On Multi-loop Computations in Polyakov's String Theory

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Abstract

Multi-loop calculations in closed bosonic string theory are considered using the Polyakov path-integral formalism. The functional determinants that enter the problem are evaluated to arbitrary string-loop order. It is demonstrated that various divergences arising in the definition of such determinants can be eliminated by appropriate renormalizations.

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1. Introduction

The Polyakov formulation of string theory ¹⁾ has the merit of manifestly preserving both Lorentz and world-sheet reparametrization invariance. It is now also known how to generate S-matrix elements for arbitrary (low lying) external particle states in this framework ²⁾. In the critical dimension and on mass-shell (where local two dimensional Weyl invariance is recovered) these agree with the corresponding Koba-Nielsen amplitudes of the dual model and may thus be assumed to have correct factorization properties.

However if this covariant string theory is to be unitary, loop amplitudes have to be included as well. This is because the tree level S-matrix (which is the sum of poles only) is non-trivial and therefore cannot be unitary by itself. The hope is that including loops will restore unitarity. In the path-integral formulation of the theory this means summing over all compact, oriented two manifolds Σ of arbitrary genus.

So far, partial results have been obtained at the one string loop level ³⁾; in particular modular invariance (i.e. under global reparametrizations or diffeomorphisms not homotopic to the identity) has been shown to be preserved. Much less is known about the general multi-loop amplitude.

To give a more precise meaning to summing over topologies, consider the functional integral for a 'free' string

$$\sum_{g=2}^{\infty} (\lambda)^{(g-1)} \int [D_{\Sigma} q_{ab}] D x^{\mu} \exp -1/2 \int_{\Sigma} d^2 \sigma \sqrt{q} q^{ab} \partial_a x^{\mu} \partial_b x^{\mu} ,$$

(Σ, q_{ab}) is a two dimensional compact (without boundary), oriented manifold of genus g , carrying a metric q_{ab} of Euclidean signature. The parameter λ weights the contribution of surfaces Σ according to their genus; equivalently it is a loop counting parameter for the string theory. The measure $[Dq_{ab}]$ is some appropriate measure on the space of all metrics, $\text{Riem.}\Sigma$. So that only gauge inequivalent configurations contribute to the path-integral it should be taken to be the measure on the orbit space

$$\mathcal{M} = \text{Riem.}\Sigma / (\text{Diff}(\Sigma) \otimes \text{Weyl}) ,$$

$\text{Diff}(\Sigma) \otimes \text{Weyl}$ being the semi-direct product of the group of diffeomorphisms $f: \Sigma \rightarrow \Sigma$ and of Weyl rescalings of the metric tensor q .

If $\text{Diff}(\Sigma)$ includes only connected diffeomorphisms (i.e. those that can be built up from the identity by a series of infinitesimal ones) $\mathcal{M}$ is referred to as Teichmüller space; for $g \geq 2$ at least, for which there are no conformal Killing vectors, it is a manifold. If both connected as well as disconnected diffeomorphisms are included in $\text{Diff}(\Sigma)$, $\mathcal{M}$ is the moduli space of the Riemann surface Σ — it is in general not a manifold.

Following Polyakov (1; see also Ref.4) one may write a natural norm in the infinite dimensional space $\text{Riem.}\Sigma$ of all metrics q

$$\|\delta q\|^2 = \int d^2 \sigma \sqrt{q} (q^{ac} q^{bd} + q^{ab} q^{cd}) \delta q_{ab} \delta q_{cd}$$

where δq is an arbitrary infinitesimal variation in the metric at a given point q . This decomposition can be taken to be

$$\delta q_{ab} = \delta \phi q_{ab} + \nabla_a \xi_b + \nabla_b \xi_a - q_{ab} \nabla^c \xi_c + h_{ab} ;$$

it is orthogonal provided the symmetric tensor h_{ab} is both traceless and transverse. The norm is reparametrization invariant in the sense that

$$G[X, Y] := \int d^2\sigma \sqrt{q} (q^{ac} q^{bd} + c q^{ab} q^{cd}) X_{ab} Y_{cd}$$

has isometries given by

$$0 = \mathcal{L}_\xi G[X, Y] := \int d^2\sigma (\delta G / \delta q \mathcal{L}_\xi q + \delta G / \delta X \mathcal{L}_\xi X + \delta G / \delta Y \mathcal{L}_\xi Y),$$

where $\mathcal{L}_\xi(q, X, Y)$ denotes the usual transformations of (q, X, Y) under infinitesimal co-ordinate changes.

Thus provided that a reparametrization invariant regularization scheme is used in defining the string functional integral, it is clear that invariance under connected diffeomorphisms at least will be maintained.

Preserving Weyl 5) and global (i.e. disconnected) diffeomorphism invariance 6) is a much more subtle question. However both the latter are crucial (in the critical dimension at any rate) for the self-consistency of the underlying string theory.

Moore and Nelson (5; see also 7) have demonstrated the decoupling in the critical dimension of the conformal mode ϕ (in the decomposition $q_{ab} = e^{\phi} \hat{q}_{ab}$, where $\hat{q}_{ab}$ is a metric of constant curvature) and hence the preservation of Weyl invariance to all string loops for the generating functional. In particular they show that the string functional integral can be cast into the form equation (1) of the following section. We shall take this as our starting point.

Here we are concerned with some further aspects of evaluating this expression. Namely, the evaluation of the functional determinants resulting from the gaussian integral over the string variables $x^\mu(\sigma)$ for arbitrary

multiply connected 2-manifolds Σ . The harder problem of evaluating the contribution of the finite dimensional determinants that enter the multi-loop problem, namely the Teichmüller measure, will only be mentioned in passing in the present paper. We hope to return to this elsewhere.

In the next section, after isolating the relevant factors of the multi-loop functional integral, the problem of computing in the general case the determinants of the scalar Laplacian and Faddeev-Popov ghost contribution is set up. Some pertinent results of the theory of Riemann surfaces are gone into in section III, following which the determinants are evaluated using a trace formula for compact, multiply-connected 2-manifolds due to Selberg. Sources of ultraviolet and infrared divergences are identified and it is shown that these can be removed by appropriate subtractions.

In an appendix we have included an intuitive discussion of the Selberg formula. We conclude in section IV.

After most of this work was completed, our attention was drawn to Ref.8) in which similar techniques are employed.

II. Laplacian and Faddeev-Popov Operators

The path-integral for string amplitudes can be cast into the following form with all operators constructed with respect to a constant curvature metric q (see Ref.5) whose notation we follow). We have then, to evaluate for $g \geq 2$ the expression

$$6(g-1) \int \prod_{i=1}^g d\tau_i \left[\frac{\text{Det}'(-P^\dagger P)}{\det H(P^\dagger)} \right]^{1/2} \left[\frac{\text{Det}' \Delta}{\int_\Sigma d^2 \sigma \sqrt{q}} \right]^{-d/2} \det \langle \psi | \chi \rangle L(p_1, \dots, p_k) \quad (1)$$

The factor L represents the external momentum dependent contribution of the vertex operators to the amplitude and will not concern us here. The operator P acts on vector fields V and is defined to be:

$$(Pv)_{ab} = \nabla_a v_b + \nabla_b v_a - q_{ab} \nabla^c v_c \quad (2)$$

$P^\dagger$ is its adjoint with respect to the norm:

$$\|v\|^2 := \langle v, v \rangle = \int d^2\sigma \sqrt{q} q_{ab} v^a v^b \quad (3)$$

For any symmetric, traceless tensor T_{ab} ,

$$(P^\dagger T)_a = \nabla^b T_{ab}, \text{ so that}$$

$$(P^\dagger P)_a = \nabla^b [\nabla_a v_b + \nabla_b v_a - q_{ab} \nabla^c v_c] := -\mathcal{L}^a_b v^b \quad (4)$$

If $\psi^{(l)}$ are a set of $6(g-1)$ solutions to the equations

$$q^{ab} h_{ab} = 0 \quad \text{and} \quad \nabla^a h_{ab} = 0, \quad (5)$$

i.e. h_{ab} is symmetric and transverse-traceless, the (finite dimensional) matrix $H(P^\dagger)$ is given by:

$$\begin{aligned} H_{lm}(P^\dagger) &= \langle \psi_{(l)}, \psi_{(m)} \rangle \\ &= \int d^2\sigma \sqrt{q} q_{ac} q_{bd} \psi_{(l)}^{ab} \psi_{(m)}^{cd}. \end{aligned} \quad (6)$$

The functions $\chi^{(l)}_{ab}$ are however defined somewhat differently. Let $\hat{q}(\tau)$ be a set of points S on $\text{Riem.}\Sigma$ such that (i) no two points in S are related by a Weyl or connected co-ordinate transformation, (ii) any $q \in \text{Riem.}\Sigma$ is gauge equivalent to a point in S . The space S is referred to as a 'slice'. One then defines

$$\chi^{(l)}_{ab}(\tau) = \partial \hat{q}_{ab}(\tau) / \partial \tau^l - 1/2 \hat{q}_{ab} \hat{q}^{cd} \partial \hat{q}_{cd}(\tau) / \partial \tau^l \quad (7)$$

As Moore and Nelson point out, it is not necessary that at a given point $\hat{q} \in S$ the $\chi^{(l)}(\tau)$ be identical to the ψ 's; in fact in general

$$\nabla^a \chi^{(l)}_{ab} \neq 0.$$

Therefore the matrix

$$\langle \psi_{(l)} | \chi_{(m)} \rangle = \int_{\Sigma} d^2\sigma \sqrt{q} q_{ac} q_{bd} \psi_{(l)}^{ab} \chi_{(m)}^{cd} \quad (8)$$

will not in general be unity.

The integration region $\mathcal{Z}$ of τ (real) in equation (1) is over those points $\hat{q}(\tau)$ that are not related by a disconnected diffeomorphism.

In order to calculate the expression (1), the various determinants have to be evaluated. Here we shall compute only the infinite dimensional determinants in this equation. The finite dimensional ones appear much harder to obtain in any useful (i.e. sufficiently explicit) form.

As was mentioned in the introduction, the authors of Ref.5) have demonstrated that the expression (1) is in fact independent of the conformal mode ϕ and consequently invariant under Weyl rescalings of q . Every Riemann metric on Σ of genus $g \geq 2$ is equivalent under a Weyl transformation to one of constant negative curvature. Furthermore, this correspondence is unique. We shall assume this freedom has been used to set $R(\hat{q}) =$ negative constant.

Selberg 9) has shown how to evaluate the trace of invariant differential operators on such a manifold (see also Refs. 10) and 11). For the case of the Laplacian this is relatively simple. However before discussing this in detail we will show how the $\text{Det}(P^\dagger P)$ factor may be related to $\text{Det}'\Delta$ (the prime denotes that zero modes of the operator have been removed).

First of all, one observes that $P^\dagger P$ has no zero modes for $g \geq 2$. To see this note that for the compact manifolds under consideration,

$$\int d^2\sigma \sqrt{q} (v^b \mathcal{L}^c_b v_c) = \int d^2\sigma \sqrt{q} \phi^{ab} \phi_{ab} \geq 0$$

where $\phi_{ab} = \nabla_a v_b + \nabla_b v_a - q_{ab} \nabla^c v_c$, with equality for $(Pv)_{ab} = 0$ only. The zero modes are the conformal Killing vectors. But for $g \geq 2$ there are none.

In fact, $\mathcal{L}$ is a positive self-adjoint operator on the Hilbert space V of vector fields on Σ with norm given by equation (3). So $\mathcal{L}$ has a complete set of eigenstates.

The scalar Laplacian $\Delta := -q^{ab} \nabla_a \nabla_b$ is a positive semi-definite operator, but acts on the Hilbert space $\mathcal{H}$ of scalar fields with norm:

$$\|\phi\|^2 = \int d^2\sigma \sqrt{q} \phi^2.$$

Thus, roughly speaking, V is 'twice as big' as $\mathcal{H}$. We shall set up a one-to-one correspondence between eigenstates of $\mathcal{L}$ in V and Δ in $\mathcal{H} \oplus \mathcal{H}$.

Unlike $\mathcal{L}$, Δ does have zero modes: constant functions are zero modes of the Laplacian. A map from $\mathcal{H} \oplus \mathcal{H}$ into V is given by

$$\nabla: \mathcal{H} \oplus \mathcal{H} \rightarrow V$$

under which

$$v^a = \nabla^a \phi + \epsilon^{ab} \nabla_b \psi.$$

The kernel of this map is the set of constant functions. Conversely, any v^a can be decomposed as a Hodge decomposition:

$$v^a = \nabla^a \phi + \epsilon^{ab} \nabla_b \psi + \omega^a$$

with ϕ, ψ being unique up to constants and ω a harmonic form

$$\nabla^a \omega_a = 0$$

$$\nabla_a \omega_b - \nabla_b \omega_a = 0.$$

Let us decompose the spaces $\mathcal{X}$ and V into orthogonal subspaces:

$$\begin{aligned} V &= V^1 \oplus H_1 \\ \mathcal{X} &= \mathcal{X}^1 \oplus H_0 \end{aligned}$$

H_1 being the space of harmonic forms (this has dimension $2g$ for genus g) and H_0 the space of constant functions.

The Hilbert spaces V^1 and $\mathcal{X}^1 \oplus \mathcal{X}^1$ are isomorphic. To a (ϕ, ψ) we associate as above $v_a = \nabla_a \phi + \varepsilon_{ab} \nabla^b \psi$. The image

$$T: \mathcal{X}^1 \oplus \mathcal{X}^1 \rightarrow V^1$$

of this map is all of V^1 since every element can be so decomposed. The kernel is trivial because the only constant function in $\mathcal{X}^1$ is zero. However this map is not unitary since

$$\begin{aligned} \|(\phi, \psi)\|^2 &:= \int d^2\sigma \sqrt{q} \phi^2 + \int d^2\sigma \sqrt{q} \psi^2 \\ &\neq \int d^2\sigma \sqrt{q} [|\nabla_a \phi|^2 + |\nabla_a \psi|^2] \\ &:= \|T(\phi, \psi)\|^2 . \end{aligned}$$

T is an invertible one-one map. The following operator identity may be proved by a straightforward computation¹.

$$\Delta = T^{-1}(\mathcal{L} + 2R)T \tag{9}$$

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¹ We are using conventions in which the scalar curvature corresponding to the Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$ is $R = -1$. $R_{abd}{}^c = 1/2 (\partial_b \Gamma_{ac}{}^d - \partial_a \Gamma_{bc}{}^d - \Gamma_{bc}{}^e \Gamma_{ae}{}^d + \Gamma_{ac}{}^e \Gamma_{be}{}^d)$, $R_{ab} = R_{acb}{}^c$ and $R_{ab} = 1/2 q_{ab} R$.

This equation holds within the spaces $\mathcal{H}^1 \otimes \mathcal{H}^1$ and V^1 . From the identity (9) it is clear that

$$2 \operatorname{Tr}_1 \exp -t (\Delta - 2R) = \operatorname{Tr}_1 \exp -t \mathcal{L} , \quad (10)$$

where Tr_1 means that the trace is restricted to $\mathcal{H}^1 \otimes \mathcal{H}^1$ and V^1 respectively². Since $V = V^1 \oplus H_1$ is an orthogonal decomposition, one has

$$\operatorname{Tr} \exp -t \mathcal{L} = \operatorname{Tr}_1 \exp -t \mathcal{L} + \operatorname{Tr}_1 \exp -t \mathcal{L} ,$$

Tr_1 being the trace restricted to H_1 . We shall also need the following trace identity

$$\operatorname{Tr}_1 \exp -t \mathcal{L} = 2g \exp +t 2R . \quad (11)$$

To prove this consider $\omega \in H_1$, i.e.

$$\nabla^a \omega_a \quad \text{and} \quad \nabla_a \omega_b - \nabla_b \omega_a = 0 ,$$

whence $\nabla^a [\nabla_a \omega_b + \nabla_b \omega_a] = 0$. Now, from equation (4)

$$\begin{aligned} (\mathcal{L} \omega)_a &= -\nabla^b [\nabla_a \omega_b + \nabla_b \omega_a - q_{ab} \nabla^c \omega_c] \\ &= -\nabla^2 \omega_a - (\nabla^b \nabla_a - \nabla_a \nabla^b) \omega_b \\ &= -2R \omega_a . \end{aligned}$$

Therefore we have that

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² This means that in the L.H.S. of (10) zero modes of Δ and not obviously of $(\Delta - 2R)$ are to be omitted from the sum.

$$(\exp\{-t \mathcal{L}\} \omega)_a = (\exp\{+t 2R\} \omega)_a ,$$

and recalling that $\dim H_1 = 2g$, the desired result, equation (11), follows. Hence one has the trace relation

$$\begin{aligned} \text{Tr} \exp\{-t \mathcal{L}\} &= 2 \text{Tr}_1 \exp\{-t (\Delta - 2R)\} + 2g \exp\{+t R\} \\ &= 2 \text{Tr} \exp\{-t (\Delta - 2R)\} - 2 \exp\{+t 2R\} + 2g \exp\{+t 2R\}. \end{aligned} \quad (12)$$

This allows one to relate the calculation of $\text{Det} \mathcal{L}$ to that of $\text{Det} \Delta$. A heat kernel regularization may be employed to define these quantities.

$$\begin{aligned} -\ln \text{Det}_\epsilon \mathcal{L} &= \int_\epsilon^\infty dt/t \text{Tr} \exp\{-t \mathcal{L}\} \\ &= 2 \int_\epsilon^\infty dt/t \text{Tr} \exp\{-t (\Delta - 2R)\} \\ &\quad - 2(g-1) \int_\epsilon^\infty dt/t \exp\{+t 2R\} , \end{aligned} \quad (13)$$

whence

$$\text{Det}_\epsilon \mathcal{L} = [\text{Det}_\epsilon (\Delta - 2R)]^2 \exp[2(g-1)\ln(2R\epsilon) + \dots] . \quad (14)$$

We shall compute these expressions in the following section.

III. Calculation of the Determinants

The functional determinants, viz. $\text{Det}'\Delta$ and $\text{Det}P^\dagger P$ can be evaluated using the Selberg trace formula (9,10) for invariant operators defined on multiply connected compact spaces. Using the results of the previous section the two computations may in fact be combined. A non-rigorous discussion of the Selberg formula is contained in an appendix; however to proceed it will be necessary at this point to introduce some terminology.

For Riemann surfaces Σ of $g \geq 2$, $\Sigma = U/\Gamma$ where U is the complex upper half plane and Γ is a strictly hyperbolic Fuchsian group (see for instance Ref.10),12). Γ is a discrete, properly discontinuous subgroup of $\text{PSL}(2,\mathbb{R})$. The Poincaré metric on the upper half plane, $ds^2 = (Imz)^{-2} dz dz^*$, has $\text{SL}(2,\mathbb{R})$ as an isometry which acts as

$$P: z \rightarrow \omega = (az + b)/(cz + d) ,$$

with a,b,c,d real and $ad - bc = 1$. Any such map can be brought to a 'normal' form

$$(\omega - \xi)/(\omega - \eta) = N(P) (z - \xi)/(z - \eta) ,$$

the multiplier $N(P)$ satisfying $N^{1/2} + N^{-1/2} = a + d$. If $|a + d| > 2$, such an element P is said to be hyperbolic. A hyperbolic Fuchsian group is a Γ all of whose elements P are hyperbolic.

Two Riemann surfaces Σ and Σ' are to be identified if Γ is conjugate to Γ' in $\text{SL}(2,\mathbb{R})$ i.e. if $\Gamma' = g\Gamma g^{-1}$ for $g \in \text{SL}(2,\mathbb{R})$. Let us denote by K_g the space of such equivalence classes of subgroups of $\text{SL}(2,\mathbb{R})$, for genus g .

In this language then, the problem is to find $\text{Det } \Delta$ on Σ as a function defined on K_g . The Selberg trace formula does precisely this.

$$\begin{aligned} \text{Tr } e^{-t(\Delta+a|R|)} &= |R| \mu(\Sigma)/4\pi \int_{-\infty}^{\infty} k e^{(k^2+1/4)tR+taR} \tanh(\pi k) dk \\ &+ \sum_{\{P\}} \frac{\ln N(P_0)}{N^{1/2}(P)-N^{-1/2}(P)} \frac{e^{tR(a+1/4)}}{\sqrt{(4\pi t|R|)}} \frac{e^{[\ln N(P)]^2/(4tR)}}{(15) \end{aligned}$$

In this equation $\mu(\Sigma)$ is the volume of the surface Σ , a is a non-negative constant and the scalar curvature $R < 0$. From the Gauss Bonnet theorem in two dimensions one knows that $|R| \mu(\Sigma) = 4\pi(g-1)$. The remaining notation is as follows:

Σ is a sum over conjugacy classes P in Γ .

$\{P\}$

$N(P)$ is the multiplier of P and $N(P) > 1$.

P_0 is a 'primitive' element of Γ in the sense that all the inconjugate elements can be expressed as (positive) powers of the P_0 's.

For a Riemann surface of genus g , Γ will have $2g$ generators. On the upper half plane Σ is represented as the interior of a $4g$ sided closed polygon whose opposite sides are identified under the action of Γ .

There are potentially both ultraviolet as well as infrared divergences to be encountered upon evaluating the infinite dimensional determinants. The latter has its origin in the zero modes of the Laplacian operator. A heat kernel regularization suffices to isolate the UV infinities, while we shall use what essentially amounts to introducing a mass to handle the infrared difficulty.

A similar infrared divergence arises at one-loop for the same reason (see for instance equation A.4 of Ref.3). This 'infinity' is universal in the sense that it does not appear to depend on genus and can in fact be removed by an infinite subtraction already implicit in a careful definition of the trace to account for the zero mode term. Consider the one-loop case. The regulated determinant can be defined to be

$$- \ln \text{Det}' \Delta = \lim_{m^2 \rightarrow 0} - \ln \text{Det}_\epsilon^{-1}(\Delta + m^2) = \int_\epsilon^\infty dt/t e^{-m^2 t} \left\{ \sum_{\vec{n}=-\infty}^\infty \exp(-n_a n_b q^{ab} t) - 1 \right\} \quad (16)$$

The matrix q^{ab} is given by
$$q^{ab} = 1/\tau_2 \begin{bmatrix} |\tau|^2 & -\text{Re}\tau \\ -\text{Re}\tau & 1 \end{bmatrix}$$

where τ is the complex moduli parameter for the torus. The zero mode of the operator Δ has been explicitly subtracted out. For finite ϵ and m^2 both the integral and sum are convergent. Therefore, using the Poisson summation formula, one can split equation (16) as:

$$\begin{aligned} - \ln \text{Det}_\epsilon^{-1}(\Delta + m^2) &= \int_\epsilon^{\frac{1}{\mu^2}} \frac{dt}{t} e^{-m^2 t} \left[\sum_{\vec{n} \neq 0} \frac{e^{-\frac{n_a n_b q^{ab}}{t}}}{4\pi t} + \frac{1}{4\pi t} - 1 \right] \\ &+ \int_{\frac{1}{\mu^2}}^\infty \frac{dt}{t} e^{-m^2 t} \left[\sum_{\vec{n}=-\infty}^\infty e^{-\frac{n_a n_b q^{ab}}{t}} - 1 \right] \end{aligned} \quad (17)$$

in which an arbitrary mass scale μ has been introduced. Now, the first integral in (17) has a leading (UV) singularity of $1/\epsilon$ as $t \rightarrow 0$ but independent of the infrared cut-off m . Consider next the second integral in equa-

tion (17). Only the zero mode term ($n_a = 0$) in the sum is divergent in the limit $t \rightarrow \infty$, $m \rightarrow 0$ and behaves as $\ln(m\mu)^2$. The regulated singularity is of course cancelled by the 'counterterm'

$$\begin{aligned} & \infty \\ & - \int dt/t \exp -m^2 t, \\ & -1/\mu^2 \end{aligned}$$

possibly leaving behind finite μ dependent pieces. The UV and infrared limits have been disentangled and the latter is seen to be spurious provided the trace is defined correctly.

Returning now to the multi-loop problem, the regulated expression to be evaluated is:

$$\begin{aligned} F_\epsilon(a) &:= -\ln \text{Det}_\epsilon^{-1}(\Delta - aR) = \epsilon \int_0^\infty dt/t \text{Tr}_1 e^{-t(\Delta - aR)} \\ &= \epsilon \int_0^\infty dt/t e^{-ta|R|} (\text{Tr} e^{-t\Delta} - 1) \end{aligned} \quad (18)$$

Again, for finite ϵ and a , both the integration and summation are convergent and their order can be interchanged; the expression can be rewritten as:

$$\begin{aligned} F_\epsilon(a) &= \int_\epsilon^{1/\mu^2} \frac{dt}{t} e^{-t(a + \frac{1}{4})|R|} \left[\sum_{\{P\}} \frac{\ln N(P_0)}{N^{\frac{1}{2}}(P) - N^{-\frac{1}{2}}(\bar{P})} \frac{e^{-[\ln N(P)]^2 / 4t|R|}}{\sqrt{(4\pi t|R|)}} \right] \\ &+ \int_{\frac{1}{\mu^2}}^\infty \frac{dt}{t} e^{-a|R|t} \left[\sum_{\{P\}} \frac{\ln N(P_0)}{N^{\frac{1}{2}}(P) - N^{-\frac{1}{2}}(\bar{P})} e^{-\frac{t|R|}{4}} \frac{e^{-[\ln N(P)]^2 / 4t|R|}}{\sqrt{(4\pi t|R|)}} - 1 \right] \\ &+ \frac{|R| \mu(\Sigma)}{4\pi} \int_{-\infty}^\infty k e^{-t(k^2 + \frac{1}{4})|R|} \tanh(\pi k) dk \end{aligned}$$

$$-\int_{\epsilon}^{\frac{1}{\mu^2}} \frac{dt}{t} e^{-a|R|t} + \frac{|R| \mu(\Sigma)}{4\pi} \int_{\epsilon}^{\frac{1}{\mu^2}} \frac{dt}{t} e^{-a|R|t} \int_{-\infty}^{\infty} k e^{-t(k^2 + \frac{1}{4})|R|} \tanh(\pi k) dk$$

(19)

where an arbitrary mass scale μ has once again been introduced. We shall examine the various integrals in turn. The integral in the first line of (19) is convergent as $t \rightarrow 0$ since for hyperbolic Γ 's $N(P) > 1$; this is true independently of a . So we can safely take the limit of both $\epsilon \rightarrow 0$ and $a \rightarrow 0$ in this term.

Next consider each of the three integrals in the second and third lines of equation (19). The last of these, proportional to the volume of Σ is finite as $t \rightarrow \infty$, even if $a \rightarrow 0$ simultaneously. Naively, the same appears to be true about the first integral as well; however in the limit of $a \rightarrow 0$ the sum over conjugacy classes $\{P\}$ ceases to converge — hence the integration and summation cannot be interchanged. This difficulty is removed by subtracting the '1' term before taking the limit $a \rightarrow 0$, the sum of both these integrals being finite in the limit.

Thus the only divergences to be met are in the last two terms of equation (19). In particular, the penultimate term diverges as $t \rightarrow 0$ like $\ln(\epsilon/\mu^2)$; independently of a . This is in accordance with the remarks made previously regarding the universality of this particular infinite counterterm. It appears to be independent of the genus. (There will in addition be finite, a dependent terms of course. In fact for the ghost determinant evaluated in this manner, the entire such 'counterterm' is finite.) It is interesting that

the originally infrared zero mode divergence has been converted into an independent ultraviolet one.

The final term in equation (19) is UV divergent as $t \rightarrow 0$. A simple scaling argument shows that it behaves like

$$|R| \mu(\Sigma)/4\pi (1/\epsilon\mu + \dots) = (g-1) (1/\epsilon\mu + \dots) \quad (20)$$

Since it is proportional to $\mu(\Sigma)$ or $(g-1)$ it can in principle be absorbed by a multiplicative renormalization of the loop counting parameter λ .

To reiterate, it is important that the remnant of the zero mode infinity is genus independent. It can be eliminated by a universal subtraction implemented as an overall genus independent renormalization of the infinite dimensional determinants.

One may proceed by defining a completely finite expression for F by adding two counterterms:

$$\begin{aligned} \tilde{F}_\epsilon(a, \mu) = & F_\epsilon(a) + \epsilon \int^{1/\mu^2} dt/t e^{-a|R|t} \\ & - (g-1) \epsilon \int^{1/\mu^2} dt/t e^{-a|R|t} - \int_{-\infty}^{\infty} dk e^{-t(k^2+1/4)|R|} \tanh(\pi k) dk \end{aligned} \quad (21)$$

The limits $\epsilon \rightarrow 0$ and $a \rightarrow 0$ can now be taken such that $\tilde{F}(\mu^2)$ is both finite and independent of the regulators ϵ and a . There will in general be μ dependent, finite counterterms left over as usual, though to evaluate them explicitly is quite hard.

Having understood the infinity structure of F , we return to equation (19) and evaluate $F_\epsilon(a)$ as a function of the cut-offs.

$$\begin{aligned}
F_\varepsilon(a) = & (g-1) \int_\varepsilon^\infty \frac{dt}{t} e^{-a|R|t} \int_{-\infty}^\infty k e^{-t(k^2 + \frac{1}{4})|R|} \tanh(\pi k) dk \\
& + \int_\varepsilon^\infty \frac{dt}{t} e^{-a|R|t} \left[\sum_{\{P\}} \frac{\ln N(P_0)}{N^{\frac{1}{2}}(P) - N^{-\frac{1}{2}}(P)} e^{-\frac{t}{4}} \frac{e^{-\frac{[\ln N(P)]^2}{4t|R|}}}{\sqrt{(4\pi t|R|)}} - 1 \right]
\end{aligned}
\tag{22}$$

The first term can be easily evaluated (up to terms non-singular as $\varepsilon \rightarrow 0$); we concentrate on the second. Define

$$X = \sum_{\{P\}} \frac{\ln N(P_0)}{N^{\frac{1}{2}}(P) - N^{-\frac{1}{2}}(P)} \int_0^\infty \frac{dt}{t} \frac{e^{-t|R|(a + \frac{1}{4})}}{\sqrt{(4\pi t|R|)}} e^{-\frac{[\ln N(P)]^2}{4t|R|}}
\tag{23}$$

In (22), ε has been set to zero; for non-zero a this is allowed since the integral and sum converge. The integral over t is straightforward to evaluate and yields:

$$X(\alpha) = \sum_{\{P\}} \frac{\ln N(P_0)}{N^{\frac{1}{2}}(P) - N^{-\frac{1}{2}}(P)} \frac{1}{\ln N(P)} e^{-\alpha |\ln N(P)|}
\tag{24}$$

where $\alpha = \sqrt{a + 1/4}$. Using a property of primitive elements of Γ viz. that $N(P) = N(P_0^n) = N^n(P_0)$,

$$X(\alpha) = \sum_{\{P_0\}} \sum_{n=1}^\infty \frac{1}{n} \frac{N(P_0)^{-\alpha n}}{N^{\frac{1}{2}}(P_0) - N^{-\frac{1}{2}}(P_0)}
\tag{25}$$

The summations over n can be rewritten using the identity

$$\sum_{n=1}^{\infty} \frac{x^{-\alpha n}}{n(x^{n/2} - x^{-n/2})} = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} \frac{x^{-n(\alpha + \frac{1}{2} + r)}}{n} = - \sum_{r=0}^{\infty} \ln[1 - x^{-(\alpha + \frac{1}{2} + r)}] \quad (26)$$

With (26), $X(\alpha)$ assumes the form:

$$\begin{aligned} X(\alpha) &= - \sum_{\{P_0\}} \sum_{r=0}^{\infty} \ln[1 - N(P_0)^{-(\alpha + \frac{1}{2} + r)}] \\ &= - \ln \prod_{\{P_0\}} \prod_{r=0}^{\infty} [1 - N(P_0)^{-(\alpha + \frac{1}{2} + r)}] \\ &= - \ln Z(\alpha + \frac{1}{2}), \end{aligned} \quad (27)$$

where $Z(s)$ is the Selberg zeta-function (10), defined as

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} [1 - N(P_0)^{-s-k}]$$

It is an entire function, and converges for $\text{Re}(s) > 1$; outside this region it can be defined by analytic continuation. $Z(1)$ has a zero of multiplicity one. Combining equations (22), (27) and (20) we have

$$F_\varepsilon(a) = -\ln \text{Det}_\varepsilon^{-1}(\Delta - aR) = (g-1) \left[\frac{1}{\varepsilon |R|} + \dots \right] \\ - \ln Z(\alpha + \frac{1}{2}) - \int_\varepsilon^\infty \frac{dt}{t} e^{-a|R|t} \quad (28)$$

Since $\alpha = \sqrt{a + 1/4}$, one sees that as mentioned above the zero mode infinity ($\alpha \rightarrow 1/2$) manifests itself here as a lack of convergence of the Selberg product. Restoring the counterterms of equation (21),

$$\tilde{F}_\varepsilon(a, \mu^2) = (g-1) \left[\frac{1}{\varepsilon |R|} + \dots \right] - \ln Z(\alpha + \frac{1}{2}) \\ - \int_\varepsilon^\infty \frac{dt}{t} e^{-a|R|t} + \int_\varepsilon^{\frac{1}{\mu^2}} \frac{dt}{t} e^{-a|R|t} \\ - (g-1) \int_\varepsilon^{\frac{1}{\mu^2}} \frac{dt}{t} e^{-a|R|t} \int_{-\infty}^\infty k e^{-t(k^2 + \frac{1}{4})|R|} \tanh(\pi k) dk \\ = (g-1) f(\varepsilon, a, \mu) - \ln Z(\alpha + \frac{1}{2}) + \left(\ln \frac{a|R|}{\mu^2} + \text{finite terms} \right) \quad (29)$$

Taylor expanding the zeta-function about its zero at $a=0$ ($\alpha=1/2$)

$$\lim_{a \rightarrow 0} \ln [Z(\alpha + 1/2)/a] = \lim_{a \rightarrow 0} \ln [Z'(1) + a Z''(1)/2! + O(a^2)] \\ = \ln Z'(1) + O(a) ,$$

and upon inserting this into (29),

$$\tilde{F}_\epsilon(a, \mu^2) = -\ln Z'(1) + (g-1) f(\epsilon, a, \mu) + k(a, \mu)$$

where f and k are smooth functions of the regulators ϵ and a . Therefore we can now take the limits $\epsilon \rightarrow 0^+$ and $a \rightarrow 0^+$ respectively whence a finite determinant for the scalar Laplacian is obtained; namely

$$\text{Det}'\Delta = \frac{\tilde{F}}{e} = Z'(1) e^{-[(g-1)f(\mu) + k(\mu)]} \quad (30)$$

For the case of the ghost operator the relevant limit is $a \rightarrow 2$ (cf. equation (14) of the previous section). There is no zero mode, the concomitant infrared divergence is absent and the determinant can be evaluated straightforwardly in the presence of the proper time regulator alone. We simply state the result:

$$\ln \text{Det}_\epsilon(\Delta - 2R) = \ln Z(2) - (g-1) [1/(\epsilon R)] + \dots \quad (31)$$

Inserting this into equation (14) gives the regulated Faddeev-Popov determinant:

$$[\text{Det}_\epsilon \mathcal{D}]^{1/2} = Z(2) e^{-(g-1)[1/(\epsilon R) + \ln(2|R|\epsilon) + h(\epsilon, \mu)]} \quad (32)$$

where h is a smooth function of ϵ . The infinities in the limit $\epsilon \rightarrow 0^+$ can be removed as before by subtracting an infinite counterterm proportional to $\mu(\Sigma)$ or equivalently by a multiplicative renormalization of the loop counting parameter λ , even if a somewhat unconventional one.

Combining equations (30) and (32) in the critical dimension $d=26$, and inserting the product into (1) one has

$$\sum_{g=2}^{\infty} \lambda_R^{(g-1)} \int \prod_{i=1}^6 d\tau_i \frac{\det\langle\psi|\chi\rangle}{\det H(P^\dagger)} \frac{Z(2) [Z'(1)]^{-13}}{[\int d^2\sigma \sqrt{q}]^{-13}} L(p_1, \dots, p_k) \quad (33)$$

where $Z(2) = \prod_{\{P_0\}} \prod_{n=2}^{\infty} [1 - N(P_0)^{-n}]$. $Z'(1)$ does not appear to have any particularly simple form. We have also absorbed in λ_R finite renormalizations (the f , k and h terms above) which as usual are implicit in any subtraction scheme. These could be computed with additional labour, but are unlikely to be of any immediate interest.

This completes the evaluation of the infinite dimensional determinants in equation (1) of section II.

These expressions may be compared with existing dual model calculations (13,14). However at this stage any detailed comparison would be premature; the finite dimensional determinants are also needed.

IV. Concluding remarks

In this paper we have computed some of the various determinants necessary to calculate string amplitudes, namely the infinite dimensional ones to arbitrary string-loop order ($g \geq 2$). We have endeavoured to demonstrate that the divergences that arise in their definition can be consistently removed. Those originating in an infrared zero mode problem, by a careful definition of the traces involved, are seen to be spurious and in fact genus

independent. The ultraviolet infinity can evidently be eliminated by a renormalization of the genus or loop counting parameter of the string theory.

It is perhaps to be emphasized that the infinities dealt with in this paper have more to do with a sensible definition of the string generating functional itself in the Polyakov approach rather than with any 'physical' ultraviolet finiteness or otherwise of amplitudes or Green functions³. Certainly the former is antecedent to any eventual proof of the latter. In any case, since a coherent set of Feynman-like rules to arbitrary loop order have yet to be formulated for strings, clarity as to what divergences are to be expected and how they should be handled is helpful. This is also necessary for any investigation of unitarity of this formulation of string theory.

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Note Added:

We have been apprised of a recent Columbia preprint by D'Hoker and Phong (CU-TP-323) evidently covering similar ground. Mandelstam has also discussed similar questions in his Berkeley preprint UCB-PTH-85/47; October 1985.

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³ For instance, the divergences we have removed here may factor out when amplitudes are properly normalized.

Appendix

We shall attempt an intuitive description of the Selberg formula used in the text. Consider the quantum mechanics of a point particle moving on a compact, multiply connected surface. In particular, consider the probability that the particle returns to its starting point after a time t . This will be given by the partition function $\text{Tr } e^{-Ht}$, with $H = \Delta$.

The Selberg formula relates the eigenvalues of the Laplacian to the length of closed geodesics on Σ . In other words, the eigenvalues of the Hamiltonian to the classical action for closed trajectories.

That this is possible for integrable systems follows from WKB quantization. It is remarkable that such a direct relation between classical and quantum dynamics exists here, because the system is definitely not integrable. In fact, the geodesics (classical trajectories) diverge from each other always because of the negative curvature. Yet they must stay within a finite volume. So the classical motion is ergodic. Gutzwiller has studied semi-classical methods in such systems in a series of important papers (15).

A path-integral representation can be written for the partition function in terms of an action

$$S = \int_0^t g_{ab}(q) \dot{q}^a \dot{q}^b d\tau, \quad q^a \in \Sigma$$

namely

$$\text{Tr } e^{-\Delta t} = \int_{q^a(t)=q^a(0)} Dq e^{-S - t/4}. \quad (\text{A.1})$$

The extra term in the exponential is a 'correction' term due to the curvature of the manifold on which the particle moves. We can rewrite the path-integral on the covering space of Σ ($g \geq 2$), that is to say on the upper half plane:

$$\text{Tr } e^{-\Delta t} \sim \sum_{\{P\}} \int_{q(0)=q}^{q(t)=P(q)} Dq e^{-S - t/4}, \quad q^a \in U \quad (\text{A.2})$$

where $\Sigma = U/\Gamma$. Γ is the fundamental group of Σ . $\{P\}$ denotes conjugacy classes of elements in Γ . We should not count two $P, P' \in \Gamma$ which are conjugate i.e.

$$P' = QPQ^{-1}, \quad Q \in \Gamma$$

as being separate contributions. For, a given closed curve C in U/Γ may be thought of as belonging to $\pi_1(\Sigma; p)$ based at any point p on C . Changing the point p leads to a conjugation of the class to which C belongs. It would therefore be overcounting to include them.

Now, the path-integral (A.2) is that for the heat kernel of the Laplacian on U .

$$\Delta h = -\partial h / \partial t, \quad ,$$

with boundary condition

$$h(0; z, z') = \delta(z, z') .$$

Since U is a homogeneous space (i.e. its Killing group is transitive) we can reduce the heat equation to an ordinary differential equation by exploiting the symmetries. For instance h may be found by using harmonic analysis. One finds in these cases that the path-integral is dominated by paths close to the classical trajectory.

$$h(t; q, q') \sim \frac{e^{-\sigma(q, q')/4t - 4t}}{\sqrt{4\pi t}} \left| \frac{\partial^2 \sigma(q, q')}{\partial q^a \partial q^{b'}} \right|^{-1/2}$$

for $q \neq q'$. In every homotopic class P , there is associated a unique geodesic whose length is $\ln N(P)$; this is $\sigma(q, q')$, the hyperbolic distance between q and q' . To find h on U/Γ we may proceed by classifying the non-conjugate elements P further into 'primitive' ones P_0 and the non-primitive ones which are powers of some P_0 . One may calculate h in the usual fashion using a WKB method to recover the Selberg formula. The first term in the trace formula is the contribution of the contractible paths in U/Γ which correspond to closed paths in U .

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