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KÄHLER POTENTIALS FOR SUPERSYMMETRIC σ -MODELS
WITH BROKEN CENTRAL CHARGES

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A B S T R A C T

We extend the Itoh-Kugo-Kunitomo construction of Kähler potentials for homogeneous Kähler manifolds to the case of broken central charges. In the case of p broken central charges the most general Kähler potential contains an arbitrary function of p independent variables. The general results are illustrated with the examples $SU(n+2)/SU(n) \times U(1)$ and $SU(n+2)/SU(n)$.

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Supersymmetric σ -models [for a review and references, see Ref. 1)] may play a crucial role as effective low energy Lagrangians for quarks and leptons^{2),3)}. They relate the quantum numbers of fermions to the geometry of a coset space, and can thereby unify a linearly realized gauge group H together with matter fields, carrying the quantum numbers of broken generators, in a single group G . Therefore they severely restrict the possible representations of matter fields in grand unified theories. Of particular phenomenological interest are σ -models based on exceptional groups.

Given a physical coset space G/H one has to construct the corresponding supersymmetric σ -models. Extending the work of Coleman, Wess and Zumino⁴⁾ to non-compact groups, this problem has been studied in general by Bando, Kuramoto, Maskawa and Uehara (BKMU)⁵⁾ who related the different supersymmetric σ -models for G/H to the different complex coset spaces $G^C/\hat{H}$, where G^C is the complex extension of G and $\hat{H}$ a complex subgroup containing H^C . More recently, Itoh, Kugo and Kunitomo (IKK)⁶⁾ gave an explicit procedure for constructing the Kähler potentials for homogeneous Kähler manifolds G/H and for obtaining the necessary projection operators η_i ⁵⁾.

In general, H has the form $H = H' \times U(1)^n$, where H' is semi-simple and the number of $U(1)$ symmetries, which are generated by the central charges Q_i , equals the number of independent projection operators η_i ⁶⁾. It has been shown^{7),8)} that spontaneous symmetry breaking in supersymmetric theories does not lead to σ -models based on homogeneous spaces but rather to σ -models which contain additional scalar fields, quasi-Goldstone bosons, beyond the usual Goldstone bosons. The simplest σ -models of this type are those with broken central charges which are based on the coset spaces $G/H' \times U(1)^m$, $m \neq n$, rather than the Kähler manifold $G/H' \times U(1)^n$. For each broken $U(1)$ generator they have a chiral superfield which is a singlet with respect to H and contains one quasi-Goldstone boson. Such "novino" superfields⁹⁾ are not just an unavoidable consequence of the spontaneous symmetry breaking of the original theory, they also affect significantly the residual interactions^{9),10)} of all Goldstone superfields. In this note we extend the construction of Itoh et al. of Kähler potentials for homogeneous manifolds to σ -models of this type.

Our starting point is the construction of Kähler potentials for homogeneous manifolds which requires the knowledge of the complex subgroups $\hat{H}$ of G^C ⁵⁾. A convenient method to obtain all complex subgroups $\hat{H}$, or, equivalently, all complex structures on G/H makes use of the central charges Q_i ^{11),6)}. With

respect to a chosen linear combination of these $U(1)$ charges the generators of G^c/H^c are divided into positive ($\bar{X}_p$) and negative (X_p) generators. The generators S_a of the semi-simple subgroup H' , the central charges Q_i and the generators X_p form the Lie algebra of $\hat{H}$, the generators $\bar{X}_p$ correspond to $G^c/\hat{H}$.

The general Kähler potential in the case of broken central charges can be easily obtained once the Kähler potential for the corresponding homogeneous manifold G/H is known. Here one starts in the standard representation from the variable

$$\xi(\varphi) = e^{\varphi \cdot \bar{X}}, \quad \varphi \cdot \bar{X} = \varphi^p \bar{X}_p, \quad (1)$$

where $\bar{X}_p$ are the generators of $G^c/\hat{H}$ and φ^p are the Goldstone superfields. The corresponding non-linear realization is defined through

$$g \xi(\varphi) = \xi(\varphi') \hat{h}(\varphi; g), \quad g \in G^c, \quad (2a)$$

with

$$\hat{h} = e^{\alpha \cdot X} e^{\beta \cdot S} e^{i\gamma \cdot Q}, \quad \alpha = \alpha(\varphi; g) \text{ etc.}, \quad (2b)$$

where X , S and iQ are the generators of $\hat{H}$. As the generators X_p and S_a are traceless one has the important identity⁵⁾

$$\det_{\eta_i} \hat{h} = e^{i \text{tr}(\eta_i \gamma \cdot Q)}, \quad (3)$$

where, in the case of n central charges, the n projection operators η_i are defined through⁵⁾

$$\eta_i = \eta_i^\dagger, \quad \eta_i^2 = \eta_i, \quad (4a)$$

$$\hat{h} \eta_i = \eta_i \hat{h} \eta_i, \quad \text{for all } \hat{h} \in \hat{H}. \quad (4b)$$

From Eq. (3) one obtains immediately

$$\det_{\eta_i} (\xi'^\dagger \xi') = \det_{\eta_i} (\xi'^\dagger \xi) e^{i \text{tr}[\eta_i (\gamma'^* - \gamma) \cdot Q]}, \quad (5)$$

which gives for the most general Kähler potential the BKMU expression⁵⁾

$$K = \sum_{i=1}^n f_i^2 \ln \det_{\eta_i} (\zeta^\dagger \zeta) \quad . \quad (6)$$

Let us now consider the case where only $m < n$ central charges $\tilde{Q}_i$ are unbroken whereas $n-m$ central charges R_k are broken and associated with chiral superfields Φ_k . Clearly, $\tilde{H} = H/U(1)^{n-m}$ is again a complex subgroup of G^c with the same projection operators η_i , and instead of $\xi(\phi)$ we have to consider the variable

$$\zeta(\tilde{\psi}, \Phi) = e^{\tilde{\psi} \cdot \bar{X} + i \Phi \cdot R} \quad . \quad (7)$$

The non-linear realization of G^c on ϕ and Φ takes the form

$$g \zeta(\tilde{\psi}, \Phi) = \zeta(\tilde{\psi}', \Phi') \tilde{h}(\tilde{\psi}, \Phi; g), \quad g \in G^c, \quad (8a)$$

with

$$\tilde{h} = e^{\tilde{\alpha} \cdot X} e^{\tilde{\beta} \cdot S} e^{i \tilde{\gamma} \cdot \tilde{Q}}, \quad \tilde{\alpha} = \tilde{\alpha}(\psi, \Phi; g), \text{ etc.} \quad (8b)$$

The non-linear realization on $\zeta(\tilde{\psi}, \Phi)$ is related to the one on $\xi(\phi)$ as follows. Since $[\bar{X}, \bar{X}] \subset \bar{X}$, $[\bar{X}, R] \subset \bar{X}$ and $[R, R] = 0$, we can write

$$\begin{aligned} \zeta(\tilde{\psi}, \Phi) &= e^{\tilde{\psi} \cdot \bar{X} + i \Phi \cdot R} \\ &= e^{\psi \cdot \bar{X}} e^{i \Phi \cdot R} = \zeta(\psi) e^{i \Phi \cdot R} \end{aligned} \quad (9)$$

Equation (9) defines implicitly the relation between $\tilde{\psi}$, Φ and ψ . We will adopt ϕ and Φ as co-ordinates in the case with broken central charges. With the help of Eq. (2) we obtain

$$\begin{aligned} g \zeta(\tilde{\psi}, \Phi) &= g \zeta(\psi) e^{i \Phi \cdot R} \\ &= \zeta(\psi') \hat{h}(\psi, g) e^{i \Phi \cdot R} \\ &= \zeta(\psi') e^{i \Phi \cdot R} \tilde{h}(\psi, \Phi) \end{aligned} \quad (10)$$

The last equation holds because $[X, R] \subset X$, $[S, R] = [\tilde{Q}, R] = 0$, and it determines the relation between $\tilde{h}$ and $\hat{h}$. From Eqs. (8) and (10) one obtains

$$\bar{\Phi}' = \bar{\Phi} + \gamma(\varphi, g) \quad (11)$$

Analogous to Eq. (5) one has for the quantity $\zeta^\dagger \zeta$ the transformation law

$$\det_{\eta_i}(\zeta'^\dagger \zeta') = \det_{\eta_i}(\zeta^\dagger \zeta) e^{i \text{tr}[\eta_i(\tilde{\gamma}^* - \tilde{\gamma}) \cdot \tilde{Q}]}, \quad (12)$$

where, contrary to (5), the phase factor involves only $m < n$ independent coefficients. Therefore $n-m$ invariant independent variables can be constructed from the n quantities (12). This implies that the most general Kähler potential is an arbitrary function of the $n-m$ invariant variables, which is only subject to the constraint of yielding properly normalized kinetic terms for the Goldstone superfields. Moreover, these variables can easily be constructed using Eq. (9) which implies

$$\det_{\eta_i}(\zeta^\dagger \zeta) = \det_{\eta_i}(\xi^\dagger \xi) e^{i \text{tr}[\eta_i(\Phi - \bar{\Phi}) \cdot R]} \quad (13)$$

where $\det_{\eta_i}(\xi^\dagger \xi)$ is obtained according to the IKK prescription. Following this method, one describes the manifold $G^C/\tilde{H}$ in terms of the co-ordinates ϕ and $\bar{\Phi}$ rather than $\tilde{\phi}$ and Φ , which is equally allowed. The knowledge of the explicit transformation relating $\tilde{\phi}$, Φ and ϕ is nowhere required.

Let us illustrate this general result by means of the "flag manifold" $SU(n+2)/SU(n) \times U(1)^2$, where one or both central charges may be broken. The manifold allows two inequivalent complex structures¹²⁾,

$$\Omega_I = \bar{n}^{-1, -n-1}(\varphi_i) + \bar{n}^{-n-1, -1}(\chi_i) + \bar{n}^{n, -n}(\varphi') \quad (14a)$$

$$\Omega_{II} = \bar{n}^{-1, -n-1}(\varphi_i) + \bar{n}^{n+1, 1}(\chi_i) + \bar{n}^{n, -n}(\varphi') \quad (14b)$$

where the two indicated charges are $Q_I = \text{diag}(1, \dots, 1, -n, 0)$ and $Q_{II} = \text{diag}(1, \dots, 1, 0, -n)$. Ω_I and Ω_{II} correspond to the defining charges $Q_I = -Q_2$ and $Q_{II} = Q_1 - Q_2$, respectively.

According to the IKK prescription⁶⁾ the operators η_i are obtained as follows. One splits the fundamental representation of $SU(n+2)$ into irreducible representations of $SU(n) \times U(1)^2$, i.e.,

$$\tilde{n+2} = \tilde{n}^{1,1} + \tilde{1}^{-n,0} + \tilde{1}^{0,-n}, \quad (15)$$

and constructs for $\Omega_I(\Omega_{II})$ the projection operators onto the subspaces which contain all vectors whose charge $Q_I(Q_{II})$ is larger than a chosen eigenvalue of $Q_I(Q_{II})$ occurring in the decomposition (15). This procedure leads for the two complex structures to the block-diagonal projection operators:

$$\Omega_I: \quad \mathcal{M}_1 = \begin{pmatrix} 0 & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad (16)$$

$$\Omega_{II}: \quad \mathcal{M}_1 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}. \quad (17)$$

It is now straightforward to compute the quantities

$$U_i = \det_{\mathcal{M}_i}(\xi^\dagger \xi) \quad (18)$$

for Ω_I and Ω_{II} . We find the results:

$$\Omega_I: \quad U_1 = 1 + (\bar{\psi} + \frac{1}{2}\bar{\psi}'\bar{\chi}) \cdot (\psi + \frac{1}{2}\psi'\chi) + \bar{\psi}'\psi', \quad (19a)$$

$$U_2 = 1 + \bar{\chi}\chi + (\bar{\psi} - \frac{1}{2}\bar{\psi}'\bar{\chi}) \cdot (\psi - \frac{1}{2}\psi'\chi) + \bar{\chi}\chi\bar{\psi}\psi - \bar{\psi}\chi\bar{\chi}\psi; \quad (19b)$$

$$\Omega_{II}: \quad U_1 = 1 + \bar{\psi}\psi + (\bar{\psi}' + \frac{1}{2}\bar{\chi}\bar{\psi})(\psi' + \frac{1}{2}\chi\psi), \quad (20a)$$

$$U_2 = 1 + \bar{\chi}\chi + (\bar{\psi}' - \frac{1}{2}\bar{\chi}\bar{\psi})(\psi' - \frac{1}{2}\chi\psi). \quad (20b)$$

The most general Kähler potential for the homogeneous manifold $SU(n+2)/SU(n) \times U(1)^2$ reads

$$K_h = f_1^2 \ln U_1 + f_2^2 \ln U_2. \quad (21)$$

Let us now consider the case where $Q = Q_1$ is unbroken and $R = Q_2$ is broken. The quantities U_i are replaced by

$$V_i = \det_{\eta_i} (\zeta^\dagger \zeta) \quad . \quad (22)$$

From Eq. (13) one easily obtains:

$$\Omega_I: \quad V_{1,2} = U_{1,2} e^{-in(\Phi - \bar{\Phi})} \quad , \quad (23)$$

$$\Omega_{II}: \quad V_1 = U_1 e^{-in(\Phi - \bar{\Phi})} \quad , \quad V_2 = U_2 \quad . \quad (24)$$

Now we have to consider the transformation Eq. (12) for the quantities V_i . For Ω_I and Ω_{II} one has

$$\text{tr}(\eta_1 Q_1) = 0 \quad , \quad \text{tr}(\eta_2 Q_1) \neq 0 \quad . \quad (25)$$

Thus in both cases V_1 is an invariant and one obtains for the most general Kähler potential

$$K = K_h + F(V_1) \quad , \quad (26)$$

where F is an arbitrary function.

In the case where $Q = Q_2$ is unbroken and $R = Q_1$ is broken, one has

$$\Omega_I: \quad V_1 = U_1 \quad , \quad V_2 = U_2 e^{-in(\Phi - \bar{\Phi})} \quad , \quad (27)$$

$$\text{tr}(\eta_1 Q_2) = -n \quad , \quad \text{tr}(\eta_2 Q_2) = -n \quad . \quad (28)$$

$$\Omega_{II}: \quad V_1 = U_1 \quad , \quad V_2 = U_2 e^{in(\Phi - \bar{\Phi})} \quad , \quad (29)$$

$$\text{tr}(\eta_1 Q_1) = -n \quad , \quad \text{tr}(\eta_2 Q_1) = 0 \quad . \quad (30)$$

This yields the Kähler potentials

$$\Omega_I: \quad K = K_h + F\left(\frac{V_1}{V_2}\right) \quad , \quad (31)$$

$$\Omega_{\mathbb{I}} : K = K_h + F(V_2) \quad (32)$$

Finally, in the case where both central charges are broken, one obtains

$$\Omega_{\mathbb{I}} : V_1 = U_1 e^{-in(\Phi_2 - \bar{\Phi}_1)} \quad (33a)$$

$$V_2 = U_2 e^{-in(\Phi_1 + \Phi_2 - \bar{\Phi}_1 - \bar{\Phi}_2)} \quad (33b)$$

$$\Omega_{\mathbb{I}} : V_1 = U_1 e^{-in(\Phi_2 - \bar{\Phi}_1)} \quad (34a)$$

$$V_2 = U_2 e^{in(\Phi_1 - \bar{\Phi}_1)} \quad (34b)$$

The most general Kähler potential contains an arbitrary function of both variables:

$$K = K_h + F(V_1, V_2) \quad (35)$$

Given the Kähler potential of a σ -model for a homogeneous manifold G/H according to the prescription of Itoh, Kugo and Kunitomo, we have shown how to obtain the general Kähler potential for σ -models containing in addition the Goldstone bosons of p broken central charges and p quasi-Goldstone bosons. The result is very simple: for appropriately chosen fields, the BKMU variables $\det_{\eta_i}(\xi^\dagger \xi)$ are modified by phase factors involving the Goldstone superfields of the broken $U(1)$ generators; in the case of p broken central charges the most general Kähler potential is the sum of the Kähler potential for the homogeneous manifold G/H and an arbitrary function of p independent variables constructed from the modified BKMU variables $\det_{\eta_i}(\zeta^\dagger \zeta)$.

As we explained in the Introduction, supersymmetric σ -models with broken central charges are more likely to be of physical relevance than σ -models based on homogeneous Kähler manifolds. We hope that our results will be useful for further investigations of this interesting variant of supersymmetric σ -models.

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