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DYNAMICAL MASS GENERATION IN THREE-DIMENSIONAL YANG-MILLS THEORY

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A B S T R A C T

We calculate an effective potential for three-dimensional Yang-Mills theory to two loops. The result implies that a gauge invariant part of the field strength has a non-zero vacuum expectation value, ϕ . The two-point function of this field, calculated to one loop, shows that two polarization states propagate with a mass proportional to $\sqrt{\phi}$. In contrast to the usual Higgs mechanism the remaining degrees of freedom do not propagate. This dynamical generation of a "glueball" mass renders the already ultra-violet finite theory infra-red finite and stabilizes vortices associated with the centre of the group.

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1. - INTRODUCTION

The simplicity of pure Yang-Mills theories in three dimensions makes them attractive toy models in the search for confining mechanisms. 't Hooft has argued that quarks may be permanently confined by the presence of field configurations with non-trivial topological charges associated with the centre of the gauge group¹⁾. In Feynman's picture the gauge invariance of the wave functional implies a finite energy for any excitation above the ground state²⁾, and thus the dynamical generation of a glueball mass. Three-dimensional Yang-Mills theories are also of physical interest because they are the high temperature limit of four-dimensional theories³⁾.

In three dimensions the square of the gauge coupling constant has the dimension of mass so that pure gauge theories are super-renormalizable in the ultra-violet but divergent in the infra-red. Some of these divergences may be treated by resummation⁴⁾ or by exploiting gauge invariance⁵⁾, however, at sufficiently high order gauge invariant singularities occur in the loop expansion, signalling its breakdown. The failure of the perturbation expansion in the infra-red need not imply that the theory is inconsistent or does not exist because the problem would be cured by dynamical mass generation.

Topological mass terms with quantized coefficients may be added to the Yang-Mills Lagrangian in three dimensions⁶⁾. These are interesting because of their novelty, and also because they might enable the difficulties of the perturbation expansion to be avoided⁷⁾. Their origin has been considered by a number of authors⁸⁾, but we shall not discuss them here. For a review of three-dimensional gauge theories, see Ref. 9).

We will study a pure Yang-Mills theory in three dimensions based on the group $SU(2)$ which is the simplest model. We will show that at the one-loop level when the theory is expanded about a particular field configuration it describes two massive modes and the remaining degrees of freedom do not propagate. The configuration is characterized by a constant, ϕ , that is invariant under both gauge transformations and spatial rotations. The magnitude of ϕ determines the mass of the excitations and is itself chosen so as to minimize an effective potential which we calculate to two-loop accuracy. Thus glueballs emerge as massive, gauge invariant oscillations about a configuration with energy lower than that of the perturbative vacuum.

2. - AN EFFECTIVE POTENTIAL IN THE ONE-LOOP APPROXIMATION

If the addition of a term $\sigma F[\phi]$ to the action of a field theory results in a new Hamiltonian that is also linear in the external source, i.e.,

$$H \rightarrow H + \sigma \tilde{F}[\phi] \equiv H_\sigma,$$

then the Euclidean functional integral

$$\int \mathcal{D}\phi \exp - (S[\phi] + \sigma F[\phi]) \sim \exp - V \varepsilon_\sigma$$

in the limit of a large space-time volume V . ε_σ is the energy density corresponding to the minimum eigenvalue of H_σ , E_σ . Furthermore, if $|E_\sigma\rangle$ is the corresponding eigenstate

$$E'_\sigma \equiv \frac{\partial E_\sigma}{\partial \sigma} = \langle E_\sigma | \tilde{F}[\phi] | E_\sigma \rangle \equiv \tilde{F}_c,$$

$$\langle E_\sigma | H | E_\sigma \rangle = E_\sigma - J E'_\sigma \equiv W$$

W is the lowest expectation value of the Hamiltonian for a state in which $\tilde{F}_c$ is the expectation value of $\tilde{F}$. Minimizing W with respect to $\tilde{F}_c$ provides a variational estimate of the vacuum energy. For a gauge theory this programme is complicated by the requirement that the terms added to the action should be gauge invariant as well as respect the usual symmetries of Lorentz and translation invariance that we expect of the vacuum.

In three Euclidean dimensions it is convenient to work with the dual of the field strength which is given in terms of the $SU(2)$ gauge field, A , as

$$F_i^A = \epsilon_{ijk} \partial_j A_k^A + \frac{1}{2} g \epsilon_{ijk} \epsilon^{ABC} A_j^B A_k^C \quad (2.1)$$

The space-time index, i , and the colour index A both take on the values 1,2,3 so that the components may be assembled into a 3×3 matrix F . Under space-time rotations the matrix transforms as

$$F \rightarrow F \Lambda \quad (2.2)$$

where Λ is an element of $SO(3)$ and under gauge transformations F transforms as

$$F \rightarrow G F \quad (2.3)$$

where $G(x)$ is an element of (colour) $SO(3)$. Because F transforms locally under gauge transformations it is straightforward to decompose it into gauge variant and invariant pieces. The invariant piece may then be used to construct an effective potential. The real symmetric matrix FF^T may be diagonalized by a gauge transformation, H , so

$$H F F^T H^T = \text{diag} (\lambda_1, \lambda_2, \lambda_3) \quad (2.4)$$

where the λ are positive or zero. Thus F must have the form

$$F = H^T \cdot \text{diag} (\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}) \cdot \Omega \quad (2.5)$$

with Ω an $SO(3)$ matrix. This may be written as

$$F = R f \quad (2.6)$$

$$R = H^T \Omega, \quad f = \Omega^T \cdot \text{diag} (\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}) \cdot \Omega, \quad (2.7)$$

so that R is an $SO(3)$ matrix and f a symmetric matrix. To agree with (2.2) and (2.3) R and f transform under space-time rotations as

$$R \rightarrow R \Lambda, \quad f \rightarrow \Lambda^T f \Lambda, \quad (2.8)$$

and under gauge transformations as

$$R \rightarrow G R, \quad f \rightarrow f. \quad (2.9)$$

Thus f is the required gauge invariant part of the dual field strength.

The parametrization (2.6) is a three-dimensional analogue of that used by Izergin et al.¹⁰⁾ in an attempt to circumvent the Gribov ambiguity. In the usual approach to the path integral quantization of Yang-Mills theories, the integration over the gauge orbit of a given gauge field A is not performed explicitly because all points on the orbit are physically equivalent, having the same Boltzmann weight. Instead a representative member of the orbit is chosen by imposing a gauge condition and it is these representatives that are integrated

over. Gribov¹¹⁾ showed that the condition $\partial_i A_i = 0$ does not choose a unique element of each gauge orbit. In the first part of Ref. 10), Izergin et al., motivated by the formulation of Goldstone and Jackiw¹²⁾, use the constraint that R be a constant matrix to pick a representative from each gauge orbit. However, such a condition is too restrictive because there exist allowed configurations of R which cannot be transformed to a constant by a continuous gauge transformation. R is an element of $SO(3)$ which is not simply connected, $\pi_1(SO(3)) = Z_2$. There is thus the possibility of a configuration in which R is continuous at each point on c , a closed curve in space-time, but as the curve is travelled along once R follows a non-contractable path in group space. Were R continuous, then, by shrinking the spatial curve to zero the path in group space would also vary only continuously. This implies that at the point at which the spatial curve has shrunk to zero R still runs over all the values of a non-contractable path in group space and so must in fact be singular. Such a singularity in R will not spoil continuity of the field strength F if f vanishes whenever R becomes singular. Whenever R has such a singularity f must vanish on a closed curve which loops through c . Such field configurations are the vortices associated with the centre of the gauge group described in Ref. 1). Because only continuous gauge transformations are allowed in the theory, singularities in R cannot be transformed away, they contain physical information. Topological obstructions to the choice of global gauge conditions are a general feature of Yang-Mills theories¹³⁾.

We can now return to the construction of an effective potential. The Euclidean action is

$$S_F = \int d^3x \frac{1}{2} F_i^A F_i^A. \quad (2.10)$$

The addition of a term

$$S_\sigma \equiv \sigma \int d^3x f_i^i \quad (2.11)$$

preserves gauge invariance as well as invariance under spatial rotations and translations. Furthermore the corresponding Hamiltonian is also linear in the source σ so that the functional integral given by

$$\exp - U(\sigma) = \int \mathcal{D}A \exp - (S_F + S_\sigma), \quad (2.12)$$

has the physical interpretation described earlier. The addition of S_σ is analogous to placing the system in an external field. Savvidy considered a similar problem in the four-dimensional theory¹⁴⁾. He calculated an effective potential by expanding the theory about a covariantly constant field strength and showed that it was minimized away from the perturbative vacuum. This potential was later shown to have an imaginary part indicating that the configuration is unstable¹⁵⁾. This is perhaps not so surprising because the configuration breaks Lorentz invariance and cannot be expected to characterize the vacuum. Our external field does not break rotational invariance, neither does it result in a covariantly constant field strength, it is not an analogue of Savvidy's.

The functional integral (2.12) may be treated in the conventional way by introducing gauge fixing, and Faddeev-Popov ghosts. Instead we will employ a different, although equivalent, approach, consisting of a change of integration variable. This we do in two stages. Firstly we change variables from A to F . From (2.1) a variation δA induces a variation in the dual field strength

$$\begin{aligned}\delta F_i^A &= \epsilon_{ijk} \partial_j \delta A_k^A + g \epsilon_{ijk} \epsilon^{ABC} A_j^B \delta A_k^C \\ &\equiv \Gamma(\partial, A)_i^A{}^C \delta A_k^C\end{aligned}\tag{2.13}$$

so that the measure $\mathcal{D}A$ becomes

$$\mathcal{D}A = \mathcal{D}F \cdot (\text{Det } \Gamma/\mu)^{-1}\tag{2.14}$$

where μ is an arbitrary constant with the dimensions of mass introduced to make the determinant dimensionless. As the second step we use the decomposition (2.6) to transform from F to R and f . This is a local change of variables and the Jacobian is readily evaluated. At each point of space-time we have

$$dF = \{(\text{tr } f)^3 - \text{tr}(f^3)\} df dR\tag{2.15}$$

where dR is the Haar measure on $SO(3)$, and df is induced by the metric on variations of f

$$\| \delta f \|^2 = \text{tr}(\delta f \delta f)$$

In the trivial topological sector where R is non-singular, A is related to the configuration, $\bar{A}$, that gives a symmetrical dual field strength by the gauge transformation $G = R$.

$$A_i^A = R^{AB} \bar{A}_i^B - \frac{1}{2g} \epsilon^{ABC} R^{DE} \partial_i R^{CE}, \quad (2.16)$$

$$f_i^r = \epsilon_{ijk} \partial_j \bar{A}_k^r + \frac{g}{2} \epsilon_{ijk} \epsilon^{rst} \bar{A}_j^s \bar{A}_k^t. \quad (2.17)$$

Thus the R dependence of Γ may be factored out

$$\Gamma(\partial, A)_{ik}^{AC} = R^{AB} \Gamma(\partial, \bar{A})_{ik}^{BD} R^{CD},$$

and if ψ is an eigenfunction of $\Gamma(\bar{A})$ belonging to eigenvalue λ , then $R\psi$ is an eigenfunction of $\Gamma(A)$ also belonging to λ so that the product of eigenvalues, $\text{Det } \Gamma$ is independent of R. Restricting our attention to this vortex free sector we can express (2.12) as

$$\exp - U(\sigma) = \int \mathcal{D}f \mathcal{D}R \cdot \Delta \cdot (\text{Det } \Gamma / \mu)^{-1} \cdot \exp - \int d^3x \text{tr} \left(\frac{1}{2} ff + \sigma f \right), \quad (2.18)$$

with

$$\Delta = \prod_x \{(\text{tr } f)^3 - \text{tr}(f^3)\} / \mu^{9/2}$$

The integrand is independent of R so this integration may be done trivially. Absorbing the resulting constant into the normalization of the functional integral gives

$$\exp - U(\sigma) = \int \mathcal{D}f \cdot \Delta \cdot (\text{Det } \Gamma(\bar{A}) / \mu)^{-1} \cdot \exp - \int d^3x \text{tr} \left(\frac{1}{2} ff + \sigma f \right). \quad (2.19)$$

We now evaluate this expression to one-loop accuracy. To keep track of the order of the approximation, Planck's constant may be reinstated in the exponent, i.e.,

$$S_F + S_\sigma \rightarrow \frac{1}{\hbar} (S_F + S_\sigma).$$

Separating f into a "classical" and "quantum" part

$$f_j^i = -\sigma \delta_j^i + \sqrt{\hbar} \bar{f}_j^i \quad (2.20)$$

results in a series expansion for $\bar{A}$ satisfying (2.17),

$$\bar{A}(\bar{f}) = \sum_{r=0}^{\infty} \sqrt{\kappa}^r \bar{A}^{(r)}(\bar{f}) \quad (2.21)$$

The order of the approximation is determined by how many terms we retain in the evaluation of $\Delta/\text{Det } \Gamma$. To one-loop order we keep only the first terms (and thus neglect terms of order $\hbar^2$ in the expansion of u), obtaining

$$\exp - U(\sigma)/\hbar \simeq \int D\bar{f} \Delta_0 \cdot (\text{Det } \Gamma_0/\mu)^{-1} \exp \left(- \int d^3x \text{tr } \frac{1}{2} \bar{f} \bar{f} + \frac{3}{2} \frac{\sigma^2 V}{\kappa} \right)$$

where

$$\Delta_0 = \prod_x 24 \sigma^3 / \mu^{9/2},$$

$$\Gamma_0 = \Gamma(\bar{A}^{(0)}),$$

and V is the volume of space-time. The $\bar{f}$ integration is now Gaussian so that

$$\exp - U(\sigma)/\hbar \simeq \Delta_0 \cdot (\text{Det } \Gamma_0/\mu)^{-1} \exp \frac{3}{2} \frac{\sigma^2 V}{\kappa} \quad (2.22)$$

and it remains only to evaluate the determinant. From (2.17) a constant $\bar{A}^{(0)}$ may be found provided

$$-\sigma \delta_i^r = \frac{1}{2} g \epsilon_{ijk} \epsilon^{rst} \bar{A}^{(0)}{}_j^s \bar{A}^{(0)}{}_k^t$$

which is satisfied by

$$\bar{A}^{(0)}{}_j^i = \pm i \sqrt{\sigma/g} \delta_j^i \equiv A_{\pm j}^i \quad (2.23)$$

Since we have two solutions the relationship between f and $\bar{A}$ is not one-to-one. A given value of f gives a unique value of $\bar{A}$ only locally, i.e., within the neighbourhood of A_+ or A_- . Expression (2.22) should properly include a summation over the two neighbourhoods. That both A_+ and A_- are imaginary when σ is positive presents no problem, it only indicates that the contour of integration of A should be rotated to the imaginary axis to go through the saddle-points of (2.12). As will become clear later we will only be interested in the case of positive σ [see (2.24)]. Note that to obtain (2.23) we have exploited the non-Abelian character of the theory, this contrasts with the work of Refs. 14) and 15) where an embedding of an Abelian configuration was used, i.e., in an

appropriate gauge all the components of the gauge field and field strength point in the same, constant, direction in colour space.

Both solutions (2.23) give the same value to $\text{Det}\Gamma/\mu$. Substituting them into Γ gives

$$\Gamma_0(\partial, A_\pm)_{ik}^{AC} = \delta^{AC} \epsilon_{ijk} \partial_j \pm im \delta_{ik}^{AC} \equiv \Gamma_\pm,$$

for $m = \sqrt{og}$.

It is straightforward to work out the determinant of the Fourier transform of Γ_0 , $\Gamma_0(ip, A_\pm)$, because although this is a 9×9 matrix it has a simple action on a basis formed by taking Cartesian products of the three eigenvectors of $\epsilon_{ijk} p_j$.

The result is

$$\det \Gamma_0(ip, A_\pm) = \mp 2im^7 (p^2 + m^2) \quad (2.24)$$

The determinant of the operator $\Gamma(\partial)$ is formally the product of this expression over all possible momenta, and so its logarithm is

$$\ln \text{Det} \Gamma_0 / \mu = - \int \frac{d^3p d^3x}{(2\pi)^3} \ln \left(\frac{\bar{\mu}^2}{p^2 + m^2} \right),$$

$$\bar{\mu}^2 = \pm i\mu^9 / (2m^7). \quad (2.25)$$

To give a regularized definition of this otherwise meaningless expression introduce the function

$$\zeta(s) = \int \frac{d^3p d^3x}{(2\pi)^3} \left(\frac{\bar{\mu}^2}{p^2 + m^2} \right)^s \quad (2.26)$$

which exists for $s > 3/2$ (and a finite volume of space-time). If $\zeta(s)$ is analytically continued to smaller values of s then we can define

$$\ln (\text{Det} \Gamma_0 / \mu)_{\text{reg.}} = \lim_{s \rightarrow 0} - \zeta'(s) \quad (2.27)$$

$\zeta(s)$ is readily evaluated as

$$\xi(s) = \frac{m^3 V}{8\sqrt{\pi}^3} \left(\frac{\bar{\mu}}{m} \right)^s \cdot \frac{\Gamma(s-3/2)}{\Gamma(s)} \quad (2.28)$$

For small s $\Gamma(s) \sim 1/s$ so

$$-\xi'(0) = -\frac{m^3 V}{6\pi} \quad (2.29)$$

Note that no ultra-violet singularity has appeared. The regularized determinant is finite and also independent of the scale μ . Since μ is arbitrary we can choose

$$24\sigma^3 = \mu^{9/2}$$

so that

$$\Delta_0 = 1$$

without the need to regularize Δ_0 (although for any other value of μ the regularized Δ_0 will still be one). We finally obtain as the one-loop approximation for $U(\sigma)$

$$U(\sigma) \simeq U_1(\sigma) = -V \left(\frac{3}{2} \sigma^2 + \kappa \frac{\sqrt{\sigma g}^3}{6\pi} \right) \quad (2.30)$$

where we have dropped terms independent of σ . u may be written in terms of a density $u(\sigma)$

$$U_1(\sigma) = V \cdot u(\sigma)$$

The effective potential, V , is expressed in terms of the derivative of u

$$-3\varphi \equiv \frac{\partial u(\sigma)}{\partial \sigma}$$

as

$$u(\sigma) + 3\sigma\varphi = v(\varphi) \quad (2.31)$$

and is, according to the physical interpretation given earlier, the minimum energy density for states in which the expectation value of f_j^i is $-\phi\delta_j^i$. To one-loop accuracy

$$v(\varphi) = \frac{3}{2} \varphi^2 - \kappa \frac{\sqrt{9\varphi}^3}{6\pi}, \quad \varphi \geq 0. \quad (2.32)$$

3. - THE PROPAGATOR FOR f

In the last section we calculated, to one-loop accuracy, an effective potential V which is minimized away from what would be the usual perturbative vacuum $\phi = 0$. The minimum of (2.32) occurs at

$$\phi = \frac{\hbar^2 g^3}{(12\pi)^2} \quad (3.1)$$

In the vicinity of this value the two terms in $V(\phi)$ are of the same order of magnitude even though in the spirit of the approximation the second term has been treated as a small perturbation to the first. Furthermore, if the one-loop correction is of the same size as the classical result the higher loop terms, which we have so far neglected, may also be significant. The only ways to find out are either to explicitly compute them, or search for some bound on their magnitude. In the next section we will calculate the two-loop contribution to $V(\phi)$ and discover that the minimum still occurs away from vanishing ϕ . Here we will investigate one of the implications of assuming that this qualitative feature holds true to all orders.

The particle spectrum may be obtained by analyzing the time ordered expectation value in the trial ground state $|E_\sigma\rangle$ of the product of the physical (i.e., gauge invariant) fields f . The analytic continuation to Euclidean space of this is:

$$\begin{aligned} T \langle E_\sigma | f_j^i(x) f_s^r(y) | E_\sigma \rangle &\equiv F_{js}^{ir}(x-y) = \\ e^{VE_\sigma} \int \mathcal{D}f \cdot \Delta \cdot (\text{Det } \Gamma(\bar{A})/\mu)^{-1} & f_j^i(x) f_s^r(y) \\ \times \exp - \int d^3x \text{tr} \left(\frac{1}{2} f f + \sigma f \right) & \quad (3.2) \end{aligned}$$

To evaluate this we use the standard trick of inserting a source term j into the action, and calculate to one-loop accuracy

$$Z[j] \equiv \quad (3.3)$$

$$\frac{1}{2} \int d^3x \Delta \cdot (\text{Det } \Gamma(\bar{A})/\mu)^{-1} \exp - \frac{1}{\hbar} \int d^3x \text{tr} \left(\frac{1}{2} f f + [\sigma + j] f \right)$$

Proceeding as before, f may be separated into a "classical" and "quantum" part

$$f_j^i = -\sigma \delta_j^i - j_j^i + \hbar \bar{f}_j^i \quad (3.4)$$

To obtain F , Z must be differentiated twice with respect to the source j which is afterwards set to zero, so we are interested in expanding Z in powers of j only to second order. Thus j may be regarded as a small correction to $\sigma \mathbb{1}$ and there are again two saddle-points in $\bar{A}$ close to $A_{\pm}$ given by (2.23), $\tilde{A}_{\pm}$. Both these saddle points must be summed over so that, after integrating over $\bar{f}$,

$$Z[j] = \frac{1}{2} \sum_{\pm} \Delta \cdot (\text{Det } \Gamma(\tilde{A}_{\pm}))^{-1} \exp \frac{1}{2\hbar} \int d^3x \text{tr} \{ (j + \sigma \mathbb{1})^2 \} \quad (3.5)$$

If we write the difference between $A_{\pm}$ and $\tilde{A}_{\pm}$ as $a_{\pm}$ then

$$\tilde{A}_{\pm} = A_{\pm} + a_{\pm} ,$$

$$j_j^i = \Gamma_{\pm js}^{ir} a_{s\pm}^r + \frac{1}{2} g \epsilon^{irs} \epsilon_{jkl} a_{k\pm}^r a_{l\pm}^s . \quad (3.6)$$

Using the notation $a_{\pm}^*$ for the matrix

$$(a_{\pm}^*)_{jl}^{is} = \epsilon^{irs} \epsilon_{jkl} a_{s\pm}^r$$

enables (3.6) to be written more compactly as

$$j = \Gamma_{\pm} a_{\pm} + \frac{1}{2} g a_{\pm}^* a_{\pm} .$$

This may be solved for a as a series in powers of j

$$a_{\pm} = \Gamma_{\pm}^{-1} j - \frac{1}{2} g (\Gamma_{\pm}^{-1} j)^* \Gamma_{\pm}^{-1} j + O(j^3) . \quad (3.7)$$

In terms of $a_{\pm}$ the operator $\Gamma(\tilde{A})$ is

$$\Gamma(\tilde{A}_{\pm}) = \Gamma_{\pm} + g a_{\pm}^*$$

so that the reciprocal of the determinant may be expressed as

$$\begin{aligned} (\text{Det } \Gamma(\tilde{A}_{\pm}))^{-1} &= \\ (\text{Det } \Gamma_{\pm})^{-1} \exp(-g \text{Tr}(\Gamma_{\pm}^{-1} a_{\pm}^*) + \frac{1}{2} g^2 \text{Tr}(\Gamma_{\pm}^{-1} a_{\pm}^* \Gamma_{\pm}^{-1} a_{\pm}^*)) & \end{aligned} \quad (3.8)$$

to second order in j .

To evaluate the traces we first take the Fourier transforms of the operators. For $\Gamma_{\pm}^{-1}$ the Fourier transform may be shown to be

$$\begin{aligned} \Gamma^{-1}(ip, A_{\pm})_{rs}^{ij} &\equiv \tilde{\Gamma}_{\pm}^{-1}{}_{rs}^{ij} = \\ &= \frac{i}{2m} (\delta_r^i \delta_s^j - 2 \delta_s^i \delta_r^j + \delta^{ij} \delta_{rs}) \\ &+ \frac{i}{2m^2} (p_j \epsilon_{irs} - p_r \epsilon_{ijs}) \\ &\pm \frac{i}{2m^3(p^2+m^2)} (\pm m \epsilon_{iqj} p_q + p_i p_j + m^2 \delta_{ij}) \times \\ &\quad (\pm m \epsilon_{rts} p_t + p_r p_s + m^2 \delta_{rs}) \end{aligned} \quad (3.9)$$

Thus the first trace in (3.8) contains integrals of the form

$$\int d^3p \cdot \frac{Q(p)}{p^2+m^2} \quad (3.10)$$

whilst the second consists of integrals

$$\int d^3p \cdot \frac{1}{p^2+m^2} \cdot \frac{1}{(p+q)^2+m^2} \cdot \tilde{Q}(p) \quad (3.11)$$

where Q and $\tilde{Q}$ are polynomials in p . Clearly these expressions stand in need of regularization. We continue to use the method described earlier. In principle the determinant of $\Gamma(\tilde{A}_{\pm})$ is regularized by evaluating the eigenvalues, α , as a perturbation expansion about those of $\Gamma_{\pm}$ which are labelled by momentum p . For

each value of p the product of these eigenvalues is formed and a ζ function constructed as

$$\zeta(s) = \int \frac{d^3p d^3x}{(2\pi)^3} \left(\frac{\mu^9}{\pi \alpha(p)} \right)^s$$

The regularized determinant is then defined as before, (2.27). In practice this procedure amounts to defining regularized versions of (3.10) and (3.11) as the limits as s approaches one of the analytic continuations of

$$\int d^3p \cdot \frac{Q(p)}{(p^2 + m^2)^s}, \quad \int d^3p \cdot \frac{1}{(p^2 + m^2)^s} \cdot \frac{1}{(q+p)^2 + m^2} \cdot \tilde{Q}(p)$$

respectively. In particular this leads to the regularized definition of

$$\int d^3p (p^2 + m^2)^n,$$

(for n an integer greater than or equal to zero) as

$$\lim_{s \rightarrow -n} m^{3-2s} \sqrt{\pi}^3 \Gamma(s - 3/2) / \Gamma(s) = 0.$$

For n equal to zero we obtain the same convention as in dimensional regularization that

$$\delta^{(3)}(0)_{\text{reg}} = 0.$$

This is the reason we could set $\Delta_0 = 1$ earlier, Δ is the product of local quantities at each point in space-time so that its logarithm is proportional to $\delta^{(3)}(0)$. Again we can set $\Delta = 1$ in (3.5). Also we see that the traces of purely local operators such as

$$\text{Tr} (\partial^2 - m^2)$$

make no contribution. Using these rules to give meaning to the traces in (3.8) we obtain, after some work,

$$\begin{aligned}
 & -g \operatorname{Tr} (\Gamma_{\pm}^{-1} a_{\pm}^*)_{\text{reg}} + \frac{1}{2} g^2 \operatorname{Tr} (\Gamma_{\pm}^{-1} a_{\pm}^* \Gamma_{\pm} a_{\pm}^*)_{\text{reg}} \\
 & = -\frac{m g}{12\pi} \int d^3x \operatorname{tr} j(x) + \frac{m g^2}{24\pi} \int \frac{d^3p}{(2\pi)^3} \tilde{j}(-p) \tilde{G}_{\pm}(p) \tilde{j}(p) \\
 & + O(j^3)
 \end{aligned} \tag{3.12}$$

Here $\tilde{j}(p)$ is the Fourier transform of $j(x)$

$$j(x) = \int \frac{d^3p}{(2\pi)^3} e^{i p \cdot x} \tilde{j}(p)$$

and $\tilde{G}(p)$ is given by

$$\tilde{G}_{\pm} = \tilde{\Gamma}_{\pm}^{-1} \tilde{\Omega}_{\pm} \tilde{\Gamma}_{\pm}^{-1}$$

where $\Omega_{\pm}$ is quite a complicated matrix function of p

$$\begin{aligned}
 \tilde{\Omega}_{\pm}(p)_{rs}^{ij} &= \delta_s^i \delta_r^j - \delta_r^i \delta_s^j \pm \xi(p) \delta^{ij} \epsilon_{rst} p_t \\
 & \pm \eta(p) \delta_{rs} \epsilon^{ijk} p_k + \kappa(p) \epsilon_{ist} \epsilon_{rjk} p_t p_k \\
 & + \varrho(p) \epsilon_{ijk} p_k \epsilon_{rst} p_t \\
 & + \mu(p) (p_i p_j - p^2 \delta_{ij}) (p_r p_s - p^2 \delta_{rs}) ,
 \end{aligned} \tag{3.13}$$

$$\begin{aligned}
 \xi &= \alpha + \frac{1}{m} , \quad \eta = \alpha - \frac{1}{m} , \quad \kappa = 2\beta p^2 + 4/p^2 \\
 \varrho &= -\beta p^2 - \gamma p^2 + 4/p^2 , \quad \mu = 3\beta + \gamma
 \end{aligned}$$

$$\alpha = -\frac{3}{8m^2} (2m d^2 p^3 \theta [p^2 - 4m^2] + \frac{2}{3} m^4 p^2 + \frac{4}{3} m^6 + \frac{1}{4} m^2 p^4)$$

$$\beta = -\frac{6d^2}{m^2} \left(\frac{1}{4} d^2 p^3 \theta + \frac{5}{24} m^3 + \frac{1}{32} m p^2 \right)$$

$$\gamma = \frac{6d^2}{m^4} (1 + p\theta/(2m))$$

$$\theta = \frac{1}{4} + \frac{m^2}{p^2} , \quad \sin \theta = 1/(2\sqrt{2}) \tag{3.14}$$

To the level of approximation we have been working, i.e., to one-loop and second order in j , our expression for $Z[j]$ becomes

$$Z[j] = \exp(-V\mathcal{E}_T) \cdot \exp\left(\frac{1}{\hbar} \int d^3x \, b \left(\frac{1}{2} j^2 + \varphi j\right)\right) \\ \times \frac{i}{2} \sum_{\pm} \exp\left(\frac{m g^2}{24\pi} \int \frac{d^3p}{(2\pi)^3} j(-p) C_{\pm}(p) j(p)\right) \quad (3.15)$$

where we have eliminated σ in favour of ϕ , so that

$$m = \sqrt{\varphi g}$$

From $Z[j]$ we obtain the two-point function F as

$$F_{js}^{ir}(x-y) = \hbar^2 \left(\frac{\delta}{\delta j(x)_j} \frac{\delta}{\delta j(y)_s} \exp(V\mathcal{E}_T) \cdot Z[j] \right)_{j=0} \\ = \delta_j^i \delta_s^r \varphi^2 + \delta(x-y) \delta^{ir} \delta_{js} + C_1(x-y)_{js}^{ir}$$

where

$$C_1(x-y)_{js}^{ir} = \hbar^2 \sum_{\pm} \frac{m g^2}{24\pi} C_{\pm}(x-y)_{js}^{ir} \quad (3.16)$$

and

$$C_{\pm}(x) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \tilde{C}_{\pm}(p)$$

The first two terms in F constitute the classical result, the ϕ^2 term being due to our ansatz that in the trial vacuum $\langle f_j^i \rangle = -\phi \delta_j^i$. At the classical level f does not propagate. The last term is the one-loop correction and describes the propagation of quanta associated with f . These are the only quanta in the theory if we restrict ourselves to the trivial topological sector where there are no Z_2 vortices. To obtain the masses of these particles we look for the poles of the eigenvalues of $\tilde{G}$ in the complex p^2 -plane. $\tilde{Q}_{\pm}$ have no poles. Even though (3.14) is replete with terms that diverge at vanishing p^2 these all cancel in $\tilde{Q}_{\pm}$. The only poles come from $\Gamma_{\pm}^{-1}$, and according to (3.9) these occur only for

$$p^2 = -m^2 = -\varphi g \quad (3.17)$$

We can construct the eigenvalues of $\tilde{G}$ by forming a product basis out of the eigenvectors of $\varepsilon_{ijk} p_k$, $\{\hat{p}, e^1, e^2\}$ where

$$\begin{aligned} |p| \hat{p}_j &= p_j, \quad \epsilon_{ijk} p_k e_j^1 = i|p| e_i^1, \\ \epsilon_{ijk} p_k e_j^2 &= -i|p| e_i^2. \end{aligned} \quad (3.18)$$

The products defined as

$$A_s^i \equiv e_i^s e_j^s, \quad s = 1, 2$$

are eigenvectors of $\tilde{\Gamma}_\pm$ so that

$$\tilde{\Gamma}_\pm^{-1} A_s = (\mp im - (-)^s |p|)^{-1} A_s,$$

and these are the only eigenvectors of $\tilde{\Gamma}_\pm^{-1}$ belonging to eigenvalues that have poles in the complex p -plane. Furthermore, the A_s are eigenvectors of $\tilde{Q}_\pm$ and thus of $\tilde{G}_\pm$,

$$\begin{aligned} \tilde{G}_\pm A_s &= (im \pm (-)^s |p|)^{-2} \cdot (-7 \pm (-)^s 2i\alpha |p| + 2(\beta + \gamma) p^4) A_s \\ &\equiv \lambda_\pm^s A_s, \end{aligned} \quad (3.19)$$

which implies that

$$\tilde{G}(p) A_s = \frac{\hbar^2 m g^2}{24\pi} \left(\sum_\pm \lambda_\pm^s \right) A_s \equiv \hbar^2 \lambda A_s,$$

$$\begin{aligned} \lambda &= m g^2 \{ (p^2 - m^2) (-7 + 2(\beta + \gamma) p^4) - 4 m p^2 \alpha \} (p^2 + m^2)^{-2} / (12\pi) \\ &\equiv \Phi(p^2) / (p^2 + m^2)^2 \end{aligned} \quad (3.20)$$

Φ is non-singular so that λ has a double pole at $p^2 = -m^2$. The continuation of Φ to negative values of p^2 is straightforward because θ occurs in the combination $\theta|p|$ and we can take

$$\theta|p| = -m \ln \sqrt{3} \quad \text{at } p^2 = -m^2 \quad (3.21)$$

leading to

$$\Phi(-m^2) = \frac{m^3 g^2}{12\pi} \left(\frac{97}{64} + \frac{2187}{128} \ln \sqrt{3} \right), \quad \Phi'(-m^2) = - \frac{13 m g^2}{12\pi}. \quad (3.22)$$

λ is the only eigenvalue of $\tilde{G}$ with a pole in p^2 . This means that out of the six possible polarization states of f only two, the A_s , propagate. These have mass m whilst the others are infinitely heavy. Of course we have only worked to one-loop accuracy and we have seen that no modes propagate at the classical level. Nonetheless, the picture that emerges at the one-loop level of dynamical mass generation by radiative corrections is qualitatively different from the usual Higgs mechanism approach to mass generation in gauge theories.

4. - THE TWO-LOOP CORRECTION TO THE EFFECTIVE POTENTIAL

The minimum of the effective potential calculated to one-loop accuracy occurs away from what would be the perturbative vacuum. We will now show that this feature is preserved when the two-loop corrections are included.

Expanding the functional integral (2.19) to two loops requires a knowledge of $\text{Det } \Gamma$ expanded about the saddle points $A_{\pm}$ to second order in $\sqrt{\kappa} \bar{f}$

$$f_j^i = -\sigma \delta_j^i + \sqrt{\kappa} \bar{f}_j^i.$$

In the last section we obtained $\text{Det } \Gamma$ to second order in j where

$$f_j^i = -\sigma \delta_j^i - j_j^i$$

so by replacing j by $-\sqrt{\kappa} \bar{f}$ in (3.12) we get

$$\begin{aligned} (\text{Det } \Gamma / \mu)^{-1} &= (\text{Det } \Gamma_{\pm} / \mu)^{-1} \times \\ \exp \left\{ \frac{m g \sqrt{\kappa}}{12\pi} \int d^3x \bar{f}_s^s + \frac{m g^2 \kappa}{24\pi} \int \frac{d^3p}{(2\pi)^3} \tilde{\bar{f}}(-p) \tilde{G}_{\pm}(p) \tilde{\bar{f}}(p) + O(\kappa^3) \right\}. \end{aligned} \quad (4.1)$$

The two-loop correction to U , U_2 where

$$U(\sigma) = U_1(\sigma) + \hbar^2 U_2(\sigma) + O(\hbar^3),$$

is given by

$$\begin{aligned} \exp -\hbar U_2 &= \\ \frac{1}{2} \sum_{\pm} \int \mathcal{D}\bar{f} \exp \left(- \int d^3x \operatorname{tr} \left(\frac{1}{2} \bar{f} \bar{f} - \frac{mg\sqrt{\hbar}}{12\pi} \bar{f} \right) + \frac{mg^2\hbar}{24\pi} \int \frac{d^3p}{(2\pi)^3} \bar{f} \tilde{G}_{\pm} \bar{f} \right) \\ &= \frac{1}{2} \sum_{\pm} \exp \left(\frac{3m^2g^2\hbar}{2(12\pi)^2} V + \frac{mg^2\hbar}{24\pi} V \int \frac{d^3p}{(2\pi)^3} \operatorname{tr} \tilde{G}_{\pm}(p) \right) \end{aligned}$$

As before we have taken Δ to be unity.

The momentum integral may be given a regularized definition using the method described earlier. The only eigenvalues of $\tilde{G}_{\pm}$ which contribute are $\lambda_{\pm}^S$ because the others correspond to local operators whose regularized integrals may be written as sums of integrals of the type

$$\left(\int d^3p (p^2 + m^2)^n \right)_{\text{reg}} = 0, \quad n \geq 0.$$

Thus

$$\begin{aligned} \tau(\sigma) &\equiv \frac{mg^2}{24\pi} \left(\int \frac{d^3p}{(2\pi)^3} \operatorname{tr} \tilde{G}_{\pm}(p) \right)_{\text{reg}} \\ &= \left(\int \frac{d^3p}{(2\pi)^3} \cdot \frac{\Phi(p^2)}{(p^2 + m^2)^2} \right)_{\text{reg}} \end{aligned}$$

The integration over the angular part of p is straightforward and the integration over the length of p may be extended to the whole of the real line as the integrand is a function of p^2 , thus

$$\tau(\sigma) = \frac{1}{4\pi^2} \left(\int_{-\infty}^{\infty} dp \cdot \frac{p^2 \Phi(p^2)}{(p^2 + m^2)^2} \right)_{\text{reg}}.$$

The contour of integration may be closed in the upper half of the complex p -plane because the regularization procedure serves to eliminate the contribution from the circle at infinity. The regularized expression for τ is finally given by the contour integral around the pole at $p = im$

$$\begin{aligned}\tau(\sigma) &= \frac{1}{4\pi^2} \oint dp \cdot \frac{p^2 \Phi(p^2)}{(p^2 + m^2)^2} \\ &= \frac{1}{8\pi m} \Phi(-m^2) - \frac{m}{4\pi} \Phi'(-m^2) = g^3 \sigma \kappa / \pi\end{aligned}$$

where $\kappa = (3522+2187 \ln 3)/(12288\pi) \approx 1/8$. The two-loop expression for the density U/V is thus

$$u(\sigma) = -\frac{3}{2} \sigma^2 - \frac{\hbar}{6\pi} \sqrt{\sigma g}^3 - \frac{3\hbar^2}{2} \cdot \frac{1}{(12\pi)^2} \sigma g^3 - \frac{\hbar^2 \kappa}{\pi} \sigma g^3$$

from which the effective potential is obtained as the Legendre transform

$$V(\varphi) = u(\sigma) - \sigma u'(\sigma), \quad \varphi = -u'(\sigma)/3$$

so that to two-loop accuracy

$$V(\varphi) = \frac{3}{2} \varphi^2 - \frac{\hbar}{6\pi} \sqrt{g\varphi}^3 - \frac{\hbar^2}{\pi} \kappa g^3 \varphi. \quad (4.2)$$

This potential is minimized away from the origin, however, it is clear from the magnitudes of the coefficients that we cannot expect this expression to be reliable near its minimum.

In Section 2 we showed that the f_j^1 field propagates with a mass whose square is proportional to the value of ϕ that minimizes the effective potential. Thus the one-loop approximation for $V(\phi)$ gives a mass

$$m = g^2 / (12\pi) \approx 0.03 g^2 \quad (4.3)$$

whilst (4.2) implies

$$m \approx 0.13 g^2 \quad (4.4)$$

i.e., the two-loop correction leads to a four-fold increase in the estimate of the mass. It is possible that resummation techniques may yield a more reliable series.

5. - CONCLUSION

In three Euclidean dimensions the physical degrees of freedom of SU(2) pure Yang-Mills theory consist of a real symmetric tensor f_j^i , a gauge invariant part of the dual field strength, and variables describing the vortices associated with the centre of the group. In the vortex free sector the vacuum expectation value of f , $\langle f_j^i \rangle = -\phi \delta_j^i$, may be non-zero. The two-point function of f calculated to one loop in Section 3 implies that only two of the polarization states of f propagate with mass $m = \sqrt{\phi}g$, the rest do not propagate. In the usual treatment of the theory the three-by-three matrix A_1^A has three independent modes after the gauge conditions $A_3^A = 0$, $\partial_i A_1^A = 0$, have been imposed and all three would be expected to propagate. The dynamical generation of non-zero $\langle f \rangle$ has thus lead to the suppression of one mode. This is quite unlike the usual Higgs mechanism where the gauge fields either acquire a finite mass or remain massless, however higher order corrections may change this picture and unfreeze the non-propagating mode .

The effective potential, $V(\phi)$, calculated to two loops

$$V(\varphi) = \frac{3}{2} \varphi^2 - \frac{1}{6\pi} \sqrt{g\varphi}^3 - \frac{\kappa}{\pi} g^3 \varphi, \quad \kappa \approx \frac{1}{8}$$

supports the non-vanishing of $\langle f \rangle$. It is minimized away from the origin, although in the region of the minimum the loop expansion used to obtain $V(\phi)$ appears not to be very reliable. Resummation techniques may improve the accuracy of the expansion. Nonetheless a non-trivial minimum of $V(\phi)$ implies the dynamical generation of a "glueball" mass by radiative corrections, so that the theory is both infra-red and ultra-violet finite. It also implies the stability of vortices. Although these are associated with topologically non-trivial field configurations they can unwind in a domain where f_j^i vanishes. Such domains are suppressed if V is not minimized at $\phi = 0$. A condensation of such vortices may lead to permanent quark confinement¹⁾.

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