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INFRA-RED FINITENESS OF NORMALIZATION CONSTANTS OF
CURRENTS IN LATTICE GAUGE THEORIES

Giuseppe Curci^{*)}
CERN -- Geneva

A B S T R A C T

We present a simple proof of the infra-red finiteness of the renormalization of the strength of the flavour currents induced by lattice artifacts.

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*) On leave of absence from INFN, Sezione di Pisa Italy.

Recently the chiral symmetry limit of lattice QCD with Wilson fermions¹⁾ has been investigated by Bochicchio et al.²⁾ in order to give a general procedure suitable for Monte Carlo techniques to explore questions related to chirality in view, for example, of the exciting possibility of computing matrix elements of the non-leptonic weak Hamiltonian³⁾.

Starting from the analysis of lattice QCD in weak coupling perturbation theory (WCPT) given by Karsten and Smith (KS)⁴⁾, Bochicchio et al. have shown how to recover in the continuum limit results consistent with current algebra predictions.

Moreover, assuming that the structure of the relevant operators remains valid beyond WCPT, they propose non-perturbative methods to obtain well-normalized vector and axial vector flavour currents which can be employed in practical Monte Carlo measurements.

The normalization of the lattice currents is a necessary step of this programme as the lattice regularization leads in general to a renormalization of the strength of the point-split currents which has to be compensated by multiplicative factors Z_V, Z_A ⁴⁾. This effect is harmless if the rescaling Z factors have a smooth limit for vanishing quark masses. If this is the case, the Z factors can only be functions of the bare coupling constant and the asymptotic freedom of the theory implies $Z \rightarrow 1$ in the continuum limit.

As stressed in Ref. 2), the proof of the absence of divergences in the quark masses to all orders in WCPT is a necessary first step to show the consistency of the scheme and, in general, of the lattice formulation as non-perturbative regularization of QCD.

In this letter, we consider the normalization of the flavour currents in massless lattice QCD and we give a simple argument for the infra-red finiteness of Z_V and Z_A based on the infra-red finiteness of the fundamental N -point Green functions. As shown by Bochicchio⁵⁾, the lattice fundamental vertices satisfy the requirements of a general theorem of Poggio and Quinn⁶⁾ on the infra-red finiteness, for non-exceptional momenta, of renormalizable (but not super-renormalizable) Euclidean field theories containing zero mass particles.

According to this theorem, after the renormalization of the fermion self-mass, the quark and boson self-energies and the three-point boson vertices

are infra-red finite at zero external momenta and the N-point vertex functions are finite even for one zero external momentum.

For our discussion we need a corollary result of this theorem: the vector and axial form factors are infra-red finite at zero transferred momentum. The proof is trivial as the infra-red power counting arguments are the same as those used to show the infra-red finiteness of the $q\bar{q}g$ vertex.

We start by recalling the Wilson action¹⁾ on a Euclidean lattice of spacing a

$$S_W = S_U + S_F$$

with S_U the standard gauge field action and

$$\begin{aligned} S_F = a^4 \sum_x \left\{ \frac{1}{2a} \sum_{\mu} \left[\bar{\psi}(x)(\gamma_{\mu} - \tau) U_{\mu}(x) \psi(x + \hat{\mu}) \right. \right. \\ \left. \left. - \bar{\psi}(x + \hat{\mu})(\gamma_{\mu} + \tau) U_{\mu}^{\dagger}(x) \psi(x) + 2\tau \bar{\psi}(x) \psi(x) \right] \right. \\ \left. + M_0 \bar{\psi}(x) \psi(x) \right\} \quad (1) \end{aligned}$$

We suppress colour and flavour indices, the Dirac matrices are Hermitean and such that $\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}$, τ is a parameter that multiply the term introduced by Wilson to solve the fermion doubling problem¹⁾. M_0 is the bare quark mass determined by imposing the vanishing of the quark self-energy at zero momentum.

Within this formulation we consider the currents

$$\begin{aligned} V_{\mu}^{\alpha}(x) &= \frac{1}{2} \left\{ \bar{\psi}(x) \frac{\lambda^{\alpha}}{2} \gamma_{\mu} U_{\mu}(x) \psi(x + \hat{\mu}) + h.c. \right\} \\ A_{\mu}^{\alpha}(x) &= \frac{1}{2} \left\{ \bar{\psi}(x) \frac{\lambda^{\alpha}}{2} \gamma_{\mu} \gamma_5 U_{\mu}(x) \psi(x + \hat{\mu}) + h.c. \right\} \end{aligned} \quad (2)$$

where λ^{α} are the flavour matrices, $\text{Tr}(\lambda^{\alpha} \lambda^{\beta}) = 2\delta^{\alpha\beta}$. These currents are

(partially) conserved for $r = 0$ ⁴⁾. Explicitly

$$\langle \Delta_\mu V_\mu^\alpha(x) \Pi \rangle = \langle D_V^\alpha(x) \Pi \rangle - \langle \delta_V^\alpha(x) \Pi \rangle \quad (3.a)$$

$$\langle \Delta_\mu A_\mu^\alpha(x) \Pi \rangle = \langle D_A^\alpha(x) \Pi \rangle - \langle \delta_A^\alpha(x) \Pi \rangle \quad (3.b)$$

where the brackets denote the partition function average, $\Delta_\mu f(x) = [f(x) - f(x-\mu)]/a$, Π stands for a product of fields and the vector and axial transformations are defined by

$$\delta_V^\alpha(x) = \frac{\lambda^\alpha}{2} \psi(x) \frac{\delta}{\delta \psi(x)} - \bar{\psi}(x) \frac{\lambda^\alpha}{2} \frac{\delta}{\delta \bar{\psi}(x)} \quad (4)$$

$$\delta_A^\alpha(x) = \frac{\lambda^\alpha}{2} \gamma_5 \psi(x) \frac{\delta}{\delta \psi(x)} + \bar{\psi}(x) \frac{\lambda^\alpha}{2} \gamma_5 \frac{\delta}{\delta \bar{\psi}(x)}$$

The symmetry breaking terms are

$$D_V^\alpha(x) = \frac{\tau}{2a} \sum_\mu \left[\left(\bar{\psi}(x) \frac{\lambda^\alpha}{2} u_\mu(x) \psi(x+\hat{\mu}) - \bar{\psi}(x+\hat{\mu}) \frac{\lambda^\alpha}{2} u_\mu^+(x) \psi(x) \right) - (x \rightarrow x-\hat{\mu}) \right] \quad (6)$$

$$D_A^\alpha(x) = 2 M_0 P^\alpha(x) + X^\alpha(x) = 2 M_0 \bar{\psi}(x) \frac{\lambda^\alpha}{2} \gamma_5 \psi(x) + X^\alpha(x)$$

where

$$X^\alpha(x) = -\frac{\tau}{2a} \sum_\mu \left[\left(\bar{\psi}(x) \frac{\lambda^\alpha}{2} \gamma_5 u_\mu(x) \psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu}) \frac{\lambda^\alpha}{2} \gamma_5 u_\mu^+(x) \psi(x) \right) + (x \rightarrow x-\hat{\mu}) - 4 \bar{\psi}(x) \frac{\lambda^\alpha}{2} \gamma_5 \psi(x) \right]$$

For $r \neq 0$ KS show, with a one-loop calculation, that $D_V^\alpha \approx (1-Z_V) \Delta_\mu V_\mu^\alpha$ and, in the chiral limit, $D_A^\alpha \approx (1-Z_A) \Delta_\mu A_\mu^\alpha$; the $\approx$ sign signifies equality up to a

term that vanishes as $a \rightarrow 0$. And they obtain the currents $Z_V V_\mu^\alpha$ and $Z_A A_\mu^\alpha$ that are conserved in the continuum chiral limit.

Let us rewrite the result of KS in a form suitable for a general discussion.

We first examine the vector current. In this case the analysis is more simple as $D_V^\alpha = \Delta_\mu R_\mu^\alpha$ where

$$R_\mu^\alpha(x) = \frac{\tau}{2} \left[\bar{\psi}(x) \frac{\lambda^\alpha}{2} U_\mu(x) \psi(x+\hat{\mu}) - \bar{\psi}(x+\hat{\mu}) \frac{\lambda^\alpha}{2} U_\mu^\dagger(x) \psi(x) \right] \quad (7)$$

We have to show that $R_\mu^\alpha \approx (1-Z_V) V_\mu^\alpha$.

Note that $R_\mu^\alpha \approx -(ra/2) \bar{\psi}(\lambda^\alpha/2) \tilde{D}_\mu \psi + O(a^2)$, so R_μ^α is an irrelevant operator that formally vanishes as $a \rightarrow 0$. However, the insertion of R_μ^α in a Green function can induce, in the loop expansion, divergences of $O(1/a)$ that can compensate the explicit a factor of the vertex. This mechanism has been discussed at length by Sharatchandra⁷⁾. We report the results adapting his analysis to our case. The insertion of an irrelevant operator of the form $a^{k \times Q}$, where Q has dimension δ , in a Green function with n_f and n_b external legs, induce a superficial singularity in a of degree $D = \delta - n_b - \frac{3}{2} n_f$. The primitively divergent diagrams have the general form $a^{k \times O(1/a^D)}$ and Sharatchandra shows that we have to subtract the first $D-k+1$ terms in the Taylor series about the external momenta to obtain an integral of the form $a^{k \times O(1/a^{k-1})}$ that vanishes as $a \rightarrow 0$. In our case R_μ^α has $\delta = 4$ and $k = 1$ and a mixing with operators with $\delta < 3$ is possible.

An inspection of the lattice symmetries shows that R_μ^α can only mix with the vector current and, in the notation of Ref. 2), the subtraction procedure of Sharatchandra is equivalent to the definition of the operator $\bar{R}_\mu^\alpha$ such that

$$\bar{R}_\mu^\alpha = R_\mu^\alpha - (1-Z_V) V_\mu^\alpha \quad (8)$$

and $\langle \bar{R}_\mu^\alpha \rangle \approx 0$.

We can write (3.a) in the form

$$Z_V \Delta_\mu \langle V_\mu^\alpha(x) \Pi \rangle + \langle \delta_V^\alpha(x) \Pi \rangle \approx 0 \quad (9)$$

that is the result of KS.

We have now to prove that this procedure does not introduce infra-red divergences.

The constant $(1-Z_V)$ can be obtained in an explicit gauge invariant way by computing the correlation $\langle R_\mu^\alpha(x) V_\nu^\beta(y) \rangle$. For our purposes it is sufficient to consider the form factor $\Gamma_{R_\mu}^{\alpha(2,0)}$. We use the notation $\Gamma_0^{(n_f, n_b)}$ for the Fourier transform of the one-particle irreducible vertex functions with n_f quark legs, n_b boson legs and one insertion of an operator O .

A one-loop calculation shows that

$$\left[\Gamma_{R_\mu}^{\alpha(2,0)}(0,0) \right]^{(1)} = (1-Z_V)^{(1)} \frac{\Delta^\alpha}{2} \gamma_\mu = a \int_{-\pi/a}^{\pi/a} d^4 \ell F_\mu^\alpha(\ell) \quad (10)$$

where $F_\mu^\alpha(\lambda\lambda) = O(\lambda^{-3})$ as $\lambda \rightarrow 0$, and $(1-Z_V)^{(1)}$ is explicitly infra-red finite.

For the multiloop discussion we note that the $(1-Z_V)V_\mu^\alpha$ term in (8) acts as an infra-red counterterm for $\bar{R}_\mu^\alpha$ and the effect of the iterative subtractions is to recover the effective infra-red degree $\bar{\delta}_R = 4$ exhibited by R_μ^α at the tree level. As the other subdiagrams in $\Gamma_{\bar{R}_\mu}^{\alpha(2,0)}$ disjoint from the insertion have at most logarithmic singularities in a we expect, after the infra-red subtractions up to n^{th} loop, an effective degree of infra-red convergence for $[\Gamma_{\bar{R}_\mu}^{\alpha(2,0)}]^{n+1}$ given by $D = 4-3 = 1$. In loose notation, our guess is that

$$\left[\Gamma_{\bar{R}_\mu}^{\alpha(2,0)}(0,0) \right]^{n+1} \propto a \int_{-\pi/a}^{\pi/a} d^4 \ell \times O \left(\ln(a\ell)^n / \ell^3 \right) \quad (11)$$

with $(1-Z_V)^{n+1}$ infra-red finite.

An explicit check on the Feynman graphs of $[\Gamma_{\bar{R}_\mu}^{\alpha(2,0)}]^{(2)}$ confirms our guess up to the two-loop level.

We do not continue here the study of the infra-red finiteness of the subtractions of an irrelevant operator^{*}) as in our case we do not need such a detailed proof. It is sufficient to use the results on the infra-red finiteness of massless QCD.

Let us write the Ward identity

$$-i \hat{k}_\mu \Gamma_{V_\mu^\alpha}^{(2,0)}(p,q) = \Gamma^{(2,0)}(p) \frac{\lambda^\alpha}{2} - \frac{\lambda^\alpha}{2} \Gamma^{(2,0)}(q) - i \hat{k}_\mu \Gamma_{R_\mu^\alpha}^{(2,0)}(p,q) \quad (12)$$

where

$$\hat{k}_\mu = \frac{2}{a} \sin a \left(\frac{p-q}{2} \right)_\mu$$

From (12) and (8) we obtain

$$\Gamma_{V_\mu^\alpha}^{(2,0)}(p,p) - i \frac{\partial \Gamma^{(2,0)}(p)}{\partial p_\mu} \approx (1 - Z_V) \Gamma_{V_\mu^\alpha}^{(2,0)}(p,p) \quad (13)$$

and, as the form factor $\Gamma_{V_\mu^\alpha}^{(2,0)}(p,p)$ is infra-red finite, the left-hand side of (13) is infra-red finite at every order in the $(ap)^k (\ln ap)^j$ expansion⁵⁾. The iterative structure of (13) completes the proof of the infra-red finiteness of Z_V .

The reader will easily recognize that this is, *mutatis mutandis*, the argument of a famous appendix of a work of Preparata and Weisberger⁸⁾.

So we confirm the guess of KS on the equivalence of different (infinite) possible definitions on the lattice of the vector current, differing in their form by the contribution of irrelevant operators.

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^{*}) This result is tacitly assumed in the statements on the universality of different forms of the lattice action.

Let us now briefly examine the definition of the axial current. In this case X^α in (6) must not be a total derivative^{*}), explicitly $X^\alpha \approx \sum_{\mu} -(ra^2/8) \bar{\Psi}(\lambda^\alpha/2) \gamma_5 |\vec{D}_\mu|^2 \Psi$.

After an analysis of the possible mixings, we define an operator $\bar{X}^{\alpha 2)$, such that $\langle \bar{X}^{\alpha} \Pi \rangle \approx 0$, given by

$$\bar{X}^\alpha = X^\alpha + 2\bar{M} p^\alpha - (1 - Z_A) \Delta_\mu A_\mu^\alpha \quad (14)$$

The proof of the infra-red finiteness of Z_A is in two steps.

We write the Ward identity

$$\begin{aligned} -i \hat{k}_\mu \Gamma_{A_\mu^\alpha}^{(2,0)}(p, q) &= 2M_0 \Gamma_{p^\alpha}^{(2,0)}(p, q) + \Gamma_{X^\alpha}^{(2,0)}(p, q) \\ &- \frac{\lambda^\alpha}{2} \gamma_5 \Gamma^{(2,0)}(q) - \Gamma^{(2,0)}(p) \frac{\lambda^\alpha}{2} \gamma_5 \end{aligned} \quad (15)$$

For $k = p - q = 0$ we rewrite (15) in the form

$$2(M_0 - \bar{M}) \Gamma_{p^\alpha}^{(2,0)}(p, p) - \frac{\lambda^\alpha}{2} \gamma_5 \Gamma^{(2,0)}(p) - \Gamma^{(2,0)}(p) \frac{\lambda^\alpha}{2} \gamma_5 \approx 0 \quad (16)$$

and, as $\Gamma_{p^\alpha}^{(2,0)}(p, p)$ is infra-red finite, we obtain $M_0 - \bar{M} = 0$ by taking the $ap \rightarrow 0$ limit.

The second step is equivalent to the proof of the infra-red finiteness of Z_V as

$$\Gamma_{A_\mu^\alpha}^{(2,0)}(p, p) - i \frac{\partial \Gamma^{(2,0)}(p)}{\partial p_\mu} \frac{\lambda^\alpha}{2} \gamma_5 \approx (1 - Z_A) \Gamma_{A_\mu^\alpha}^{(2,0)}(p, p) \quad (17)$$

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*) In the singlet case this term gives the correct triangle anomaly .

and we use the same reasoning.

Starting from these results, it is possible to define products of currents. We do not report the relative discussion as it is parallel to that given in Ref. 2).

In conclusion, we have shown that the symmetry breaking terms due to lattice artifacts can be reabsorbed with finite rescalings. In this respect the lattice regularization does not differ from the usual continuum regularizations, as the BPHZ and dimensional schemes, where a finite perturbative rescaling of the axial current is a necessary step in order to reproduce the naïve axial vector Ward identities^{9),10)}.

Our technique is quite general and can be applied to other asymptotically free theories to show that the lattice artifacts vanish in the continuum limit, as, for example, to supersymmetric QCD.

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R E F E R E N C E S

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