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CP RESTRICTIONS ON QUARK MASS MATRICES

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A B S T R A C T

We study the general implications of CP invariance on quark mass matrices within the standard $SU(2) \times U(1)$ gauge theory, with an arbitrary number of fermion families. In particular, we derive a set of necessary conditions for CP invariance that the quark mass matrices must satisfy when written in the weak eigenstate basis. These conditions are expressed in terms of traces of powers of the mass matrices. The violation of any of these conditions implies CP non-conservation by the charged current weak interactions. In the case of three generations, we find that one of these conditions is also sufficient for CP invariance.

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In the standard model of electroweak interactions¹⁾, quark-mixing and CP violation are usually described by the Kobayashi-Maskawa matrix²⁾, which provides a parametrization of the charged-current interactions in the quark mass eigenstate basis. In this basis a non-zero KM phase and three non-zero mixing angles are necessary and sufficient for CP violation³⁾.

In this letter we would like to point out that there is an alternative way of testing the CP properties of a unified gauge theory, which may be particularly useful for model building. The method consists of studying the CP properties of a gauge model after spontaneous symmetry breaking in the weak eigenstate basis. The latter is the basis in which a gauge theory is originally written, with diagonal charged currents and in general with an arbitrary quark mass term. Although our method of analyzing the restrictions imposed by CP invariance has quite a general application, for definiteness we will consider the electroweak model based on $SU(2) \times U(1)$. We will derive a set of conditions necessary for CP invariance, applicable to an arbitrary number of generations. These conditions will be expressed in terms of traces involving powers of non-diagonal quark mass matrices written in the weak eigenstate basis. The violation of any of these conditions is a definite indication of CP breaking. In the case of three generations we identify a single condition which is both necessary and sufficient for CP conservation. The practical use of these conditions for constructing and testing models of CP violation stems from the fact that they can be applied to the quark mass matrices directly, without the need to diagonalize them. In the standard model with N generations, the quark mass term and the charged-current interaction term are given by:

$$\mathcal{L} = (\bar{u} \ \bar{d})_L \begin{pmatrix} M_u & 0 \\ 0 & M_d \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}_R + g \bar{u}_L \gamma_\mu d_L W^\mu + h.c. \quad (1)$$

The left- and right-handed u and d fields represent charge 2/3 (up) and -1/3 (down) quark weak eigenstates respectively, each given by an N component vector in generation space. M_u and M_d are the N-dimensional quark mass matrices, obtained after spontaneous symmetry breaking. We suggest that the simplest test of CP invariance of this Lagrangian consists in investigating whether there are CP transformations for the quark fields of the type:

$$\begin{aligned} u_L &\rightarrow U_L C u_L^* & ; & \quad u_R \rightarrow U_R^u C u_R^* \\ d_L &\rightarrow U_L C d_L^* & ; & \quad d_R \rightarrow U_R^d C d_R^* \end{aligned} \quad (2)$$

which leave the Lagrangian invariant. In (2), U_L , $U_R^{u,d}$ are N -dimensional unitary matrices and C is the Dirac charge conjugation matrix. Note that u_L and d_L have to transform in the same way, so that the charged-current interactions are left invariant. In order for the Lagrangian to be invariant under (2), the quark mass matrices have to satisfy the conditions:

$$\begin{aligned} U_L^\dagger M_u U_R^u &= M_u^* \\ U_L^\dagger M_d U_R^d &= M_d^* \end{aligned} \quad (3)$$

If we introduce now the Hermitian matrices $H_d = M_d M_d^\dagger$, $H_u = M_u M_u^\dagger$, it follows from (3) that

$$\begin{aligned} U_L^\dagger H_u U_L &= H_u^* = H_u^T \\ U_L^\dagger H_d U_L &= H_d^* = H_d^T \end{aligned} \quad (4)$$

The reverse is also true, namely if a matrix U_L satisfying Eqs. (4) exists, then one can easily show that one can find matrices U_R^u , U_R^d satisfying Eqs. (3). Therefore, one concludes that the existence of a matrix U_L satisfying both equations (4) is a necessary and sufficient condition for the Lagrangian given by (1) to be CP invariant. At this stage, it is worthwhile emphasizing that conditions (4) are independent of the particular weak basis which one chooses to write the matrices H_u , H_d . In other words, if one finds a solution of Eq. (4) in one particular basis, then there will be a solution in any other weak basis. Indeed, let $H'_{u,d}$, $H_{u,d}$ be the Hermitian quark masses in two different weak bases related by:

$$H_{u,d} = V_L^\dagger H'_{u,d} V_L \quad (5)$$

Then, if U_L is a solution of Eq. (4), there will be a matrix W satisfying the relation:

$$W^\dagger H'_{u,d} W = H'^T_{u,d} \quad (6)$$

From (4) and (5), it follows that the matrix W is given by:

$$W = V_L U_L V_L^T \quad (7)$$

It is clear that although Eqs. (4) constitute a necessary and sufficient condition for CP invariance, they are of little practical use in that form. If one attempts to evaluate invariants of both sides of Eq. (4), one obviously obtains trivial identities. In order to derive non-trivial conditions on H_u, H_d , we first note that Eqs. (4) can be multiplied by themselves an arbitrary number of times, to obtain similar equations for powers of H_u, H_d . One can then easily derive:

$$U_L^\dagger [H_u^p, H_d^q] U_L = -[H_u^p, H_d^q]^T \quad (8)$$

CP invariance implies that this equation holds for any pair of integers p, q . However, notice that if it is obeyed for all $p, q < N-1$, then it will also hold for higher powers by the Cayley-Hamilton theorem. Let us designate by $C^{p,q}$ the n -dimensional anti-Hermitian matrix $[H_u^p, H_d^q]$ and consider its characteristic equation:

$$\det(\lambda I - C^{p,q}) = 0$$

$$\lambda^N - c_1 \lambda^{N-1} + c_2 \lambda^{N-2} - \dots + (-1)^N c_N = 0 \quad (9)$$

The coefficients c_i can be easily calculated since they are the sum of the determinants of the i -rowed principal coaxial minors⁴⁾, i.e., the minors in which the indices of the rows are the same as those of the columns. Of course, c_N is the determinant of the matrix $C^{p,q}$ itself. The characteristic equation of a matrix has the important property that it is invariant both under similarity transformations and under transposition. In view of this, one concludes from Eq. (8) that CP invariance forces all odd coefficients of the characteristic equation of $C^{p,q}$ to vanish:

$$C_i = 0 \quad i = 1, 3, 5 \dots \quad (10)$$

Note that c_1 stands for the trace of $C^{p,q}$, which automatically vanishes for a commutator. Therefore the non-trivial constraints on H_u, H_d arise from (10) for $i > 3$.

Although the study of CP invariance constraints on H_d, H_u can be done through the analysis of the coefficients of the characteristic equation of $C^{p,q}$, as we have just done, for practical purposes it is more advantageous to use another set of conditions which will only involve the evaluation of traces of matrices. Indeed, if we take Eq. (8), multiply it by itself an odd number of times and evaluate traces, we obtain

$$\text{tr} [H_u^p, H_d^q]^r = 0 \quad r \text{ odd} \quad (11)$$

These are the necessary conditions for CP invariance which we were looking for. They are valid for an arbitrary number of families and the violation of any one of them guarantees the existence of CP non-conservation.

At this point, one may ask whether there is a relationship between the invariants of Eq. (12) and the coefficients of the characteristic equation of $C^{p,q}$. Indeed such a relationship exists and one can show for example that the following relationship holds:

$$c_3 = \frac{1}{3} \text{tr} [H_u^p, H_d^q]^3 \quad (12)$$

where c_3 is the third coefficient of the characteristic equation of $C^{p,q}$, as defined in Eq. (9).

Next we analyze the implications of the necessary conditions (11) in the simplest cases.

Three generations

In the case of $N = 3$, Eq. (11) provides only a single condition, with $p = q = 1$ and $r = 3$. This is due to the fact that for 3×3 matrices the traces involving higher powers may be easily shown to be proportional to the trace for $p = q = 1$. For instance:

$$\text{tr} [H_u^2, H_d]^3 = \frac{1}{3} [\text{tr} H_u^3 - \text{tr} (H_u)^3] \text{tr} [H_u, H_d]^3 \quad (13)$$

In the case of three generations one is therefore left with one independent necessary condition for CP invariance⁵⁾

$$\text{tr} [H_u, H_d]^3 = 0 \quad (14)$$

We will show next that Eq. (14) is also sufficient for CP invariance, namely it guarantees the existence of a unitary matrix U_L satisfying Eqs. (4). In order to prove this, we will explicitly construct this matrix U_L in the weak basis where one of the two quark mass matrices, say H_u , is diagonal. In this basis, Eq. (14) reads:

$$(m_t^2 - m_c^2)(m_c^2 - m_u^2)(m_t^2 - m_u^2) \text{Im} [(H'_d)_{12} (H'_d)_{23} (H'_d)_{31}] = 0 \quad (15)$$

where H'_d denotes the down quark mass matrix in the basis where H_u is diagonal and m_u, m_c, m_t are the masses of the up quarks. We will first consider that there is no degeneracy in these quark masses and will later examine the case of degeneracy. If there is no degeneracy, Eq. (15) implies a phase relation

$$\text{Im} [(H'_d)_{12} (H'_d)_{23} (H'_d)_{31}] = 0 \quad (16)$$

With this restriction, the phases of all elements of H'_d may be expressed as:

$$\text{phase} (H'_d)_{jk} = (\alpha_k - \alpha_j) \pmod{\pi} \quad (17)$$

In this case, a unitary matrix satisfying Eqs. (7) can be explicitly constructed and is given by:

$$(U_L)_{jk} = \delta_{jk} e^{2i\alpha_j} \quad (18)$$

We consider next the degenerate case. If two masses, say m_i, m_j are degenerate, then one can find a weak basis where H_u is diagonal and $(H'_d)_{ij} = 0$. In this case, the parametrization (17) is trivially satisfied and therefore a matrix U_L satisfying Eqs. (7) exists. This completes the proof that Eq. (14) is indeed a necessary and sufficient condition for CP invariance in the case of three generations.

More than three generations

We have seen that Eq. (11) constitutes a set of necessary conditions for CP invariance, applicable to an arbitrary number of generations. However, these conditions are obviously not all independent, and it is natural to ask whether it is possible to find a minimum subset of invariant conditions which would be not only necessary but also sufficient for CP invariance, for an arbitrary number of generations. From our knowledge of the analysis of quark mixing à la Kobayashi-Maskawa, we expect to have $(N-1)(N-2)/2$ independent conditions. Let us assume an arbitrary number of generations and consider again Eq. (4), in the weak basis where H_u is diagonal. In this basis and assuming H_u to be non-degenerate (the degenerate case can be easily treated) a unitary matrix U_L satisfying the first of Eqs. (4) is necessarily diagonal:

$$U_L = \text{diag.} (e^{i\delta_1} \dots e^{i\delta_N}) \quad (19)$$

In order for this matrix U_L to satisfy the second of Eqs. (4), the matrix H_d would have to be of the form

$$\text{phase } (H_d)_{ij} = \frac{1}{2} (\delta_i - \delta_j) \quad (\text{mod. } \pi) \quad (20)$$

where the δ_i are arbitrary. In other words, for an arbitrary number of generations, a necessary and sufficient condition for CP invariance is that H_d has cyclic phases, in the basis where H_u is diagonal. Since H_d is a N -dimensional Hermitian matrix, it has in general $N(N-1)/2$ independent phases and therefore Eq. (20) implies $(N-1)(N-2)/2$ constraints, as expected. In order to illustrate the difficulties one encounters in finding a minimum subset of necessary conditions which would be also sufficient for CP invariance, we will next analyze in some detail the case of four generations. The task is to look for a set of invariant conditions which imply Eq. (20) in the basis where H_u is diagonal. In this basis, Eq. (11), with $p = q = 1$, $r = 3$ and four generations, implies:

$$m_{123} I_{123} + m_{124} I_{124} + m_{134} I_{134} + m_{234} I_{234} = 0 \quad (21)$$

where

$$I_{ijk} \equiv \text{Im} (H_{ij}^d H_{jk}^d H_{ki}^d); \quad m_{ijk} \equiv (m_j^2 - m_i^2)(m_k^2 - m_j^2)(m_k^2 - m_i^2)$$

and m_i denote up quark masses. Similarly, from Eq. (11), with $p = 2$, $q = 1$, $r = 3$, one obtains:

$$m'_{123} I_{123} + m'_{124} I_{124} + m'_{134} I_{134} + m'_{234} I_{234} = 0 \quad (22)$$

where

$$m'_{ijk} = (m_j^4 - m_i^4)(m_k^4 - m_j^4)(m_k^4 - m_i^4)$$

Note that since in the case of four generations there are only three physical phases, not all the I_{ijk} are independent. The simplest way to see this is by noting that in a basis where H_u is diagonal, one still has the freedom to make a further transformation:

$$H_d \longrightarrow K^* H_d K \quad (23)$$

where K is a diagonal unitary matrix chosen so that H_u remains diagonal and $(H_d)_{ij}$ are real. One can then express the I_{ijk} as:

$$I_{123} = H_{12} H_{23} H_{31} \sin \alpha_{23} ; I_{124} = H_{12} H_{24} H_{14} \sin \alpha_{24}$$

$$I_{134} = H_{13} H_{34} H_{14} \sin \alpha_{34} ; I_{234} = H_{23} H_{34} H_{24} \sin(\alpha_{23} + \alpha_{34} - \alpha_{24}) \quad (24)$$

where H_{ij} stand for the modulus of $(H_d)_{ij}$ and α_{ij} for the respective phase. Substituting (24) into (21) and (22), one obtains two equations in terms of the three independent phases α_{23} , α_{24} , α_{34} . A third equation can be obtained by considering, for example, Eq. (11), with $p = 1$, $q = 2$, $r = 3$. However, due to the non-linearity of these three equations, one cannot conclude that they imply $\alpha_{ij} = n\pi$, as would be needed in order to prove that the three invariant conditions considered above are sufficient for CP invariance.

In conclusion, we have shown that the necessary and sufficient conditions for having CP invariance in the standard model, with an arbitrary number of generations, is that the Hermitian matrix H_d has cyclic phases in the weak basis where H_u is diagonal. We have also derived a set of invariant necessary conditions for CP invariance, for an arbitrary number of generations. These necessary invariant conditions are useful in constructing and testing models of CP breaking, since on the one hand they do not require the diagonalization of quark mass matrices and on the other hand, the violation of any one of them is a

clear indication of CP non-conservation. In the case of three generations, there is only one independent invariant necessary condition which is also sufficient for CP conservation.

The problem of finding a minimum set of invariant necessary conditions which would be also sufficient for CP invariance, for an arbitrary number of generations, remains an open question.

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