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NEW SYMMETRIES AND GHOST STRUCTURE OF COVARIANT STRING THEORIES

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ABSTRACT

It is shown that there exists an infinite set of new symmetries of the previously given covariant string formulations. These symmetries have themselves an infinite set of hidden local symmetries and so on. A new physically equivalent further extended string action is given in which the infinite set of symmetries is most easily displayed. A quantization involving gauge fixing and ghosts of the various covariant string actions is given.

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Despite the long history of string theory, a covariant formulation was not formulated. Recently¹⁾, a procedure involving the introduction of supplementary string fields was given for constructing a local gauge covariant formulation of string theories. In Ref. 1), this procedure was carried out for the free theory up to the sixth level and more recently²⁾ covariant formulations of all free string theories were found using only a finite number of supplementary fields. These results were generalized for bosonic strings³⁾ and supersymmetric strings⁴⁾ to contain an infinite number of supplementary fields and so place the generators of the Virasoro algebra on a more equal footing. An even larger covariant formulation of the open bosonic string has been given^{5),6)}. The results in Ref. 6) were, in fact, obtained by working backwards from the gauge fixed action of Refs. 7), 8) and 9).

Since the actions of Refs. 2)-6) propagate the same physical states at the classical level, one may wonder how their structure differs at the quantum level. Naive gauge fixing of the actions of Refs. 2)-4) leads to an incorrect count of the quantum degrees of freedom; as we shall explain here, this apparent paradox is explained by the fact that these actions actually have a larger gauge invariance. This larger gauge invariance turns out to have its own hidden local invariance, which leads to the phenomenon of ghosts for ghosts and restores the count of degrees of freedom to what it should be. This procedure repeats itself, and we are led to an infinite set of nested gauge invariances, and their corresponding ghosts, ghosts for ghosts, ghosts for ghosts for ghosts, etc. This remarkable set of invariances is most easily expressed with a new, enlarged, set of supplementary fields, which contains as subtheories those of Refs. 2)-6). With this enlarged set, the invariant action remains linear in the Virasoro operators. We show that upon quantization it leads to the correct count of degrees of freedom at all mass levels.

In Ref. 2), it was remarked that a "first order" formalism was best suited for supersymmetry, because fermions obey first order equations of motion. Actually, in the course of investigating the supersymmetry of the action of Ref. 4), we were led to the conclusion that, at least for elegance, one needs a formulation where there exist equations of motion that contain the superconformal operators (i.e., L 's) acting on any field which appears in the theory. We were thus also led by a different route to the results which we report in this paper. Their supersymmetric extension and the supersymmetry transformation will be discussed in a separate publication.

Finally, this new action and these invariances are written in extremely simple compact and elegant form with the introduction of two classical anticommuting degrees of freedom on the string.

Let us consider the quantization of the formulation of Ref. 3). It consists of the fields ψ , ϕ^n , ζ_m^n and the action is given by

$$\begin{aligned} & \frac{1}{2}(\psi, (L_0 - 1)\psi) + \sum_{n=1}^{\infty} [(L_n \psi, \varphi^n) + n(\varphi^n, \varphi^n)] \\ & + \sum_{n,m=1}^{\infty} [(L_n \varphi^m + (2n+m)\varphi^{n+m}, \zeta_m^n) - \frac{1}{2}(\zeta_m^n, (L_0 + n + m - 1)\zeta_m^n)] \quad (1) \end{aligned}$$

It is invariant under the gauge transformations

$$\begin{aligned} \delta \psi &= \sum_{n=1}^{\infty} L_{-n} \Lambda^n \\ \delta \varphi^n &= -(L_0 + n - 1) \Lambda^n \\ \delta \zeta_m^n &= -2L_m \Lambda^n - (2m + n) \Lambda^{n+m} \end{aligned} \quad (2)$$

Fixing the gauge by setting $\phi^n = 0$ would completely fix the parameters Λ^n . We could then introduce ghosts c^n and antighosts $\bar{c}_n$. At this stage, it is possible to count the on-shell degrees of freedom by subtracting the number of fields in c^n and $\bar{c}_n$ which propagate from those in ψ and ζ_m^n . (We note that ϕ^n is an auxiliary field which does not propagate.) This works at the first two levels in the sense that it agrees with the well-known result obtained from the partition function in 24 dimensions namely

$$\prod_{n=1}^{\infty} \frac{1}{(1-z^n)^{24}} = \sum_n T^{24}(n) z^n \quad (3)$$

where $T^{24}(n)$ is the number of on-shell states at level n . However, at the third level we find two states too few and at higher levels, the situation becomes progressively worse. This leads us to suspect that the above considerations have missed something. It is well known that when quantizing any system, one must be aware of all local symmetries or one will not introduce the corresponding ghosts and the on-shell count will not be correct.

In fact, this is the case and one can verify, up to the fourth level, that the following variations are an invariance of the action of Eq. (1)

$$\begin{aligned} \delta\psi &= 0, \quad \delta\varphi^1 = -L_2\Omega_3 - \frac{1}{4}L_3L_1\Omega_3, \quad \delta\varphi^2 = L_1\Omega_3 \\ \delta\varphi^3 &= \Omega_3 - \frac{1}{4}L_1L_1\Omega_3, \quad \delta\varphi^4 = -\frac{1}{2}L_1\Omega_3, \\ \delta\zeta'_2 &= -4\Omega_3, \quad \delta\zeta^2_1 = 2\Omega_3, \quad \delta\zeta^3_1 = -\frac{1}{2}L_1\Omega_3, \\ \delta\zeta'_3 &= \frac{3}{2}L_1\Omega_3. \end{aligned} \quad (4)$$

One may observe that these transformations now have themselves a hidden invariance given at the third level by

$$\begin{aligned} \delta\Lambda_1 &= -L_2T, \quad \delta\Lambda_2 = L_1T, \quad \delta\Lambda_3 = T, \\ \delta\Omega_3 &= (L_0+2)T. \end{aligned} \quad (5)$$

This invariance is hidden in the sense that it leaves all fields of the theory inert. The above gauge fixing and ghost discussion must now be modified. At the third level, we still take as our gauge fixing terms $\phi_1 = \phi_2 = \phi_3 = 0$. This fixes the gauges Λ_1 , Λ_2 and Λ_3 and the only invariance left is the hidden invariance T which as usual leads to an invariance of the ghost action. This invariance is fixed by setting the ghost of the extra symmetry equal to zero. We then find two commuting ghosts for ghosts. Carrying this analysis out in detail leads to the following gauge fixed and ghost action up to the third level

$$\begin{aligned} &\frac{1}{2}(\psi, (L_0-1)\psi) + \sum_{n,m=1}^2 (\zeta^m_n, (L_0+n+m-1)\zeta^m_n) \\ &+ (\bar{c}^1, -L_0c_1) + (\bar{c}_2, -(L_0+1)c_2) + (\bar{c}^3, -(L_0+2)c_3) \\ &+ (\bar{e}, (L_0+2)e) \end{aligned} \quad (6)$$

The commuting ghost e corresponds to the hidden symmetry T, while the ghost for the extra symmetry Ω_3 does not appear since it is set to zero by the δ function in the path integral. From the point of view of counting the $c_3(\bar{c}_3)$ and $e(\bar{e})$ cancel each other out and we recover the correct number of on-shell states.

At the fourth level there is no new hidden symmetry and so one only requires one ghosts and one antighost corresponding to the symmetry Λ_4 . It is easily verified that this gives the correct count at the fourth level.

The origin of the additional symmetry Ω_3 can be traced to the identity

$$[L_{-2}, L_{-1}] = -L_{-3}, \quad (7)$$

since it is this identity that allows ϕ to be inert and its equation of motion invariant under Ω_3 . At higher levels there exist many more identities and correspondingly many more new symmetries.

The situation for the finite set of supplementary fields of Ref. 2) is similar. If we consider the open bosonic string, the first non-trivial identity originates from writing $5 = 3+2$ or $5 = 4+1$. Hence starting from the identity

$$[L_{-4}, L_{-1}] - 3[L_{-3}, L_{-2}] = 0, \quad (8)$$

and writing L_{-4} and L_{-3} in terms of L_{-2} and L_{-1} we find an identity of the form

$$L_{-1} A^{-4} + L_{-2} B^{-3} = 0 \quad (9)$$

The corresponding hidden new symmetry is given by

$$\begin{aligned} \delta \psi &= 0, \quad \delta \varphi^1 = A^{-4} \Omega_5, \quad \delta \varphi^2 = B^{-3} \Omega_5, \\ \delta \zeta^1_1 &= (-L_3 + \frac{2}{3} L_{-1} L_{-2}) \Omega_5 \\ \delta \zeta^1_2 &= (2L_{-2} + \frac{2}{3} L_{-1}^2) \Omega_5 \\ \delta \zeta^2_1 &= -(2L_{-2} + L_{-1}^2) \Omega_5 \\ \delta \zeta^2_2 &= -7L_{-1} \Omega_5 \end{aligned} \quad (10)$$

Again one has a hidden symmetry given at the fifth level by

$$\delta \Lambda_1 = A^{-4} T, \quad \delta \Lambda_2 = B^{-3} T, \quad \delta \Omega_5 = (L_0 + 4) T \quad (11)$$

which leaves all the fields inert. The gauge fixing is given by $\phi_1 = \phi_2 = 0$ and for which we have respectively two ghosts and two antighosts. The hidden symmetry leads to two ghosts for ghosts. The reader may verify that this gives the correct on-shell count up to and including the fifth level. For a detailed discussion of the quantization of the finite set and the set of Ref. 3), the reader is referred to Ref. 10).

Although the new symmetries, at the level at which they first occur are rather simple, they become rather complicated at subsequent levels [see Eq. (4)].

This situation is rather reminiscent of that in Ref. 1), where similar complications were overcome by the introduction of extra fields. It is natural therefore to extend the results of Eqs. (1) and (2). The obvious way to do this is to introduce L_n 's acting on the ζ_n^m . This will require the introduction of new fields ϕ_p^{mn} . However, the knowledge of L_n on these fields requires further new fields ϕ_{rs}^{mn} . Although one could stop at this stage, one can continue this process to all orders, which we will now do.

We therefore introduce tensors ϕ_K^K with K indices upstairs and K downstairs and tensors ϕ_K^{K+1} which have K lower and $K+1$ upper indices. All indices in the same level are antisymmetrized. To construct the action, it is useful to recall the tensor calculus introduced by Banks and Peskin⁵⁾. They introduced the following operations on a tensor with a upper and b lower indices

$$\begin{aligned}
 (dC)_{[n_1 \dots n_{b+1}]}^{[m_1 \dots m_a]} &= L_{[n_1} C_{n_2 \dots n_{b+1}]^{[m_1 \dots m_a]} + a W_p^{[m_1} C_{n_2 \dots n_{b+1]}^{p m_2 \dots m_a]} \\
 &\quad - \frac{1}{2} \theta V_{[n_1 n_2}^p C_{p n_3 \dots n_{b+1}]^{[m_1 \dots m_a]} , \\
 (DC)_{[n_1 \dots n_b]}^{[m_1 \dots m_{a+1}]} &= L_{[n_1} C_{n_2 \dots n_b]^{p m_1 \dots m_{a+1}}} + \theta W_{[n_2 p}^q C_{q n_3 \dots n_b]^{p m_1 \dots m_{a+1}}} \\
 &\quad - \frac{1}{2} (a+1) V_{[n_1 n_2}^{kl} C_{kl n_3 \dots n_b]^{[m_1 \dots m_{a+1}]} , \\
 (KC)_{[n_1 \dots n_b]}^{[m_1 \dots m_a]} &= (b+m+n) C_{[n_1 \dots n_b]^{m_1 \dots m_a}} , \quad m = \sum m_i, \quad n = \sum n_i, \\
 (\eta C)_{n_1 \dots n_{b+1}}^{m_1 \dots m_{a+1}} &= \eta_{[n_2}^q C_{n_3 \dots n_{b+1]}^{q m_1 \dots m_{a+1}}} .
 \end{aligned} \tag{12}$$

The V_{mn}^p and W_{nm}^p are the structure constants that appear in the Virasoro algebra

$$\begin{aligned}
 [L_n, L_m] &= V_{nm}^p L_p , \\
 [L_{-n}, L_{-m}] &= -V_{nm}^p L_{-p} , \\
 [L_m, L_{-n}] &= W_{mn}^p L_p + W_{nm}^p L_{-p} + \eta_{mn} L(m) ,
 \end{aligned}$$

where

$$L(m) = 2L_0 + \frac{13}{8}(m^2 - 1).$$

These operators satisfy the relations

$$d^2 = D^2 = 0 = dD - Dd - 2\gamma K. \quad (13)$$

The equations of motion corresponding to the action of Eq. (1) can be written in the form

$$\begin{aligned} K\psi_0^0 + D\psi_0^1 &= 0 \\ d\psi_0^0 - 2\gamma\psi_0^1 + D\psi_1^1 &= 0 \\ d\psi_0^1 - K\psi_1^1 &= 0, \end{aligned} \quad (14)$$

while the invariance becomes

$$\delta\psi_0^0 = D\Lambda_0^0, \quad \delta\psi_1^1 = -d\Lambda_0^1, \quad \delta\psi_0^1 = -K\Lambda_0^0. \quad (15)$$

The extension to the infinite set of tensors discussed above is obvious. The first two equations are as before and the next few equations become

$$\begin{aligned} K\psi_1^1 - d\psi_0^1 + D\psi_1^2 &= 0, \\ d\psi_1^1 + 2\gamma\psi_1^2 + D\psi_2^2 &= 0, \\ K\psi_2^2 - d\psi_1^2 + D\psi_2^3 &= 0, \end{aligned} \quad (16)$$

while the general equations are

$$\begin{aligned} K \psi_n - d \psi_{n-1} + D \psi_n^{K+1} &= 0, \\ d \psi_n + 2\gamma \psi_n^{K+1} + D \psi_{n+1}^{K+1} &= 0. \end{aligned} \quad (17)$$

The corresponding action is given by

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{2} (\psi_K^K, K \psi_K^K) + (d \psi_K^K, \psi_K^{K+1}) \right. \\ \left. - (\psi_K^K, d \psi_{K-1}^K) + (\psi_K^{K+1}, \gamma \psi_K^{K+1}) \right]. \end{aligned} \quad (18)$$

This action is invariant under the following transformations which generalize those of Eq. (15).

$$\begin{aligned} \delta \psi_K^K &= -d \wedge_{K-1}^K + D \wedge_K^{K+1}, \\ \delta \psi_K^{K+1} &= -K \wedge_K^{K+1}. \end{aligned} \quad (19)$$

However, corresponding to the new symmetries found above, the action of Eq. (18) is also invariant under

$$\begin{aligned} \delta \psi_K^K &= 2\gamma \wedge_{K-1}^{K+1}, \\ \delta \psi_K^{K+1} &= D \wedge_K^{K+2} + d \wedge_{K-1}^{K+1}. \end{aligned} \quad (20)$$

Consequently we may write the full transformation as

$$\begin{aligned} \delta \psi_K^K &= -d \wedge_{K-1}^K + D \wedge_K^{K+1} + 2\gamma \wedge_{K-1}^{K+1}, \\ \delta \psi_K^{K+1} &= -K \wedge_K^{K+1} + D \wedge_K^{K+2} + d \wedge_{K-1}^{K+1}. \end{aligned} \quad (21)$$

These transformations have a hidden invariance which leaves all fields inert and which is given by

$$\begin{aligned}\Lambda_K^{K+1} &= \mathcal{D} S_{K-1}^{K+1} + d S_{K-2}^K + 2\gamma S_{K-2}^{K+1}, \\ \Lambda_{K-1}^{K+1} &= -d S_{K+2}^{K+1} + \mathcal{D} S_{K-1}^{K+2} + K S_{K-1}^{K+1}.\end{aligned}\quad (22)$$

Examining Eq. (22) we find that the right-hand side has the same structural form as the equations of motion of Eq. (17) and thus S in turn has a hidden invariance. Clearly this process goes on indefinitely, each step leading to another hidden invariance which involves a tensor with more upper and less lower indices. Starting from ϕ_K^K and ϕ_K^{K+1} we obtain the following chain of symmetries

$$\begin{aligned}\Lambda_{K-1}^K, \Lambda_{K-1}^{K+1}; S_{K-1}^{K+1}, S_{K-2}^{K+1}; T_{K-2}^{K+1}, T_{K-2}^{K+2}; \\ R_{K-2}^{K+2}, R_{K-2}^{K+3}; \text{etc} \dots\end{aligned}\quad (23)$$

Let us now gauge fix and quantize this theory. We first fix the Λ_K^{K+1} invariance by choosing our gauge fixing condition to $\phi_K^{K+1} = 0$. In fact, this then only leaves the hidden invariances since

$$d\varphi_K^K - \mathcal{D}\varphi_K^{K+1} = 0 = K(-d\Lambda_{K-1}^K + \mathcal{D}\Lambda_K^{K+1} + 2\gamma\Lambda_{K-1}^{K+1}). \quad (24)$$

At this first step, we implement the gauge fixing by using $\delta(\phi_K^{K+1})$ in the path integral and adding the ghost terms

$$\bar{c}_{K+1}^K (-K c_K^{K+1} + \mathcal{D} c_K^{K+2} + d c_{K-1}^{K+1}). \quad (25)$$

Clearly, the ghost action has a hidden invariance analogous to that given in Eq. (22). We fix this by choosing the gauge fixing term c_{K-1}^{K+1} which is implemented by putting $\delta(c_{K-1}^{K+1})$ in the path integral as well as the following ghost for ghost terms

$$\bar{c}_{K+1}^{K+1} (\mathcal{D} c_{K-1}^{K+2} - d c_{K-2}^{K+1} + K c_{K-1}^{K+1}). \quad (26)$$

This ghost again has another hidden invariance leading to ghosts for ghosts for ghosts, etc.

We may now simply implement the δ functions and set $\phi_K^{K+1}, c_{K-1}^{K+1}, \dots$, to zero. An equivalent procedure would be to write the δ functions using Lagrange multipliers. Performing the former course of action we are led to the following set of commuting fields

$$\begin{array}{ccccccc}
 \dots & \psi_4^0 & \psi_2^0 & \psi_0^0 & \psi_0^2 & \psi_0^4 & \dots \\
 & \dots & \psi_3^1 & \psi_1^1 & \psi_1^3 & \dots & \\
 & & \dots & \psi_2^2 & \psi_2^4 & \dots & \\
 & & & \vdots & \psi_3^3 & \vdots & \\
 & & & & \vdots & &
 \end{array} \quad (27)$$

The central column is the original fields, ϕ_K^K , the next columns on the left and right are the ghosts for ghosts and their corresponding antighosts for ghosts, respectively. The next column contains ghost for ghost for ghost for ghosts, etc. As far as anticommuting fields we have the following content

$$\begin{array}{ccccccc}
 \dots & \Lambda_5^0 & \Lambda_3^0 & \Lambda_1^0 & \Lambda_0^3 & \Lambda_0^5 & \dots \\
 & \dots & \Lambda_4^1 & \Lambda_2^1 & \Lambda_1^2 & \Lambda_1^4 & \dots \\
 & & \dots & \Lambda_5^2 & \Lambda_3^2 & \Lambda_2^3 & \Lambda_2^5 & \dots \\
 & & & \vdots & \vdots & \vdots & \vdots &
 \end{array} \quad (28)$$

Clearly, there is one and only one tensor of the type Λ_q^p for every p and q and the tensor is commuting if $p+q$ is even and anticommuting if $p-q$ is odd. The quantized linearized action is of the form

$$\sum_{K=0}^{\infty} (-1)^K \frac{1}{2} \psi_K^K K \psi_K^K + \sum_{K=1}^{\infty} \sum_{n=1}^K \bar{\Lambda}_{K+n}^{K-n} K \Lambda_{K-n}^{K+n}, \quad (29)$$

where we have used the same symbol for the ghosts and their corresponding invariance. Here we recognize the BRST quantized theory obtained by Siegel⁷⁾. It is straightforward to deduce the BRS change which reflects many of the properties given above.

We will now demonstrate that the formulation given above has the correct on-shell count. Consider a given tensor Λ_q^p which has c indices in common, i.e.,

$$\bigwedge_{m_1, \dots, m_a, p_1, \dots, p_c}^{m_1, \dots, m_a, p_1, \dots, p_c} \quad (30)$$

This tensor first occurs at level $M = \sum_{i=1}^a m_i + \sum_{i=1}^b n_i + 2 \sum_{i=1}^c p_i$ and is commuting or anticommuting if $\sum_{i=1}^a m_i + \sum_{i=1}^b n_i$ is even or odd, respectively. Counting anticommuting tensors as negative degrees of freedom, the number of such contributions c_M coming from the tensors of Eq. (27) and (28) at level N , is given by the power of x^M in the partition function

$$\sum_{M=0}^{\infty} c_M x^M = \prod_{n=1}^{\infty} (1 - 2x^n + x^{2n}) = \prod_{n=1}^{\infty} (1 - x^n)^2. \quad (31)$$

Each tensor contributes at level N $T^{26}(N-M)$ degrees of freedom and hence the total number of degrees of freedom is

$$\sum_{M=0}^N c_M T^{26}(N-M). \quad (32)$$

Multiplying by x^N we find that the number of degrees of freedom $p(N)$ at level N is given by

$$\begin{aligned} \sum_{N=0}^{\infty} p(N) x^N &\equiv \sum_{N=0}^{\infty} \sum_{M=0}^N c_M T^{26}(N-M) x^{N-M} x^M \\ &= \prod_{n=1}^{\infty} (1 - x^n)^2 \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{26}} \\ &= \prod_{n=1}^{\infty} (1 - x^n)^{-24} \equiv \sum_{N=0}^{\infty} T^{24}(N) x^N, \end{aligned} \quad (33)$$

which establishes the desired reduction of two dimensions in the count of degrees of freedom to obtain the same count of on-shell degrees of freedom as in the light cone formulation

We are now ready to discuss the relation of the above considerations with the other known gauge covariant formulations. Let us begin with the formulation of Ref. 5). It is stated in Ref. 5) to be the same as that of Ref. 6) although we have not checked this point.

The reader will have noticed that we have not imposed any symmetrization on the fields ψ_K^K which enter in the gauge invariant action. This is to be

contrasted with the prescription of Ref. 5) where the field ψ_K^K had to belong to a definite symmetry type corresponding to the Young tableau (with k rows). This prescription can be recovered from the present formulation by first using the "new" symmetry parameters Λ_{K-1}^{K+1} in (20) instead of Λ_K^{K+1} as was done above. The irreducible components contained in Λ_{K-1}^{K+1} correspond to the Young tableaux in the decomposition of the product

$$\Lambda_{K-1}^{K+1} : K+1 \left\{ \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} \otimes K-1 \left\{ \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} \quad (34)$$

The analogous decomposition for the field ψ_K^K corresponds to

$$\psi_K^K : K \left\{ \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} \otimes K \left\{ \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} \quad (35)$$

and contains the same components as (34) plus the additional component corresponding to the Young tableau

$$\psi_K^K \text{ restricted} : K \left\{ \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} (+ \text{traces}) \quad (36)$$

which does not occur in (34). Therefore, one can use Λ_{K-1}^{K+1} to eliminate all irreducible components of ψ_K^K with the exception of (36). In order to preserve this symmetry type, one must then require the parameters Λ_K^{K+1} to be of the type

$$\Lambda_K^{K+1} \text{ restricted} : K+1 \left\{ \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} (+ \text{traces}) \quad (37)$$

After eliminating the auxiliary field ϕ_K^{K+1} , one finds agreement with Ref. 5). The other gauge covariant formulation, as given in Refs. 2) and 3), can also be found by an appropriate gauge choice of the formulation given in this paper. There will also exist an infinite number of other gauge covariant formulations corresponding to other partial gauge choices.

Finally, we reformulate our results by introducing anticommuting

co-ordinates which were first considered by Ogawa and Kato⁹⁾ and further developed by Siegel⁷⁾ in the context of the BRST quantization of the string. Let us consider the real anticommuting co-ordinates $c(\sigma)$ and $\bar{c}(\sigma)$ which correspond to half of the ghost and antighost co-ordinates of Ref. 7). They satisfy the boundary conditions, $\bar{c}$ and $(\partial/\partial\sigma)c$ vanish at $\sigma = 0$ and $\sigma = \pi$ and so have the following expansions

$$c(\sigma) = c^0 + 2 \sum_{n=1}^{\infty} c_n \cos n\sigma, \quad (38)$$

$$\bar{c}(\sigma) = 2 \sum_{n=1}^{\infty} \bar{c}_n \sin n\sigma. \quad (39)$$

We define the creation and annihilation operators

$$\frac{\delta}{\delta c(\sigma)} = \frac{1}{2\pi} \quad \bar{c}(\sigma) = \pi^{-1/2} \sum_{n=1}^{\infty} \bar{\beta}_n e^{in\sigma} \quad (40)$$

$$- \frac{\delta}{\delta \bar{c}(\sigma)} + \frac{1}{2\pi} \quad c(\sigma) = \pi^{-1/2} \sum_{n=1}^{\infty} \beta_n e^{in\sigma} \quad (41)$$

$$\beta_0 = c^0, \quad \bar{\beta}_0 = \frac{\partial}{\partial c^0}, \quad \bar{\beta}_n^+ = \bar{\beta}_n, \quad \beta_n^+ = \beta_n, \quad (42)$$

which obey the relations

$$\{\bar{\beta}_0, \beta_0\} = 1, \quad \{\bar{\beta}_n^+, \beta_m\} = \{\bar{\beta}_n, \beta_m^+\} = \delta_{n,m}; \quad n, m \geq 1 \quad (43)$$

the remaining anticommutators vanish.

Any functional of $x^\mu(\sigma)$, $c(\sigma)$, $\bar{c}(\sigma)$ can be written in terms of Hermite polynomials and so the above operators. We define the vacuum to be

$$\bar{\beta}_n |> = \beta_n |> = 0. \quad (44)$$

The properties of the vacuum under the zero mode operators β^0 , $\bar{\beta}^0$ is discussed in Ref. 8) and we consider two vacua $|>_+$ and $|>_-$ which are defined by

$$\beta^0 |>_+ = 0, \quad |>_- = \bar{\beta}^0 |>_+, \quad (45)$$

as well as the conditions of Eq. (44). One easily deduces that

$$\bar{\beta}_0 | \gamma_- = 0. \quad (46)$$

The most general functional of $x^\mu(\sigma)$, $c(\sigma)$ and $\bar{c}(\sigma)$ may therefore be written in an oscillator basis as

$$\begin{aligned} \chi = & \sum_{\{n\}\{m\}} \psi_{n_1 \dots n_a}^{m_1 \dots m_a} \beta_1^{n_1+} \dots \beta_a^{n_a+} \bar{\beta}_{m_1}^+ \dots \bar{\beta}_{m_a}^+ | \gamma_- \\ & + \sum_{\{n\}\{m\}} \varphi_{n_1 \dots n_a}^{m_1 \dots m_{a+1}} \beta_1^{n_1+} \dots \beta_a^{n_a+} \bar{\beta}_{m_1}^+ \dots \bar{\beta}_{m_{a+1}}^+ | \gamma_+ . \end{aligned} \quad (47)$$

We observe that χ has exactly the same number of component field tensors as occurred in the quantized theory before we implemented the δ functions which set to zero the second term.

The gauge covariant theory before quantization contains the fields $\psi_{n_1 \dots n_b}^{m_1 \dots m_a}$ and $\phi_{n_1 \dots n_b}^{m_1 \dots m_{a+1}}$ with the constraint that $a = b$. This is similar to the $L_0 = \bar{L}_0$ constraint of the closed string and we may implement it by imposing the algebraic condition

$$\left[\sum_n (\beta^{n+} \bar{\beta}^n - \bar{\beta}_n^+ \beta_n) + \beta^0 \bar{\beta}^0 \right] \chi = 0 \quad (48)$$

The tensor calculus of Ref. 5) can be written in a more compact form using the above oscillators. Following Ref. 9), consider the objects

$$\begin{aligned} d &= \beta^{n+} \left[L_n + W_{mn}^p \bar{\beta}_p^+ \beta^m + \frac{1}{2} \beta^{m+} V_{mn}^p \bar{\beta}_p \right], \\ D &= \beta^m \left[L_{-m} + W_{mn}^p \beta^{n+} \bar{\beta}_p + \frac{1}{2} \bar{\beta}_p V_{mn}^p \beta^m \right], \end{aligned}$$

$$K = \beta^0 (L-1 + \sum_n n \beta_n^+ \bar{\beta}_n + n \bar{\beta}^+ \beta_n) ,$$

$$M = 2 \bar{\beta}^0 \sum_{n=1}^{\infty} n \beta_n^+ \beta_n . \quad (49)$$

When applied to the arbitrary functional of Eq. (47), they yield the operations, denoted by the same symbols of Eq. (12), up to numerical factors.

The action of Eq. (18) is now given in terms of χ , which is subject to Eq. (48), and is given by

$$\langle \chi | Q | \chi \rangle , \quad (50)$$

where

$$Q = K + d + D + M. \quad (51)$$

This action when expressed in functional form is

$$\int dx^*(\sigma) dc(\sigma) d\bar{c}(\sigma) \chi[x^*(\sigma), c(\sigma), \bar{c}(\sigma)] \tilde{Q} \chi[x^*(\sigma), c(\sigma), \bar{c}(\sigma)] , \quad (52)$$

where $\tilde{Q}$ is the expression for Q in functional form.

The transformations of ψ and ϕ , which are given in Eqs. (21) and (22), can also be written in this language. Let us consider the transformation parameters

$$\begin{aligned} \Lambda = & \sum_{\{n\}\{m\}} \Lambda_{n_1 \dots n_g}^{m_1 \dots m_g} \beta^{n_1+} \dots \beta^{n_g+} \bar{\beta}_{m_1}^+ \dots \bar{\beta}_{m_g}^+ | >_- \\ & + \sum_{\{n\}\{m\}} S_{n_1 \dots n_g}^{m_1 \dots m_g} \beta^{n_1+} \dots \beta^{n_g+} \bar{\beta}_{m_1}^+ \dots \bar{\beta}_{m_g}^+ | >_+, \end{aligned} \quad (53)$$

which is subject to the constraint

$$\left(\sum_n \beta^{n+} \bar{\beta}^n - \bar{\beta}^{n+} \beta^n + \beta^0 \bar{\beta}^0 + 1 \right) \Lambda = 0. \quad (54)$$

The transformation law of Eq. (21) is then given by

$$\delta \chi = Q \Lambda, \quad (55)$$

and the invariance of the action follows from the identity

$$Q^2 = 0. \quad (56)$$

This latter result is a consequence of the formulae of Eq. (13) when transferred to the definitions of Eq. (49). The hidden symmetries have expressions identical to Eq. (55) but the constraints of Eq. (48) and (54) are changed by integers. It is important to stress that it is the classical theory and not the quantized BRST theory which has been expressed using two extra anticommuting variables.

However when written in this way, the transition to the BRST quantized theory⁷⁾ is so obvious that we will not spell it out. The only major difference is the relaxation of the constraints of Eq. (48) and (54). The above formalism allows a shorter proof that the formulation presented in this paper contains the correct number of degrees of freedom in the quantized theory. The number of physical states at a given level is given by the number of bosonic n_B minus the number of fermionic n_F degrees of freedom. Consider

$$\begin{aligned} & \sum_{N=0}^{\infty} (n_B(N) - n_F(N)) x^N \\ & = \text{STr} x \sum_{n=1}^{\infty} n (a_n^\dagger a_n + \beta_n^\dagger \bar{\beta}_n + \bar{\beta}_n^\dagger \beta_n) \end{aligned}$$

$$\begin{aligned}
 &= \prod_{n=1}^{\infty} (1-x^n)^{-26} \prod_{n=1}^{\infty} (1-x^n)^2 \\
 &= \prod_{n=1}^{\infty} (1-x^n)^{-24}.
 \end{aligned}$$

This proof illustrates the intuitive notion that the two-fermionic degrees of freedom subtract two degrees of freedom from the 26 bosonic co-ordinates to leave the 24 "physical" degrees of freedom.

The analogous results for the superstring theories will be spelt out elsewhere¹¹⁾.

Recently, the gauge covariant interacting bosonic open string has been found¹²⁾. This is indeed most easily given for the formulation given in this paper. As one may expect in the version involving $x^\mu(\sigma)$, $c(\sigma)$ and $\bar{c}(\sigma)$, the overlap δ functions play a prominent role. This role and an operator formulation of the vertex are also given in Ref. 12).

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