MANY-PARTICLE RAPIDITY CORRELATIONS IN LIGHT-ION INTERACTIONS AT THE CERN INTERSECTING STORAGE RINGS

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ABSTRACT

Many-particle rapidity correlations have been studied for $\alpha\alpha$, dd , and pp interactions at the CERN Intersecting Storage Rings using the Split-Field Magnet detector, and basing the investigation on the distributions of rapidity gaps between particles. In the framework of a simple cluster model, it is found that all the data samples have similar features and that the events contain clusters of two different types.

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1. INTRODUCTION

Correlations between particles in high-energy interactions have been studied experimentally for about the last 10 years [1, 2]. The most frequently used method for carrying out such studies has been to compare the observed two-particle spectra with the expectations obtained from mixing one-particle spectra. The drawback of this method is that only two particles are examined simultaneously and the information about the rest of the particles in the event is disregarded. Distributions where more than two particles are considered can be studied, but the nature of the problem is such that it grows more complex as the number of particles increases. The results of such studies are thus hard to interpret.

Correlations between produced particles have also been investigated by studying the distributions of rapidity gaps between the particles [3]. When the rapidity gaps are differentiated according to the number of particles inside the gap, not only the two particles forming the gap enter the analysis, but also information about the rest of the event.

An analysis of two-particle correlations in $\alpha\alpha$, αp , and pp interactions at the CERN Intersecting Storage Rings (ISR) has already been published [2]. It was found that most of the observed inclusive correlations had their origin in the correlation between the phase-space distribution of particles and the total multiplicity of the event. Thus the analysis was carried out using semi-inclusive distributions, i.e. at fixed charged-particle multiplicities. The short-range correlations observed in this way were found to be equal for the three different data sets, indicating that the mechanisms responsible for the short-range correlations are the same for the different kinds of interactions.

The subject of this paper is, once again, the correlations between particles produced in light-ion interactions. A new method is used which combines techniques from the method of using the two-particle correlation functions and from rapidity-gap studies. This new method has recently been applied, for the first time, to studies of correlations in proton-emulsion interactions [4].

The experiment was performed at the CERN ISR using the Split-Field Magnet (SFM) detector. A description of the SFM detector and of the experimental set-up can be found elsewhere [5]. The data sets used for the analysis are listed in Table 1, together with the c.m. energies.

2. THE METHOD

As mentioned in the Introduction, rapidity correlations between produced particles are most frequently described by means of the normalized two-particle correlation function

$$R(y_1, y_2) \equiv \frac{\rho^{(2)}(y_1, y_2)}{\rho^{(1)}(y_1)\rho^{(1)}(y_2)} - 1, \quad (1)$$

where

$$\rho^{(1)}(y) = \sigma_{\text{in}}^{-1} \frac{d\sigma}{dy} \quad \text{and} \quad \rho^{(2)}(y_1, y_2) = \sigma_{\text{in}}^{-1} \frac{d^2\sigma}{dy_1 dy_2}$$

are the particle densities. In ratios such as this, the numerator represents the experimentally observed distribution, whilst the denominator represents the distribution resulting from the

single-particle spectra in the absence of correlations. Such ratios have the advantage that most of the experimental uncertainties cancel out. The same function can also be defined for semi-inclusive data, i.e. for fixed total multiplicity n :

$$R_n(y_1, y_2) \equiv \frac{\rho_n^{(2)}(y_1, y_2)}{\rho_n^{(1)}(y_1) \rho_n^{(1)}(y_2)} - 1. \quad (2)$$

As was also mentioned in the Introduction, there is a basic difference between (1) and (2). When the event multiplicity increases, the additional particles must be produced with small rapidities since the total amount of available energy is limited. Thus the width of the rapidity distributions, $\rho_n^{(1)}(y)$, must decrease as the multiplicity increases. This creates a trivial correlation which is inherent in (1) but not in (2) (see Ref. 2). One way to avoid this problem is to redefine (1) by changing the denominator to

$$\sum_n \pi(n) \rho_n^{(1)}(y_1) \rho_n^{(1)}(y_2),$$

where $\pi(n)$ is the experimentally determined multiplicity distribution. In this way, only those pairs where both particles come from events with the same multiplicities contribute. Thus the function

$$R'(y_1, y_2) \equiv \frac{\rho^{(2)}(y_1, y_2)}{\sum_n \pi(n) \rho_n^{(1)}(y_1) \rho_n^{(1)}(y_2)} - 1 \quad (3)$$

will be as easy to work with as the one defined in (1), and will not be affected by the kind of trivial correlations mentioned above. Note that

$$\rho^{(2)}(y_1, y_2) \equiv \sum_n \pi(n) \rho_n^{(2)}(y_1, y_2).$$

It is however observed that, in the central rapidity region, $R_n(y_1, y_2)$ is essentially only dependent on the rapidity difference $|y_1 - y_2|$ [2]. This is also a feature of several theoretical models; for example, the cluster models [6], where the decay of a cluster is independent of where it is produced in phase space, provided all other properties are left unchanged. We can thus change (3) into a one-dimensional ratio:

$$R(\Delta y) \equiv \frac{\int \rho^{(2)}(y, y+\Delta y) dy}{\sum_n \pi(n) \int \rho_n^{(1)}(y) \rho_n^{(1)}(y+\Delta y) dy} - 1. \quad (4)$$

Equation (4) represents the distribution of rapidity gaps Δy between the produced particles, where all such gaps contribute, regardless of the number of additional particles that are found inside the gap. There is, however, a basic difference in cases where two particles have a rapidity difference of Δy and where no other particles are found between them, compared with cases where one or more particles fall into this rapidity interval. Let us define ${}_n G_k$ as the gap length between two particles, having exactly $k-1$ particles located between them, in an event with multiplicity n . We can now form the ratio between the experimental distribution of gaps, ${}_n f_k(G)$,

and the corresponding distribution calculated from one-dimensional density distributions, ${}_n\tilde{f}_k(G)$, in exact accordance with (2). Thus

$${}_nR_k(G) \equiv \frac{{}_n\tilde{f}_k(G)}{{}_n\tilde{f}_k(G)} - 1 \quad (5)$$

defines our normalized semi-inclusive rapidity-gap correlation function of order k . The functions ${}_n\tilde{f}_k(G)$ and ${}_n\tilde{f}_k(G)$ are, for the sake of convenience, both normalized to unity. It can be shown (see Appendix) that ${}_n\tilde{f}_k(G)$ can be written as

$$\begin{aligned} {}_n\tilde{f}_k(G) &= k \binom{n}{k} \int h_n(y) h_n(y+G) \left[H_n(y+G) - H_n(y)^{k-1} \right] \\ &\times \left[1 - H_n(y+G) + H_n(y) \right]^{n-k-1} dy, \end{aligned} \quad (6)$$

where

$$h_n(y) = \rho_n^{(1)}(y)/n \quad \text{and} \quad H_n(y) \equiv \int_{y_{\min}}^y h_n(t) dt.$$

Finally, we can obtain the inclusive ratio

$$R_k(G) \equiv \frac{F_k(G)}{\tilde{F}_k(G)} - 1, \quad (7)$$

where

$$F_k(G) \equiv \frac{\sum_{n>k} \pi(n) (n-k) {}_n\tilde{f}_k(G)}{\sum_{n>k} \pi(n) (n-k)} \quad \text{and} \quad \tilde{F}_k(G) \equiv \frac{\sum_{n>k} \pi(n) (n-k) {}_n\tilde{f}_k(G)}{\sum_{n>k} \pi(n) (n-k)}, \quad (8)$$

using the same argument which led to the definition of (3). The normalized rapidity-gap correlation function, defined by (7), has all the advantages connected with the normalized two-particle correlation function, but these are now complemented by the advantage that one can keep track of the rest of the event.

3. EXPERIMENTAL RESULTS

For this analysis we have considered only charged particles produced in the central rapidity region. This selection has been made to exclude particles from the fragmentation regions where different production mechanisms and effects of energy conservation may play an important role. We have chosen a somewhat arbitrary rapidity interval to represent the central region: $|y| \leq 1.5$. As for most detectors, the SFM detector is not able to identify particles in the whole region of phase space; thus the pion mass has been assigned to all particles when calculating their rapidities. The rapidity interval is small enough to exclude any influence of leading-particle

phenomena, and the amount of heavier particles, i.e. protons and kaons, is expected to be small. All particles produced with $|y| \geq 1.5$ have been excluded, and the event multiplicity n has been redefined accordingly, i.e. only particles inside the interval are counted.

In Fig. 1 the gap distributions $F_k(G)$ as measured in dd interactions are shown for orders up to $k = 8$. Also indicated in the figures are the functions $\tilde{F}_k(G)$ given by (6) and (8). One observes a significant change of shape as the order increases, especially near $G \approx 0$. This behaviour is well understood, since the probability of finding other particles, between the two particles forming the gap, must decrease as the length of the gap decreases. The fast drop at $G \approx 3$ is just a reflection of the chosen rapidity interval.

When we compare the functions $F_k(G)$ and $\tilde{F}_k(G)$ we find good agreement for all k -values over several orders of magnitude. However, small but significant deviations are seen. There are always more small gaps than expected, and fewer larger gaps. This behaviour is typical for particles which tend to cluster.

In Fig. 2 we show the normalized rapidity-gap correlation function $R_k(G)$, given by (7) for the same data and for the same orders as in Fig. 1. The main feature of $R_k(G)$ is a peak for small G on a more or less flat background. As the order increases, we see that the peak becomes more prominent, indicating a strong signal for correlations not only between neighbouring particles but also between particles where several other particles are produced in the vicinity.

In order to understand the behaviour of $R_k(G)$ in a more quantitative way, we have performed a Monte Carlo calculation based on a simple cluster model where clusters are allowed to decay, giving birth to charged particles. The concept of ‘cluster’ should be understood in a very broad sense, i.e. a group of particles with some correlated properties. A cluster at rapidity Y gives rise to K charged particles at rapidities y_i so that y_i are uncorrelated and Gaussian-distributed with mean value Y and width δ . If an event contains more than one cluster, the clusters are assumed to be uncorrelated. The multiplicity of each cluster is assumed to be Poisson distributed with mean value $\langle K \rangle$. As long as the width of the cluster is small, the rapidity distribution of clusters will be approximately the same as the one observed for the final particles. The two free parameters of the model are $\langle K \rangle$ and δ .

In the Monte Carlo calculation, the clusters were produced in the rapidity interval $|y| \leq 2.5$, allowing those which were centred outside the rapidity window ($|y| \leq 1.5$) to contribute. Since the rapidity distribution of the cluster was determined by the experimental distribution $\varrho_n^{(1)}(y)$, the total multiplicity had to be known beforehand. The multiplicity distribution of the generated events was required to be the same as that of the measured events. Clusters were produced one by one, and the resulting particles constituted an event. As soon as the total number of particles created was larger than or equal to the considered event multiplicity, the event was terminated, and the generation of the next event followed. Whenever the last cluster was too large to fit into the event, particles were randomly discarded until the event multiplicity was reached. The introduction of this cut results in an average cluster size that is somewhat smaller than $\langle K \rangle$. The cut, together with the generated fraction of clusters with no charged particles, has led us to calculate $\langle K_{\text{eff}} \rangle$, which is the average number of charged particles in those clusters where at least one charged particle is produced inside the rapidity window studied. The created events were analysed in the same way as the experimental data.

The parameters $\langle K \rangle$ and δ are fitted to the data. This is done independently for each order. For most orders two different minima are found, one with $\langle K \rangle \approx 5$ and one with $\langle K \rangle \approx 1$. For low

orders the first minimum ($\langle K \rangle \approx 5$) has the smallest χ^2 . As the order increases, the χ^2 at the first minimum increases and the χ^2 at the second minimum decreases. Somewhere between orders 3 and 4 the global minimum makes a jump from one place to the other. It is important to note that only the relative χ^2 -values at the minima change, but the values of the parameters stay the same within errors. The results of the procedure are summarized in Fig. 3, where the best fits are shown as a function of the order k . Full dots indicate the global minima, open dots the additional local ones. For orders 1, 7, and 8 only one minimum is found. (One of the minima is so dominant that the other one disappears.) Also indicated in the figure are the χ^2 -values for the two minima. When the parameters are fitted to all orders simultaneously, a reasonable χ^2 cannot be obtained.

The curves shown in Fig. 2 were calculated with the parameters obtained when fitted to order 1 (solid curve) and to order 8 (dashed curve). One observes how, for increasing orders, the data points seem to move gradually from one curve to the other. The interpretation of this behaviour must be that clusters of two different types are competing. Small, narrow clusters cannot be responsible for the behaviour of the correlations of order 1, since also an increasing amount of large gaps separating such clusters would be observed (dashed curve in Fig. 2a). On the other hand, only narrow clusters, occasionally overlapping, can explain the order 8 behaviour. A large, wide cluster is not able to produce nine particles in a narrow rapidity region (solid curve in Fig. 2h).

As the low-order distributions are dominated by low-multiplicity events, and high-order distributions are dominated by high-multiplicity events, one might expect that the effect is multiplicity-dependent. For this reason, different multiplicity bins have been investigated separately. Only for the largest bin, $n \geq 16$, is there a clear tendency towards small clusters, both at order 1 and at order 8. The rest of the bins show a behaviour similar to that of the inclusive data. Thus it seems as if two cluster types are present in the same events. If a single event were to contain clusters of one type only, the effect from the two cluster types would be additive, i.e. the contributions from the two types would be the same at all orders. This possibility is clearly ruled out by the behaviour of the data.

When the method is applied to the four other data sets, the same pattern is observed. In Fig. 4 the normalized rapidity-gap correlation functions for orders 1 and 8 are shown for $\alpha\alpha$ and for pp data at three different energies. The best fits obtained by the Monte Carlo calculation are also indicated. The parameters obtained for the different data sets are summarized in Table 2.

4. SUMMARY AND CONCLUSIONS

In this paper we have investigated distributions of rapidity gaps between particles produced in $\alpha\alpha$, dd, and pp interactions at ISR energies, using as the main tool the normalized rapidity-gap correlation function defined by (7).

Signals for short-range correlations (similar for all data sets) are clearly seen. The signal grows as a function of increasing order, i.e. of increasing number of particles between the two particles defining the gap.

In the framework of a simple cluster model, the events are found to be composed of two types of clusters; one type consists of large, wide clusters; the other, of small, narrow ones. The parameters $\langle K \rangle$ and δ obtained from fits to the different data sets are summarized in Table 2. When the three data sets with the same $\sqrt{s_{NN}}$ are compared, the parameters for each cluster type

are almost the same within the statistical errors. When the different sets of pp data are compared, we find that there is a tendency for the clusters to grow in size as the energy increases.

One can interpret the two different cluster types as signals for two different mechanisms responsible for particle production. The fragmentation of quarks or strings should give rise to rather large, wide clusters, whereas the subsequent decay of low-mass resonances should result in smaller clusters. This interpretation could be checked by investigating correlations in more sophisticated models, where these two mechanisms are considered.

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Table 1

Data sets used in this work

Interaction	$\sqrt{s}$ (GeV)	$\sqrt{s_{NN}}$ (GeV)	No. of events
$\alpha\alpha$	124.6	31.2	108,773
dd	62.3	31.2	105,839
pp	62.2		66,879
	44.0		48,416
	30.4		43,730

Table 2

Parameters obtained for the different data sets

Interaction	$\sqrt{s_{NN}}$ (GeV)	Order	$\langle K \rangle$	δ	$\langle K_{eff} \rangle$
$\alpha\alpha$	31.2	1	4.15 ± 0.27	0.85 ± 0.02	3.74
		8	0.56 ± 0.10	0.30 ± 0.03	1.29
dd	31.2	1	4.27 ± 0.39	0.79 ± 0.03	3.73
		8	0.74 ± 0.12	0.20 ± 0.04	1.37
pp	62.2	1	5.00 ± 0.40	0.91 ± 0.05	4.18
		8	1.23 ± 0.22	0.39 ± 0.03	1.67
	44.0	1	4.69 ± 0.36	0.94 ± 0.06	3.96
		8	1.27 ± 0.10	0.42 ± 0.02	1.70
	30.4	1	4.46 ± 0.17	0.89 ± 0.04	3.76
		8	0.61 ± 0.17	0.26 ± 0.04	1.30

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Figure captions

- Fig. 1: Rapidity-gap distributions in dd interactions at $\sqrt{s_{NN}} = 31.2$ GeV for the first eight gap-orders. The curves correspond to the calculations, based on single-particle spectra, given by (6) and (8).
- Fig. 2: Normalized rapidity-gap correlation function for the same data and the same orders as in Fig. 1. The curves show the results from the model calculations, described in the text, for two different parameter set-ups.
- Fig. 3: The values of the parameters in the model calculation for which χ^2 has a minimum, as a function of the gap order. Global minima are indicated by full dots, local minima by open dots. The corresponding χ^2 -values are indicated by the curves. The calculations are based on the same data as in Figs. 1 and 2.
- Fig. 4: Normalized rapidity-gap correlation functions for $\alpha\alpha$ interactions at $\sqrt{s_{NN}} = 31.2$ GeV (a, b), and for pp interactions at $\sqrt{s} = 62.2$ GeV (c, d), 44.0 GeV (e, f), and 30.4 GeV (g, h). The functions are given for the 1st and 8th orders. The curves are results from the model calculations described in the text.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

APPENDIX A

Let x_i be N uncorrelated observations of a random variable distributed according to some known function $h(x)$. We order the N observations, so that $x_1 \leq x_2 \leq \dots \leq x_n \leq \dots \leq x_N$. The probability density function $g_n(x)$ of x_n will follow:

$$g_n(x) dx = n \binom{N}{n} h(x) \left\{ H(x) \right\}^{n-1} \left\{ 1 - H(x) \right\}^{N-n} dx ,$$

where

$$H(x) = \int_{-\infty}^x h(t) dt , \quad \text{with } H(\infty) = 1 ,$$

since $N - n$ observations have $x \geq x_n$ and $n - 1$ observations have $x \leq x_n$, and since

$$\frac{N!}{(n-1)!(N-n)!} = n \binom{N}{n} .$$

The corresponding two-dimensional distribution for $x_n = x$ and $x_{n+k} = x'$ will be

$$\begin{aligned} g_{n,k}(x, x') dx dx' &= nk \binom{N}{k} \binom{N-k}{n} h(x) h(x') [H(x)]^{n-1} \\ &\times [H(x') - H(x)]^{k-1} [1 - H(x')]^{N-n-k} dx dx' \end{aligned}$$

for $k = 1, 2, \dots, N - n$. Note that

$$nk \binom{N}{k} \binom{N-k}{n} = \frac{N!}{(n-1)!(k-1)!(N-n-k)!}$$

With $\Delta = x_{n+k} - x_n$ we obtain

$$\begin{aligned} f_{n,k}(\Delta) d\Delta &= d\Delta \int_{-\infty}^{+\infty} g_{n,k}(x, x+\Delta) dx = d\Delta nk \binom{N}{k} \binom{N-k}{n} \\ &\times \int_{-\infty}^{+\infty} h(x) h(x+\Delta) [H(x)]^{n-1} [H(x+\Delta) - H(x)]^{k-1} [1 - H(x+\Delta)]^{N-n-k} dx . \end{aligned}$$

Finally, by summing over all n , where $1 \leq n \leq N - k$ (and dividing with $N - k$ to normalize the integral of the resulting function to unity), we obtain the distribution of Δ :

$$\begin{aligned} f_k(\Delta) d\Delta &= \frac{k d\Delta}{N - k} \binom{N}{k} \int_{-\infty}^{+\infty} h(x) h(x+\Delta) [H(x+\Delta) - H(x)]^{k-1} \\ &\times \sum_{n=1}^{N-k} n \binom{N-k}{n} [H(x)]^{n-1} [1 - H(x+\Delta)]^{N-n-k} dx . \end{aligned}$$

But, since

$$\sum_{n=1}^N n \binom{N}{n} A^{n-1} B^{N-n} = N(A+B)^{N-1} ,$$

we obtain

$$\begin{aligned} f_k(\Delta) \, d\Delta &= k \binom{N}{k} d\Delta \int_{-\infty}^{+\infty} h(x)h(x+\Delta) [H(x+\Delta) - H(x)]^{k-1} \\ &\quad \times [1 - H(x+\Delta) + H(x)]^{N-k-1} dx , \end{aligned}$$

which we recognize as (6).