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NEW COMPACTIFICATIONS OF $N = 1$, $d = 11$ SUPERGRAVITY

B. Biran^{*)}

CERN -- Geneva

and

Ph. Spindel

Université de l'Etat à Mons, Belgium

ABSTRACT

We derive the explicit expression of the supersymmetric spinor of the G_2 invariant compactification of $N = 1$, $d = 11$ supergravity. Moreover, we study the generalizations of this solution using six-dimensional homogeneous spaces and show that there exist only two candidates, with respectively an $SO(5)$ or $SU(3)$ internal symmetry group.

^{*)} On leave of absence from Université Libre de Bruxelles, Brussels, Belgium.

1. INTRODUCTION

During the last years, a number of exact solutions of the field equations of eleven-dimensional supergravity [1] have been found. The main class of these solutions rests on the Freund-Rubin ansatz [2]. Their geometry consists of a Riemannian product of a 4-dimensional anti-de Sitter space with an Euclidean 7-dimensional Einstein space while the matter field is taken proportional to the volume 4-form of the AdS subspace. Moreover if the 7-dimensional subspace admits Killing spinors, it is well known that on the one hand they are related to the residual supersymmetries when the matter field is suitably defined and, on the other hand, they allow us to build Englert solutions [3], i.e. solutions with non-vanishing components of the matter field on the internal subspace.

The previous solutions are not the most general ones with an $SO(3,2)$ invariant metric [4]. They are given by

$$ds_{11}^2 = \Delta^{-1} ds_4^2 + ds_7^2 \quad (1.1)$$

where the 4-dimensional anti-de Sitter geometry is now rescaled with a non-vanishing arbitrary function of the internal co-ordinates. Such a non-trivial solution -- with an $SO(3,2) \times SO(7)$ isometry group -- was first obtained by de Wit and Nicolai [5]. The 7-dimensional part of their metric is conformal to the one of an $SO(7)$ invariant ellipsoid embedded in a flat 8-dimensional Euclidean space. It reads as

$$ds_7^2 = -N^2(\alpha) d\alpha^2 + f^2(\alpha) d\Omega_6^2 \quad (1.2)$$

where $d\Omega_6^2$ is the metric of the unit round 6-sphere. (The signature of the 11-dimensional metric we use is -9). Notice the appearance of an arbitrary function $N(\alpha)$, analogue to the lapse function commonly used in cosmology, which simplifies greatly the integration of the field equations when it is chosen proportional to Δ^{-1} . Soon after the discovery of this solution, de Wit, Nicolai and Warner (WNW) obtained a new solution [6] with similar geometry but non-vanishing components of the matter field on the internal space. Here, though the geometry is still $SO(7)$ invariant [7], the appearance of the matter field breaks the invariance of

the solution to G_2 . This solution corresponds to the G_2 critical point [8] of the scalar potential of the 4-dimensional de Wit-Nicolai theory [9]. Accordingly it must have $N = 1$ supersymmetry. As this is the first example of a solution with residual supersymmetry and non-vanishing matter field in the internal dimensions, we have found it interesting to present an explicit expression of the supersymmetric spinor in a framework which goes beyond our previous analysis of these questions [3, 10].

We shall also discuss the possible generalizations of the WNW solution by considering in arbitrary homogeneous space instead of a 6-sphere. This had led us to two new solutions with $SU(3)$ and $SO(5)$ internal invariance groups.

2. THE RESIDUAL SUPERSYMMETRY OF THE WNW SOLUTION

For the sake of completeness we collect below the expressions of the metric and matter field of the WNW solution. Our signature convention is $(+,-\dots-)$. Eleven-dimensional co-ordinates will be labelled with capital latin letters, while small greek letters will refer to the 4-dimensional subspace, small latin letters to the 6-dimensional subspace and the index 7 will correspond to the co-ordinate α of the internal space. Indices topped by a wedge will label components with respect to an orthonormal (co)frame. We shall make use of a real representation of the 8×8 Dirac matrices satisfying the usual Clifford algebra on the 7-dimensional subspace and such that $\Gamma^{\hat{1}} \dots \Gamma^{\hat{7}} = -\mathbf{1}$. Finally we shall denote by $\overset{\circ}{\nabla}_{\mu}$ or $\overset{\circ}{\nabla}_m$ the components of the covariant derivatives built from the Levi-Civita connection on ds^2_{AdS} or $d\Omega_6^2$.

As stated previously the WNW metric is given by Eqs. (1.1, 1.2) where the factor in front of the anti-de Sitter metric is

$$\Delta^{-1} = \left(1 - \frac{2}{3} \sin^2 \alpha\right)^{2/3} \quad (2.1)$$

and, in a co-ordinate system such that

$$N^2 = \lambda^2 \Delta^{-1} \quad (2.2)$$

the scaling factor in front of the 6-dimensional metric is

$$\rho^2 = \lambda^2 \sin^2 \alpha \cdot \Delta^{1/2} \quad (2.3)$$

The constant λ^2 measures the mean curvature of the internal space. It is related to the inverse radius m_4 of the AdS subspace by the relation

$$m_4^2 \lambda^2 = \frac{8}{5} \quad (2.4)$$

The matter field components are

$$\begin{aligned} F_{\mu\nu\rho\sigma} &= f \overset{\circ}{\gamma}_{\mu\nu\rho\sigma} \\ F_{mnpq} &= 4 f_1 \bar{\chi} \Gamma_{mnpq} \chi \\ F_{mnp7} &= \sigma \left(\partial_\alpha f_1 \bar{\chi} \Gamma_{mnp} \chi + \sqrt{2} f N \Delta^2 f_1 \bar{\chi} \Gamma_7 \Gamma_{mnp} \chi \right) \end{aligned} \quad (2.5)$$

where $\overset{\circ}{\eta}$ is the AdS volume 4-form, f a constant related to the four dimensional curvature by

$$f^2 = \frac{5}{12} m_4^2 \quad (2.6)$$

and χ a Killing spinor on S_6 , i.e. a spinor obeying the equation

$$\overset{\circ}{\nabla}_m \chi = \frac{\sigma}{2} \Gamma_m \chi \quad (2.7)$$

with $\sigma = \pm 1$.

The dependence of the matter field on α is given by

$$f_1 = \frac{\lambda^3}{\sqrt{30}} \frac{\sin^4 \alpha}{\left(1 - \frac{2}{3} \sin^2 \alpha\right)} \quad (2.8)$$

The residual supersymmetry of the WNW solution is obtained by requiring that the supersymmetry variation of the $d = 11$ gravitino field vanishes, i.e. by solving the equations

$$\nabla_M \epsilon + \frac{2\sqrt{2}}{(4!)^2} i \left(\Gamma^{NPQR}{}_M - 8 \Gamma^{PQR} \delta_M^N \right) F_{NPQR} \epsilon = 0 \quad (2.9)$$

These equations split into three subsets:

$$\begin{aligned} & \left[\Delta^{\frac{1}{2}} \bar{\nabla}_\mu - \frac{\sqrt{2}}{6} f \Delta^2 \gamma_5 \gamma_\mu + \frac{1}{4} \frac{(\partial_\alpha \Delta)}{N \Delta} \gamma_\mu \gamma_5 \Gamma_\alpha \right] \varepsilon \\ & + \frac{8\sqrt{2}}{(4!)^2} i \gamma_5 \left[f^{-4} f_{\mu\nu} \Gamma^{\alpha bcd} \bar{\chi} \Gamma_{\alpha bcd} \chi + \sigma N^{-1} f^{-3} (\partial_\alpha f) \Gamma_\alpha \Gamma^{\alpha bc} \bar{\chi} \Gamma_{\alpha bc} \chi \right. \\ & \left. + \sigma \sqrt{2} f^{-3} f \Delta^2 f_{\mu\nu} \Gamma_\alpha \Gamma^{\alpha bc} \bar{\chi} \Gamma_\alpha \Gamma_{\alpha bc} \chi \right] \varepsilon = 0 \end{aligned} \quad (2.10)$$

$$\begin{aligned} & \left[f^{-1} \bar{\nabla}_m - \frac{\sqrt{2}}{12} f \Delta^2 \Gamma_m - \frac{1}{2} N^{-1} f^{-1} (\partial_\alpha f) \Gamma_m \Gamma_\alpha \right] \varepsilon \\ & + \frac{8\sqrt{2}}{(4!)^2} i \gamma_5 \left[f^{-4} f_{\mu\nu} \Gamma_m^{\mu\nu\rho\tau} \bar{\chi} \Gamma_{\mu\nu\rho\tau} \chi + \sigma \frac{(\partial_\alpha f)}{N f^3} \Gamma_\alpha^{\mu\nu\rho\tau} \bar{\chi} \Gamma_{\mu\nu\rho\tau} \chi \right. \\ & + \sigma \sqrt{2} f^{-3} f \Delta^2 f_{\mu\nu} \Gamma_\alpha \Gamma^{\mu\nu\rho\tau} \bar{\chi} \Gamma_\alpha \Gamma_{\mu\nu\rho\tau} \chi - 8 f^{-4} f_{\mu\nu} \Gamma^{\mu\nu\rho\tau} \bar{\chi} \Gamma_{\mu\nu\rho\tau} \chi \\ & \left. + \sigma 6 \frac{(\partial_\alpha f)}{N f^3} \Gamma_\alpha^{\mu\nu\rho\tau} \bar{\chi} \Gamma_{\mu\nu\rho\tau} \chi + \sigma 6 \sqrt{2} f^{-3} f \Delta^2 f_{\mu\nu} \Gamma_\alpha \Gamma^{\mu\nu\rho\tau} \bar{\chi} \Gamma_\alpha \Gamma_{\mu\nu\rho\tau} \chi \right] \varepsilon = 0 \end{aligned} \quad (2.11)$$

$$\begin{aligned} & \left[N^{-1} \partial_\alpha - \frac{\sqrt{2}}{12} f \Delta^2 \Gamma_\alpha \right] \varepsilon + \frac{8\sqrt{2}}{(4!)^2} i \gamma_5 \left[f^{-4} f_{\mu\nu} \Gamma_\alpha^{\alpha bcd} \bar{\chi} \Gamma_{\alpha bcd} \chi \right. \\ & \left. + \sigma 2 \frac{(\partial_\alpha f)}{N f^3} \Gamma^{\alpha bc} \bar{\chi} \Gamma_{\alpha bc} \chi + \sigma 2 \sqrt{2} f f^{-3} \Delta^2 f_{\mu\nu} \Gamma_\alpha^{\alpha bc} \bar{\chi} \Gamma_\alpha \Gamma_{\alpha bc} \chi \right] \varepsilon = 0 \end{aligned} \quad (2.12)$$

In order to go ahead, we make an ansatz for ε . We assume that it can be expressed as

$$\varepsilon(x^\mu, y^m, \alpha) = \varepsilon_1(x, \alpha) \otimes \chi(\gamma) + \varepsilon_2(x, \alpha) \otimes \Gamma_\alpha \chi(\gamma) \quad (2.13)$$

Moreover, without loss of generality, we may take $\sigma = +1$. Indeed, we can always reverse the sign of σ by substituting $\Gamma_\gamma \chi$ for χ without changing the matter field components. With the ansatz (2.13), Eq. (2.11) becomes a homogeneous linear system of four equations (see Appendix A) coupling together the chiral components of the 4-dimensional spinors ε_1 and ε_2 . This system is overdetermined but nevertheless self-consistent when

$$f \lambda = \mp \frac{\sqrt{2}}{\sqrt{3}} \quad (2.14)$$

It leads then to the relations

$$E_z = \cos \alpha [A_{\mp} E_2 + B_{\mp} i \gamma_5 E_2] \quad (2.15)$$

with

$$A_{\mp} = \frac{8 \sin^2 \alpha - 3}{3 \Delta^{-3/4} - 3 \frac{\sqrt{2}}{2} f \lambda \sin \alpha} \quad (2.16)$$

$$B_{\mp} = -2\sqrt{15} \frac{\sin \alpha \Delta^{-3/4}}{3 \Delta^{-3/4} - 3 \frac{\sqrt{2}}{2} f \lambda \sin \alpha}$$

Let us notice that for $f\lambda$ positive these expressions become singular on the equator: $\alpha = \pi/2$. Actually this singularity is spurious (see Appendix A) but we shall restrict ourselves here to the case $f\lambda$ negative.

The next step is to solve Eq. (2.10). To this end we express E_2 in terms of AdS Killing spinors as

$$E_2(x^\mu, \alpha) = \mu(\alpha) \tilde{E}(x^\mu) + \nu(\alpha) i \gamma_5 \tilde{E}(x^\mu) \quad (2.17)$$

where

$$\nabla_\mu \tilde{E}(x) = \frac{m_4}{2} \gamma_5 \gamma_\mu \tilde{E}(x) \quad (2.18)$$

Here again we have the freedom to choose the sign of m_4 . Indeed $\tilde{E}$ and $\gamma_5 \tilde{E}$ verify Killing equations with opposite eigenvalues. Accordingly we fix our convention by requiring

$$m_4 \lambda = 2 \frac{\sqrt{2}}{\sqrt{5}} \quad (2.19)$$

Equation (2.10) leads now to four equations (all equivalent) which give us ν in terms of μ as

$$\nu = \frac{\sqrt{15}}{5} \mu \frac{\sqrt{3} \Delta^{-3/4} (5 + 3\sqrt{30} \Delta^{-3/2}) + \sin \alpha (25 - 20 \sin^2 \alpha + \sqrt{30})}{3 \Delta^{-3/4} (8 \sin^2 \alpha - 7) + \sqrt{3} \sin \alpha} \quad (2.20)$$

The last unknown function μ is obtained by inserting all the previous expressions into Eq. (2.12). This gives us four first-order homogeneous differential equations. Integrating one of them, we obtain (up to an arbitrary multiplicative constant)

$$\mu = \Delta^{7/8} \left[\frac{\sqrt{3}}{3} \sin \alpha + \Delta^{-3/4} \right]^{1/2} \left[\frac{3\sqrt{10} + 4\sqrt{3}}{2\sqrt{2}} \sin \alpha - \Delta^{-3/4} \right] \quad (2.21)$$

and we have verified by direct substitution that the three remaining equations are also satisfied.

Accordingly we have shown that the WNW solution has $N = 1$ residual supersymmetry whatever the signs of the amplitudes of the matter fields are. This property remains true for the solutions built in the next section. This deserves some attention because it has been proved [11] that for Freund-Rubin solutions built on 7-dimensional coset spaces other than S_7 , the sign of the amplitude f of the 4-dimensional matter field components may be an obstruction to the existence of residual supersymmetries.

3. TWO OTHER SOLUTIONS

In the previous section we focused our attention on S_6 as a 6-dimensional subspace. However the only properties needed were:

- 1) the existence of an Euclidean Einsteinian metric with positive curvature,
- 2) the compactness of the space and
- 3) the existence of a Killing spinor^{*)}.

Accordingly any spaces V_6 with these properties will define a new WNM solution.

On the sphere S_6 the integrability conditions for the Killing equations (2.7) are identically satisfied. As a consequence these equations admit 8 independent solutions χ_I^σ ($I=1 \dots 8$) for each sign of the eigenvalue σ . These two sets of solutions are not independent but related via a multiplication by a Γ_7 matrix. This reflects the uniqueness of the 8-dimensional irreducible representation of $SO(7)$ or the uniqueness of the class of inequivalent representations of the Dirac matrices on a V_6 space. In the following we restrict ourselves to $\sigma = +1$.

*) Actually these properties are not all independent. If an n -dimensional Euclidean space possess a Killing spinor field χ such that $\nabla_a \chi = m/2 \Gamma_a \chi$, its metric is automatically Einsteinian: $R_{ab} = (n-1)m^2 g_{ab}$ and if $m \neq 0$ the space is compact.

From the 8 Killing spinors we may build on the one hand Killing vectors

$$K_{[I\bar{J}]}^a = \bar{\chi}_I \Gamma^a \chi_{\bar{J}} \quad (3.1)$$

among which only 21 are linearly independent, and on the other hand conformal Killing vectors

$$L_{[I\bar{J}]}^a = \bar{\chi}_I \Gamma_I^a \Gamma_{\bar{J}} \chi_{\bar{J}} \quad (3.2)$$

$$\bar{\nabla}^{(a} L_{[I\bar{J}]}^{b)} = g^{ab} \bar{\chi}_I \Gamma_{\bar{J}} \chi_{\bar{J}} \quad (3.3)$$

among which only 7 are linearly independent.

Notice that if $L_{[I\bar{J}]}^a$ is non-identically zero, it is a non-isometric conformal Killing vector as

$$L_{[I\bar{J}]}^a = \bar{\nabla}^a (\bar{\chi}_I \Gamma_{\bar{J}} \chi_{\bar{J}}) \quad (3.4)$$

As a consequence we deduce a *uniqueness theorem*: on a V_6 space other than S_6 there exists at most one Killing spinor field. Indeed suppose there are two, say χ_1 and χ_2 . As the only compact Einstein space possessing non-isometric conformal Killing vectors are the spheres [12], $\bar{\chi}_1 \Gamma_7 \chi_2$ must be zero. Accordingly the vector $\chi_1 \Gamma_7 \Gamma_a \chi_2$ vanishes identically and its norm must also be zero. But on the other hand a direct computation shows us that

$$\bar{\chi}_1 \Gamma_7 \Gamma_a \chi_2 \bar{\chi}_2 \Gamma^a \Gamma_7 \chi_1 = -(\bar{\chi}_1 \chi_1)(\bar{\chi}_2 \chi_2) + (\bar{\chi}_1 \chi_2)^2 + (\bar{\chi}_1 \Gamma_7 \chi_2)^2 \quad (3.5)$$

which leads to a contradiction except if χ_1 and χ_2 are linearly dependent.

In the framework of the WNW solution the symmetry of the full solution is G_2 while the geometry is $SO(7)$ invariant. This symmetry breaking is particular to the 6-sphere. On all others V_6 , if a Killing spinor χ exists, it is invariant with respect to the full isometry group of the space. Indeed if $\vec{K}$ is a Killing vector, the Lie derivative of χ along $\vec{K}$ [13]:

$$\mathcal{L}_{\vec{K}} \chi = K^a \bar{\nabla}_a \chi + \frac{1}{4} \bar{\nabla}^a K^b \Gamma_{ab} \chi \quad (3.6)$$

defines in general a new Killing spinor orthogonal to the first one

$$\bar{\chi} \mathcal{L}_{\vec{K}} \chi = 0 \quad (3.7)$$

However, as on a V_6 ($\neq S_6$) the space of Killing spinors is at most one-dimensional, we deduce that the Killing spinor is invariant with respect to the isometry group. Accordingly the invariance of the corresponding WNW solution is the full isometry group. In the exceptional case of S_6 we may easily verify that the G_2 subgroup of invariance of the solution is the one with respect to which the spinor field remains invariant.

In order to go ahead we shall restrict our analysis to homogeneous spaces G/H which provide compactification of $d = 11$ supergravity via a WNW solution. These spaces have to satisfy the three criteria listed previously. The first one implies that H is a compact Lie group and, taking into account the second one, G must also be compact. Moreover a standard structure theorem teaches us that if we assume G/H connected, then the Lie algebra of G is obtained from a sum of simple Lie algebras and an Abelian algebra whose total dimension should be between 6 and 21. Finally the requirement of a Killing spinor field implies -- if the space is not Ricci flat -- that Riemannian product spaces are not allowed. Indeed let us suppose that $V_n = V_m \otimes V'_{n-m}$ and that the metric tensor is a sum of metrics defined respectively on each factor. If a and a' are indices referring to directions tangent to V_m and V'_{n-m} , the covariant derivatives ∇_a and $\nabla_{a'}$ commute and the associated integrability condition reduces to

$$\nabla_a \nabla_{a'} \chi = 0 \quad \text{i.e. } \chi = 0 \quad (3.8)$$

The uniqueness theorem leads also to some simplification in the homogeneous spaces that we have to consider.

We assume that the spaces considered here possess a unique non-trivial Killing spinor. If on this space there exists also a Killing vector, all derivatives of which vanish at a point, at this point Eq. (3.6) reduces to

$$K^a \nabla_a \chi = 0 \quad (3.9)$$

Squaring this equality, we can factor $\bar{\chi}\chi$ out and find at the end that $K_a = 0$ and $K_{a,b} = 0$ at a point, which implies that the Killing vector field vanishes everywhere. Accordingly we have to consider only homogeneous spaces G/H with $6 \leq \dim G \leq 15$. The possible candidates as Lie algebras for G are then: $SU(4) \simeq SO(6)$, $SO(5)$, $SU(2) \simeq SO(3)$, $U(1) \simeq SO(2)$, G_2 , $SU(3)$ and products of the previous ones whose dimensions are less than or equal to 15. We shall limit ourselves to a local classification of the homogeneous spaces, without discussing the consequences of topological differences between groups owning the same Lie algebra.

Excluding spaces with S_1 factors we obtain the following list:

- 1) $SU(4)/SU(3) \times SO(2)$
- 2) $G_2/SU(3)$
- 3) $SO(5) \times SO(3)/SO(3) \times SO(3) \times SO(2)$
- 4) $SU(3) \times SU(2)/SU(2) \times U(1) \times U(1)$
- 5) $SU(2) \times SU(2) \times SU(2)/U(1) \times U(1) \times U(1)$
- 6) $SU(2) \times SU(2)$

which we now discuss in more detail.

- 1) $SO(6)/SU(3) \times SO(2)$

This space corresponds to the complex projective plane CP^3 which admits only one (up to a scale factor that we shall never take into account) homogeneous Einsteinian metric but not Killing spinor.

- 2) $G_2/SU(3)$

Here the *unique* G_2 invariant metric is also $SO(7)$ invariant and we meet again the 6-sphere. It is the existence of an almost complex structure on S_6 which is only G_2 invariant [6] that is reflected here. In terms of Killing spinors we have the following description. Starting from the 8 independent Killing spinor fields we pick up one of them, say χ_8 . The remaining seven define the 21 generators of $SO(7)$ as $K_{[ij]}^a = \bar{\chi}_i \Gamma^a \chi_j$ ($i, j = 1 \dots 7$). The 14 generators

$$K_\Lambda^a = c_\Lambda^{ij} K_{[ij]}^a \quad (\Lambda = 1, \dots, 14) \quad (3.10)$$

of the G_2 subgroup which preserves the spinor field X_8 are obtained according to Eq. (3.10) from the 14 independent solutions of the homogeneous system

$$a^k_{ij} c_{\Lambda}{}^{ij} = 0 \quad (3.11)$$

in which the a^k_{ij} denote the structure constants of the octonionic multiplication.

3) $SO(5) \times SO(3)/SO(3) \times SO(3) \times SO(2)$

Here there are three ways to embed the subgroup into the group.

a) $(SO(5)/SO(3) \times SO(3)) \times (SO(3)/SO(2))$

This leads to the product space $S_4 \times S_2$ which admits only *one* $SO(5) \times SO(3)$ invariant Einstein metric but no Killing spinor.

b) $SO(5)/SO(3) \times SO(2)$

Here we have divided out the $SO(3)$ factor of G . There still remain two ways to embed $SO(3) \times SO(2)$.

First we have the canonical embedding such that $6 \rightarrow 6$. [We recall that in order to obtain the dimension of the space of G invariant metrics, we have to consider the splitting of the Lie algebra of G into $\mathcal{K}$ (Lie algebra of H) + $\mathcal{K}$ and decompose $\mathcal{K}$ into *real* irreducible representations of $\text{Ad}H$.] This coset space admits only *one* homogeneous Einstein metric and no Killing spinors.

On the other hand we may also consider the embedding such that $6 \rightarrow 2 + 4$. In that case, the invariant metrics constitute a one-parameter family and an explicit computation (summarized in Appendix B) gives us *two* Einsteinian metrics. Moreover *one* of them allows a Killing spinor field. Here we see that the way the subgroup is embedded plays a key rôle. This can be understood by analysing the isotropy subgroup invariance equation along the lines developed in Ref. [14].

4) $SU(3) \times SU(2)/SU(2) \times U(1) \times U(1)$

This case is easier to study. If a $U(1)$ factor is considered as dividing $SU(2)$, the resulting space is $CP^2 \times S^2$ which admits *one* homogeneous Einstein metric but no Killing spinor.

An alternative is to divide out the $SU(2)$. The resulting space $SU(3)/U(1) \times U(1)$ admits invariant metrics depending on *two* parameters as $6 \rightarrow 2 + 2 + 2$. Here we have found three sets of values of the scaling factors leading to Einstein metrics. Two of them give isometric metrics without a Killing spinor field while the third one possesses one Killing spinor field (see Appendix B).

5) and 6) These two homogeneous spaces correspond to products of spheres $S^2 \times S^2 \times S^2$ or $S^3 \times S^3$, and accordingly each of them admits one Einsteinian metric without a Killing spinor.

To summarize, let us point out that all the homogeneous Einsteinian spaces obtained here provide also new compactifications of $N = 1$, $d = 11$ supergravity similar to the solution of Ref. [5].

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Appendix A

First we summarize the algebraic identities useful for handling Eqs. (2.12). They are all easily obtained once we notice that there exists a real antisymmetric representation of the Dirac matrices and that the spinors χ , $\Gamma_a \chi$ and $\Gamma_7 \chi$ define a basis for the 8-dimensional space of spinors on a V_6 . We have

$$\mathbb{1} - \chi \bar{\chi} + \Gamma_7 \chi \bar{\chi} \Gamma_7 - \Gamma_a \chi \bar{\chi} \Gamma^a = 0 \quad (\text{A.1})$$

$$\Gamma^{bcd} \chi \bar{\chi} \Gamma_{abcd} \chi = 12 \Gamma_a \chi \quad (\text{A.2})$$

$$\Gamma^{bc} \chi \bar{\chi} \Gamma_{abc} \chi = -4 \Gamma_a \chi \quad (\text{A.3})$$

$$\Gamma^{bc} \chi \bar{\chi} \Gamma_7 \Gamma_{abc} \chi = 4 \Gamma_a \Gamma_7 \chi \quad (\text{A.4})$$

where the indices a, b, c and d run from 1 to 6. All the other matrix elements we need can be deduced from these three. From the three equations (2.10-2.12) we obtain six new ones by substituting for the supersymmetric spinor $\varepsilon(x, y, \alpha)$ the ansatz (2.13) and projecting out the components of χ and $\Gamma_7 \chi$.

$$\begin{aligned} & \left[\Delta^{1/2} \bar{\nabla}_\mu - \frac{\sqrt{2}}{6} f \Delta^2 \gamma_5 \gamma_\mu + \sqrt{2} f^{-4} f_\mu \gamma_5 - \frac{2}{3} f f^{-3} \Delta^2 f_\mu \gamma_5 \right] \varepsilon_1 \\ & + \left[\frac{1}{4} N^{-1} \Delta^{-1} (2_\alpha \Delta) \gamma_\mu \gamma_5 - \frac{\sqrt{2}}{3} N^{-1} f^{-3} (2_\alpha f_\mu) \gamma_5 \right] \varepsilon_2 = 0 \end{aligned} \quad ; \quad (\text{A.5})$$

$$\begin{aligned} & \left[\Delta^{1/2} \bar{\nabla}_\mu - \frac{\sqrt{2}}{6} f \Delta^2 \gamma_5 \gamma_\mu + \sqrt{2} f^{-4} f_\mu \gamma_5 + \frac{2}{3} f f^{-3} \Delta^2 f_\mu \gamma_5 \right] \varepsilon_2 \\ & - \left[\frac{1}{4} N^{-1} \Delta^{-1} (2_\alpha \Delta) \gamma_\mu \gamma_5 + \frac{\sqrt{2}}{3} N^{-1} f^{-3} (2_\alpha f_\mu) \gamma_5 \right] \varepsilon_1 = 0 \end{aligned} \quad ; \quad (\text{A.6})$$

$$\begin{aligned} & \left[N^{-1} \partial_\alpha - 2 \frac{\sqrt{2}}{3} N^{-1} f^{-3} (2_\alpha f_\mu) \gamma_5 \right] \varepsilon_1 \\ & + \left[\frac{\sqrt{2}}{12} f \Delta^2 - \sqrt{2} f^{-4} f_\mu \gamma_5 + \frac{4}{3} f f^{-3} \Delta^2 f_\mu \gamma_5 \right] \varepsilon_2 = 0 \end{aligned} \quad ; \quad (\text{A.7})$$

$$\begin{aligned} & \left[N^{-1} \partial_\alpha + 2 \frac{\sqrt{2}}{3} N^{-1} f^{-3} (2_\alpha f_\mu) \gamma_5 \right] \varepsilon_2 \\ & - \left[\frac{\sqrt{2}}{12} f \Delta^2 - \sqrt{2} f^{-4} f_\mu \gamma_5 - \frac{4}{3} f f^{-3} \Delta^2 f_\mu \gamma_5 \right] \varepsilon_1 = 0 \end{aligned} \quad ; \quad (\text{A.8})$$

$$\begin{aligned} & \left[3 f^{-1} - \frac{\sqrt{2}}{2} f \Delta^2 - 6 \sqrt{2} f^{-4} f_\mu \gamma_5 + 2 f f^{-3} \Delta^2 f_\mu \gamma_5 \right] \varepsilon_1 \\ & + \left[3 N^{-1} f^{-1} (2_\alpha f_\mu) + \sqrt{2} N^{-1} f^{-3} (2_\alpha f_\mu) \gamma_5 \right] \varepsilon_2 = 0 \end{aligned} \quad ; \quad (\text{A.9})$$

$$\begin{aligned} & \left[3f^{-1} + \frac{\sqrt{2}}{2} f \Delta^2 + 6\sqrt{2} f^{-4} f_x i\gamma_5 + 2ff^{-3} \Delta^2 f_x i\gamma_5 \right] \varepsilon_2 \\ & + \left[3N^{-1} f^{-1} (\partial_\alpha f) - \sqrt{2} N^{-1} f^{-3} (\partial_\alpha f_1) i\gamma_5 \right] \varepsilon_1 = 0 \end{aligned} \quad (A.10)$$

Our strategy was first to solve (A.9) and (A.10). For this purpose, we express the spinor $\varepsilon_1(x, \alpha)$ in terms of $\varepsilon_2(x, \alpha)$:

$$\varepsilon_1(x, \alpha) = \cos \alpha \left[A(\alpha) + B(\alpha) i\gamma_5 \right] \varepsilon_2(x, \alpha) \quad (A.11)$$

This gives us two inhomogeneous systems of two equations for the functions A and B by projecting out the chiral components of ε_2 . For instance, (A.9) leads to

$$\left[3f^{-1} - \frac{\sqrt{2}}{2} f \Delta^2 \right] \cos \alpha \cdot A + \left[6\sqrt{2} f^{-4} f_x - 2ff^{-3} \Delta^2 f_x \right] \cos \alpha \cdot B + 3 \frac{(\partial_\alpha f)}{Nf} = 0, \quad (A.12)$$

$$\left[3f^{-1} - \frac{\sqrt{2}}{2} f \Delta^2 \right] \cos \alpha \cdot B - \left[6\sqrt{2} f^{-4} f_x - 2ff^{-3} \Delta^2 f_x \right] \cos \alpha \cdot A + \sqrt{2} \frac{(\partial_\alpha f_1)}{Nf^3} = 0.$$

The solution of this system is given by (2.16). Then we have to verify that this solution is compatible with the system coming from (A.10). Indeed this is the case if $f\lambda = \mp \sqrt{2}/\sqrt{3}$, a constraint which results from (2.4) and (2.6). So (2.16) is a self-consistent solution.

As we noticed previously, the functions A and B become singular at $\alpha = \pi/2$ for $f\lambda$ positive. Nevertheless, as we shall see later, the final expression for the supersymmetric spinor ε is not singular at this point. So we can forget about it. In fact, instead of expressing ε_1 in terms of ε_2 , we can invert (A.11) and follow the same procedure: first solve (A.10), then check the compatibility with (A.9). This leads us to

$$\varepsilon_2(x, \alpha) = \cos \alpha \left[C(\alpha) + D(\alpha) i\gamma_5 \right] \varepsilon_1(x, \alpha) \quad (A.13)$$

with

$$\begin{aligned} C(\alpha, f\lambda) &= A(\alpha, -f\lambda) \\ D(\alpha, f\lambda) &= -B(\alpha, -f\lambda) \end{aligned} \quad (A.14)$$

In this pattern, the functions C and D are singular at $\alpha = \pi/2$ for $f\lambda$ negative. So, depending on the value of $f\lambda$, we may choose one decomposition or the other to avoid this spurious singularity. But the final result must be regular and independent of our choice. From now we shall restrict ourselves to the decomposition (A.11) and discuss the other possibility later.

The next step is to solve (A.5) and (A.6). As these equations contain an AdS covariant derivative, it is appropriate to use an AdS Killing spinor $\hat{\varepsilon}(x)$ -- defined by (2.18) -- in the necessary chiral decomposition (2.17) of ε_2 . Using it, equation (A.6) gives a homogeneous system of two equations between the functions $\mu(\alpha)$ and $\nu(\alpha)$:

$$\left[\frac{1}{2} m_4 \Delta^{1/2} - \frac{\sqrt{2}}{6} f \Delta^2 + \frac{1}{4} \frac{(\partial_\alpha \Delta)}{N \Delta} \cos \alpha \cdot A - \frac{\sqrt{2}}{3} \frac{(\partial_\alpha f_1)}{N f^3} \cos \alpha \cdot B \right] \mu + \left[\sqrt{2} f^{-4} f_x + \frac{2}{3} f f^{-3} \Delta^2 f_x - \frac{1}{4} \frac{(\partial_\alpha \Delta)}{N \Delta} \cos \alpha \cdot B - \frac{\sqrt{2}}{3} \frac{(\partial_\alpha f_1)}{N f^3} \cos \alpha \cdot A \right] \nu = 0 ; \quad (A.15)$$

$$\left[\frac{1}{2} m_4 \Delta^{1/2} + \frac{\sqrt{2}}{6} f \Delta^2 - \frac{1}{4} \frac{(\partial_\alpha \Delta)}{N \Delta} \cos \alpha \cdot A + \frac{\sqrt{2}}{3} \frac{(\partial_\alpha f_1)}{N f^3} \cos \alpha \cdot B \right] \nu + \left[\sqrt{2} f^{-4} f_x + \frac{2}{3} f f^{-3} \Delta^2 f_x - \frac{1}{4} \frac{(\partial_\alpha \Delta)}{N \Delta} \cos \alpha \cdot B - \frac{\sqrt{2}}{3} \frac{(\partial_\alpha f_1)}{N f^3} \cos \alpha \cdot A \right] \mu = 0 .$$

In the same way, equation (A.7) leads to a similar homogeneous system. These four equations allows us to express ν in term of μ if all the determinants vanish. Indeed it is the case and we get

$$\nu = -\sqrt{5} \mu \frac{\Delta^{-3/4} \left(\frac{9}{2} m_4 \lambda \Delta^{-3/2} - 3 \frac{\sqrt{2}}{2} f \lambda \right) + \sin \alpha \left(5 - 4 \sin^2 \alpha - 3 \frac{\sqrt{2}}{4} f \lambda m_4 \right)}{3 \Delta^{-3/4} (7 - 8 \sin^2 \alpha) + 3 \frac{\sqrt{2}}{2} f \lambda \sin \alpha} \quad (A.16)$$

Finally, the unknown function $\mu(\alpha)$ is fixed by (A.7) and (A.8). Here again, using (A.11) and (2.17), we end up with four first-order homogeneous differential equations. For instance, equation (A.8) gives

$$N^{-1} \partial_\alpha \mu - \frac{\sqrt{2}}{12} f \Delta^2 \cos \alpha \cdot (A \mu - B \nu) - 2 \frac{\sqrt{2}}{3} N^{-1} f^{-3} (\partial_\alpha f_x) \nu - \left(\sqrt{2} f^{-4} f_x + \frac{4}{3} f f^{-3} \Delta^2 f_x \right) \cos \alpha \cdot (A \nu + B \mu) = 0 , \quad (A.17)$$

$$N^{-1} \partial_a v - \frac{\sqrt{2}}{12} f \Delta^2 \cos \alpha \cdot (A v + B \mu) + 2 \frac{\sqrt{2}}{3} N^{-1} f^{-3} (\partial_a f) \mu + (\sqrt{2} f^{-4} f_x + \frac{4}{3} f f^{-3} \Delta^2 f_x) \cos \alpha \cdot (A \mu - B v) = 0 \quad (A.18)$$

At this stage we integrate (A.17) in order to find $\mu(\alpha)$. Then we verify by direct substitution of the function $\mu(\alpha)$ that (A.18) and the two other equations coming from (A.7) are satisfied.

Up to a multiplicate constant, the general solution of (A.17) is

$$\mu = \Delta^{7/8} \left[\Delta^{-3/4} - \frac{\sqrt{2}}{2} f \lambda \sin \alpha \right]^{1/2} \left[\frac{\sin \alpha}{7} \left(\frac{5}{2} m_4 \lambda - 2 \sqrt{2} f \lambda \right) - \Delta^{-3/4} \right] \quad (A.19)$$

and the final expression for the supersymmetric spinor $\varepsilon(x, y, \alpha)$ is given by

$$\begin{aligned} \varepsilon(x, y, \alpha) = \cos \alpha \left[(A \mu - B v) + i \gamma_5 (A v + B \mu) \right] \hat{\varepsilon}(x) \otimes \chi(y) \\ + \left[\mu + i \gamma_5 v \right] \hat{\varepsilon}(x) \otimes \gamma_7 \chi(y) \end{aligned} \quad (A.20)$$

We remind that we can also use (A.13) instead of (A.11). In that case we express ε_1 in terms of an AdS Killing spinor as:

$$\varepsilon_1(x, \alpha) = \left[\tilde{\mu} + i \gamma_5 \tilde{v} \right] \hat{\varepsilon}(x) \quad (A.21)$$

rather than the decomposition (2.17). Following the same method as previously, we find

$$\begin{aligned} \varepsilon(x, y, \alpha) = \cos \alpha \left[(C \tilde{\mu} - D \tilde{v}) + i \gamma_5 (C \tilde{v} + D \tilde{\mu}) \right] \hat{\varepsilon}(x) \otimes \gamma_7 \chi(y) \\ + \left[\tilde{\mu} + i \gamma_5 \tilde{v} \right] \hat{\varepsilon}(x) \otimes \chi(y) \end{aligned} \quad (A.22)$$

with the relations (A.14) and

$$\begin{aligned} \tilde{\mu}(f \lambda, m_4 \lambda) &= \mu(-f \lambda, -m_4 \lambda) \\ \tilde{v}(f \lambda, m_4 \lambda) &= -v(-f \lambda, -m_4 \lambda) \end{aligned} \quad (A.23)$$

But the two expressions for ε have to be the same and a careful calculation shows that -- up to a global sign which is irrelevant -- this is indeed the case. So the supersymmetric spinor can be expressed uniquely in terms of the two functions μ and v :

$$\begin{aligned} \mathcal{E}(x, y, \alpha) = & - \left[\mu(-p\lambda, -m_4\lambda) - i\gamma_5 \nu(-p\lambda, -m_4\lambda) \right] \hat{\mathcal{E}}(x) \otimes \chi(y) \\ & + \left[\mu(p\lambda, m_4\lambda) + i\gamma_5 \nu(p\lambda, m_4\lambda) \right] \hat{\mathcal{E}}(x) \otimes \Gamma_7 \chi(y) \end{aligned} \quad (\text{A.24})$$

with μ and ν given by (A.19) and (A.16). It is easy to verify that this expression is regular for $\alpha = \pi/2$ in both cases $f\lambda = \mp\sqrt{2}/\sqrt{3}$. Note also that there exists a solution whatever the sign of f -- the amplitude of the matter field -- is.

Finally we can ask ourselves what would happen if we made one of the transformations: $\Delta \rightarrow -\Delta$, $\rho \rightarrow -\rho$, $N \rightarrow -N$ or $f_1 \rightarrow -f_1$. In fact, nothing, provided we make some redefinitions of ε_1 and ε_2 and some changes of sign. For instance, if we make the transformation: ρ goes to $-\rho$ together with ε_1 on ε_2 and ε_2 on $-\varepsilon_1$, all the equations remain the same, leading to the final answer $\varepsilon = \varepsilon_2 \chi - \varepsilon_1 \Gamma_7 \chi$. Similarly, for $N \rightarrow -N$, we have to replace ε_1 by $-\varepsilon_1$ and ε_2 by ε_2 . For $f_1 \rightarrow -f_1$, nothing changes if $\nu \rightarrow -\nu$. Finally, for $\Delta \rightarrow -\Delta$, the solution is obtained by changing the sign of m_4 , i.e. replacing $\hat{\mathcal{E}}$ by $\gamma_5 \hat{\mathcal{E}}$.

Appendix B

We shall briefly indicate the basic ingredients needed to construct the two solutions mentioned in the main text. For more details we refer for instance to Ref. [15].

1) $SO(5)/SO(3) \otimes SO(2)$

The generators of $SO(5)$ are denoted by $M_{[ab]}$. They obey the commutation relations

$$[M_{ab}, M_{cd}] = \delta_{ac} M_{bd} - \delta_{ad} M_{bc} + \delta_{bd} M_{ac} - \delta_{bc} M_{ad} \quad (a, b, c, d = 1 \dots 5) \quad (B.1)$$

and we choose as generators H_i of $SO(3) \times SO(2)$ the combinations

$$\begin{aligned} X_1 &= \frac{1}{2} (M_{12} + M_{34}) & ; & \quad X_2 = \frac{1}{2} (M_{13} - M_{24}) \\ X_3 &= \frac{1}{2} (M_{14} + M_{23}) & ; & \quad Y = \frac{1}{2} (M_{14} - M_{23}) \end{aligned} \quad (B.2)$$

and as coset generators K_α

$$\begin{aligned} \mu_a &= M_{a5} \quad (a = 1 \dots 4) \\ Y_5 &= \frac{1}{2} (M_{34} - M_{12}) & ; & \quad Y_6 = \frac{1}{2} (M_{13} + M_{24}) \end{aligned} \quad (B.3)$$

If L denotes a representation of the coset space, the coset coframe e^α is locally defined as

$$L^{-1} dL = \underline{e}^\alpha K_\alpha + \underline{\omega}^i H_i$$

and (up to an overall scale factor) the invariant metrics are given by

$$\underline{g} = - \sum_{\alpha, \beta=1}^4 \underline{e}^\alpha \otimes \underline{e}^\beta \delta_{\alpha\beta} - \lambda^2 \sum_{i,j=5,6} \underline{e}^i \otimes \underline{e}^j \delta_{ij} \quad (B.4)$$

Instead of this metric we may also consider a rescaled coframe

$$\begin{aligned} \varepsilon^a &= e^a \quad (a = 1, 2, 3, 4) \\ \text{but} \quad \varepsilon^b &= \lambda e^b \quad (b = 5, 6) \end{aligned} \quad (B.5)$$

With respect to this new basis we obtained as curvature 2-forms:

$$\begin{aligned}
R^1_2 &= \left(1 - \frac{3\lambda^2}{4}\right) \varepsilon^1 \wedge \varepsilon^2 + \frac{\lambda^2}{4} \varepsilon^3 \wedge \varepsilon^4, \\
R^1_3 &= \left(1 - \frac{3\lambda^2}{4}\right) \varepsilon^1 \wedge \varepsilon^3 - \frac{\lambda^2}{4} \varepsilon^2 \wedge \varepsilon^4, \\
R^1_4 &= \varepsilon^1 \wedge \varepsilon^4 - \frac{\lambda^2}{2} \varepsilon^2 \wedge \varepsilon^3 + \left(1 - \frac{\lambda^2}{2}\right) \varepsilon^5 \wedge \varepsilon^6, \\
R^2_3 &= \varepsilon^2 \wedge \varepsilon^3 - \frac{\lambda^2}{2} \varepsilon^1 \wedge \varepsilon^4 - \left(1 - \frac{\lambda^2}{2}\right) \varepsilon^5 \wedge \varepsilon^6, \\
R^2_4 &= \left(1 - \frac{3\lambda^2}{4}\right) \varepsilon^2 \wedge \varepsilon^4 - \frac{\lambda^2}{4} \varepsilon^1 \wedge \varepsilon^3, \\
R^3_4 &= \left(1 - \frac{3\lambda^2}{4}\right) \varepsilon^3 \wedge \varepsilon^4 + \frac{\lambda^2}{4} \varepsilon^1 \wedge \varepsilon^2, \\
\lambda R^5_1 &= \frac{\lambda^2}{4} \varepsilon^5 \wedge \varepsilon^1 + \frac{1}{4} (2 - \lambda^2) \varepsilon^6 \wedge \varepsilon^4, \\
\lambda R^5_2 &= \frac{\lambda^2}{4} \varepsilon^5 \wedge \varepsilon^2 - \frac{1}{4} (2 - \lambda^2) \varepsilon^6 \wedge \varepsilon^3, \\
\lambda R^5_3 &= \frac{\lambda^2}{4} \varepsilon^5 \wedge \varepsilon^3 + \frac{1}{4} (2 - \lambda^2) \varepsilon^6 \wedge \varepsilon^2, \\
\lambda R^5_4 &= \frac{\lambda^2}{4} \varepsilon^5 \wedge \varepsilon^4 - \frac{1}{4} (2 - \lambda^2) \varepsilon^6 \wedge \varepsilon^1, \\
\lambda R^6_1 &= \frac{\lambda^2}{4} \varepsilon^6 \wedge \varepsilon^1 - \frac{1}{4} (2 - \lambda^2) \varepsilon^5 \wedge \varepsilon^4, \\
\lambda R^6_2 &= \frac{\lambda^2}{4} \varepsilon^6 \wedge \varepsilon^2 + \frac{1}{4} (2 - \lambda^2) \varepsilon^5 \wedge \varepsilon^3, \\
\lambda R^6_3 &= \frac{\lambda^2}{4} \varepsilon^6 \wedge \varepsilon^3 - \frac{1}{4} (2 - \lambda^2) \varepsilon^5 \wedge \varepsilon^2, \\
\lambda R^6_4 &= \frac{\lambda^2}{4} \varepsilon^6 \wedge \varepsilon^4 + \frac{1}{4} (2 - \lambda^2) \varepsilon^5 \wedge \varepsilon^1, \\
R^5_6 &= \lambda^{-2} \varepsilon^5 \wedge \varepsilon^6 - \left(1 - \frac{\lambda^2}{2}\right) \varepsilon^2 \wedge \varepsilon^3 + \left(1 - \frac{\lambda^2}{2}\right) \varepsilon^1 \wedge \varepsilon^4.
\end{aligned} \tag{B.6}$$

In order to obtain an Einstein metric the scaling parameter must satisfy the equation

$$3 - \lambda^2 = \lambda^2 + \frac{1}{\lambda^2}$$

$$\text{i.e.} \quad \lambda^2 = 1 \quad \text{or} \quad \lambda^2 = \frac{1}{2} \tag{B.7}$$

It is easy to check that when $\lambda^2 = 1$ the integrability conditions of (2.7) admit no other solution than $\chi = 0$. When $\lambda^2 = \frac{1}{2}$ we obtain two relations

$$\Gamma_{56} \chi = \Gamma_{41} \chi = \Gamma_{23} \chi \tag{B.8}$$

These equations have two non-trivial solutions, related by a Γ_7 multiplication, which satisfy respectively the Killing spinorial equations for each sign of $m_6 = \pm \sqrt{2}/2$.

2) $SU(3)/U(1) \times U(1)$

As generators of $SU(3)$ we use the usual Gell-Mann λ_a matrices with the $U(1) \times U(1)$ subgroup generated by λ_3 and λ_8 . The coframe on the coset may be rescaled in three blocks corresponding to the indices (1,2), (4,5) and (6,7). The Ricci tensor components are diagonal and, up to an overall factor, they are given by

$$\begin{aligned} R_1^1 &= R_2^2 = - \frac{1}{\beta^2 \gamma^2} [6\beta^2 \gamma^2 - \beta^4 - \gamma^4 + 1] \\ R_4^4 &= R_5^5 = - \frac{1}{\beta^2 \gamma^2} [6\gamma^2 + \beta^4 - \gamma^4 - 1] \\ R_6^6 &= R_7^7 = - \frac{1}{\beta^2 \gamma^2} [6\beta^2 - \beta^4 + \gamma^4 - 1] \end{aligned} \quad (B.9)$$

where β and γ are rescaling factors multiplying the coset directions ($\underline{e}^4, \underline{e}^5$) and ($\underline{e}^6, \underline{e}^7$).

Imposing the Einstein conditions

$$R^a_b = S^a_b$$

we obtain three solutions

$$\begin{aligned} \beta^2 &= \gamma^2 = 1 & ; \\ \beta^2 &= 1, \gamma^2 = 2 & ; \\ \beta^2 &= \gamma^2 = \frac{1}{2} & - \end{aligned} \quad (B.10)$$

the last two being actually equivalent.

The integrability conditions can only be satisfied in the first case. They impose the conditions (after a relabelling of the Γ matrices)

$$\Gamma_{56} \chi = \Gamma_{12} \chi = \Gamma_{43} \chi \quad (B.11)$$

Just as in the previous case we may check that these conditions are satisfied by a spinor depending on one parameter once it is normalized. Then the Killing equations fix completely the solution.

Finally we would also like to indicate another property of the two geometries described here: they each admit a Penrose-Floyd tensor [16]

$$\pi_{ab} = \bar{\chi} \Gamma_a^c \Gamma_{cb} \chi \quad (\text{B.12})$$

whose relevance for the resolution of the Dirac equation is discussed in [17]

Moreover, this tensor defines an almost complex structure because

$$\pi_{ab} \pi^{bc} = -\delta_a^c \quad (\text{B.13})$$

Nevertheless this almost complex structure is non-integrable as the Nijenhuis tensor

$$\pi_a^d \pi_{[b}^c{}_{;d]} - \pi_b^d \pi_{[a}^c{}_{;d]} = -2 \bar{\chi} \Gamma_{ab}^c \chi \quad (\text{B.14})$$

is non-vanishing.

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