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Mathematical Problems in Image Processing

Partial Differential Equations and the
Calculus of Variations

Second Edition

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