $F_3(V)$ and $F_4(V)$

G: matrix M_g , polynomials $G_3(V)$ and $G_4(V)$

we wish to generate a new Lie transformation H characterized by

H: matrix M_h , polynomials $H_3(V)$ and $H_4(V)$

and representing the mapping

$$H V = G (F V)$$

In references [4,7] it is shown that this can be done by the formulas

$$M_h = M_g M_f$$

$$H_3 = G_3(V) + L_3(V)$$

$$H_4 = G_4(V) + L_4(V) + [L_3(V), G_3(V)]$$

where the polynomials L_3 and L_4 are found by substituting $M_g^{-1} V$ for V in F_3 and F_4 respectively:

$$L_3(V) = F_3(M_g^{-1} V)$$

$$L_4(V) = F_4(M_g^{-1} V)$$

9.5 Improved Evaluation without Truncation

Assume we know a factored Lie transformation

$$V(s_2) = \exp(:F_2:) \exp(:F_3:) \exp(:F_4:) V(s_1)$$

It can be performed in two steps. The first step gives an intermediate result $V^{(m)}$ by the linear transformation

$$V^{(m)} = M V(s_1) = \exp(:F_2:) V(s_1)$$

and the second step is a sequence of two Lie transformations of order 3 and 4 respectively:

$$V(s_2) = \exp(:F_3:) \exp(:F_4:) V^{(m)}$$

The second step can be derived from a generating function

$$G(q,p) = q \cdot p - F_3(q,p) + (1/2) \sum_{i=1}^3 \left\{ \frac{\partial F_3}{\partial q_i} \frac{\partial F_3}{\partial p_i} \right\} - F_4(q,p)$$

The positions q are taken at the entrance, and the momenta are taken at the exit. Using this generating function for tracking, the transformations remain symplectic for any initial conditions.

10. SUMMARY

We can state the following important rules for tracking in a conservative system:

- Always use **canonical variables**.
- Preserve **symplecticity** of the transformations.
- Use concatenation when feasible, but be careful with errors arising from truncating high-order terms.

It is important to realize that the optimal tracking method depends on the problem to be solved. If a small ring with strong nonlinearities is under study, the Lie algebraic method seems more adequate, since it allows higher order terms to be carried along. The slower execution time does not matter, if the number of magnetic elements in the ring remains small. For a large ring, it may be faster to apply thin lens tracking methods or matrix methods. If the nonlinearities are small or localized, it can however pay off to use Lie algebraic methods and to concatenate relatively large sections of the ring. The choice of the method must in any case be done carefully. The speed up tracking one wishes to concatenate as many elements as possible. This should however be done only as long as truncation does not affect the accuracy too badly. How far one can really go in this process must be left to the good judgement of the user.

References

- [1] K. L. Brown, A First- and Second-Order Matrix Theory for the Design of Beam Transport Systems and Charged Particle Spectrometers. SLAC Report No. 75, August 1972.
- [2] K. L. Brown, D. C. Carey, F. C. Iselin and F. Rothacker, TRANSPORT, A Computer Program for Designing Charged Particle Beam Transport Systems. CERN 80-4. Also published as FNAL 91 and SLAC 91.
- [3] E. D. Courant and H. S. Snyder, Theory of the Alternating-Gradient Synchrotron. Annals of Physics, Vol. 3, pp. 1-48, 1958.
- [4] D. R. Douglas, Lie Algebraic Methods for Particle Accelerator Theory. University of Maryland Ph.D. Thesis, unpublished, 1982.
- [5] D. R. Douglas and A. J. Dragt, MARYLIE, The Maryland Lie Algebraic Beam Transport and Particle Tracking Program, IEEE Transactions on Nuclear Science NS-30, pg. 2442, 1983.
- [6] D. R. Douglas, Private Communication.
- [7] A. J. Dragt, Lectures on Nonlinear Orbit Dynamics. Published in "Physics of High Energy Particle Accelerators", American Institute of Physics Conference Proceedings No. 87, R. A. Carrigan et al., editors, 1982.
- [8] A. J. Dragt and E. Forest, Computation of Nonlinear Behaviour of Hamiltonian Systems Using Lie Algebraic Methods. J. Math. Physics 24, pg. 2734, 1983.

- [9] E. Forest, Lie Algebraic Methods for Charged Particle Beams and Light Optics. University of Maryland Ph.D. Thesis, unpublished, 1984.
- [10] E. Forest and J. L. Warren, The Application of Symplectic Conditions to Second Order Transport Theory. Los Alamos, AT-6:ATN-83-17, 1984.
- [11] F. C. Iselin, MAD Reference Manual. CERN, LEP Division, to be published.
- [12] R. Ruth, A Canonical Integration Technique. Particle Accelerator Conference, 21-23 March 1983, Santa Fe, New Mexico.
- [13] R. V. Servranckx, Users Guide to the Program DIMAT. SLAC Report 270 UC-28, March 1984.
- [14] H. Wiedemann, PEP Technical Memo 230, 1981, and PEP Note 220, 1976.
- [15] A. Wrulich, Particle Tracking in Accelerators with Higher Order Multipole Fields. Lecture Notes in Physics 215, Springer, Berlin, 1984.